M E T U
Department of Mathematics

	Math 112: Discrete Math Make-up Exam	
Code : *Math 112* Acad. Year: *2009-2010* Semester : *Spring* Instructor : *KZ, AB, ÖK* Date : *14.06.2010* Time : *9.30* Duration : *110 minutes*	Last Name : Name : Student No.: Department: 6 QUESTIONS ON 4 PAGES TOTAL 60 POINTS	

1	2	3	4	5	6		

PLEASE GIVE SOLUTIONS AND REASONS, NOT ONLY ANSWERS!
ALL GRAPHS ARE LOOP-FREE AND THEY HAVE NO MULTIPLE EDGES.

1. (9 pts) 13 balls are distributed among 4 children.
(a) Is there a child who receives at least 4 balls? Why?

This is a generalization of Pigeonhole Principle. Assume otherwise. Then each child would receive at most 3 balls. Therefore only 12 balls can be distributed. Contradiction! There must be a child who receives at least 4 balls.

(b) How many distributions are there if all balls are different?

There are 4 options for each ball.

Answer: $\boxed{4^{13}}$

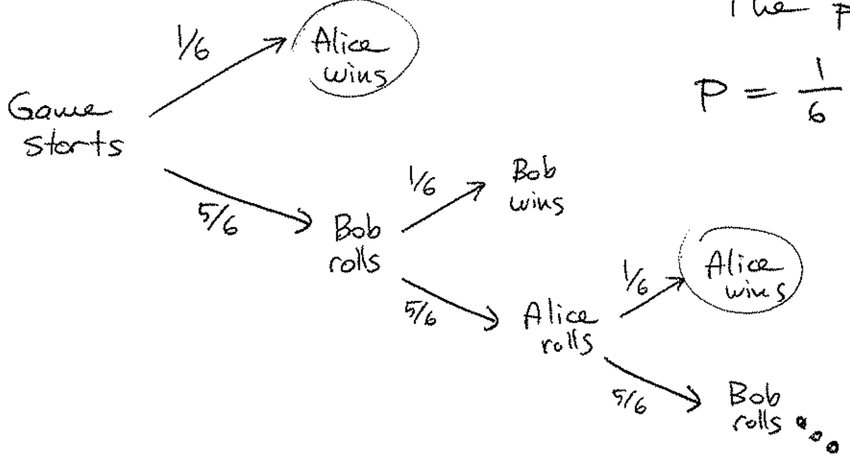
(c) How many distributions are there if there are 8 identical red balls and 5 identical blue balls?

The 8 red balls can be distributed in

$\binom{8+4-1}{8}$ different ways.

Answer: $\boxed{\binom{11}{8} \cdot \binom{8}{5}}$

2. (8 pts) Alice and Bob are taking turns to throw a die. Whoever throws a 6 first wins the game. If Alice is the first to throw the die, what is the probability that she wins the game? (Hint: You will get an infinite series as an answer, compute its value.)



The probability that Alice wins

$$P = \frac{1}{6} + \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} + \cdots$$

$$= \frac{1}{6} \cdot \sum_{n=0}^{\infty} \left(\frac{25}{36}\right)^n \qquad \sum_{n=0}^{\infty} \left(\frac{25}{36}\right)^n = \frac{1}{1-\frac{25}{36}}$$

$$= \frac{1}{6} \cdot \frac{36}{11}$$

$$= \boxed{\frac{6}{11}}$$

3. (12 pts) A license plate consists of two letters followed by three digits. Letters are chosen from the 26 letters of the English alphabeth and digits are chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) How many plates are there with no symbol repeated? (A symbol is a letter or a digit.)

___ ___   ___ ___ ___

26   25   9   8   7

Answer : $\boxed{P(26,2) \cdot P(9,3)}$

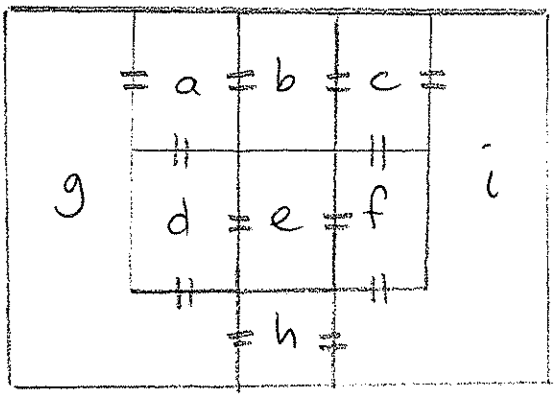
(b) How many of the plates have the letter U and the digit 2?

Let c_1 : U does not appear

c_2 : 2 does not appear

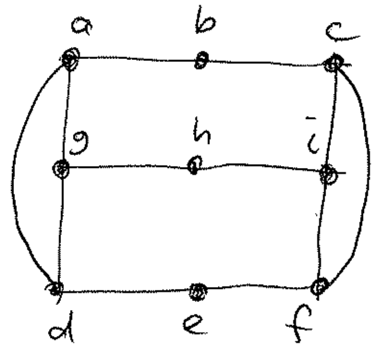
$$N(\bar{c}_1\bar{c}_2) = 26^2 \cdot 9^3 - (\underbrace{25^2 \cdot 9^3}_{\text{U does not appear}} + \underbrace{26^2 \cdot 8^3}_{\text{2 does not appear}}) + \underbrace{25^2 \cdot 8^3}_{\text{U and 2 does not appear.}}$$

4 (10 pts) The plan of a house is given below. Determine whether it is possible to take a walk in the house and visit every room exactly once, and return to the room where you started.



Let's consider the following graph

G



Now the question reduces to finding a Hamilton cycle in G.

For details see your textbook.
(Example 11.27 Grimaldi)

5. (5 pts) Prove that there is no graph with at least 2 vertices, in which every vertex has a different degree.

Let $G=(V,E)$ with $|V|=n$. Assume every vertex has a different degree. The maximum possible degree of a vertex is $n-1$. This forces that the degree of vertices are as follows:

$0, 1, 2, ---, n-1$

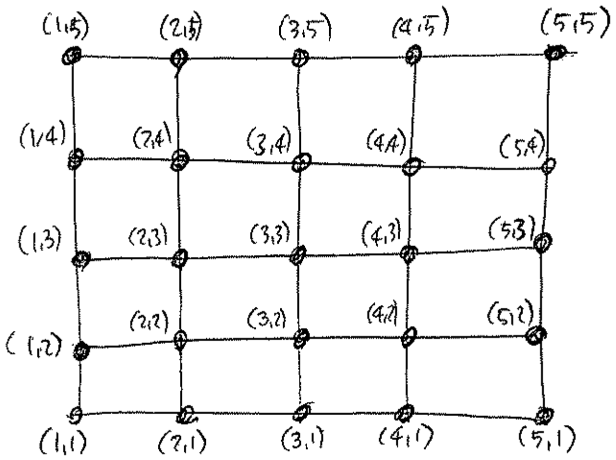
isolated vertex

a vertex connected to all other vertices

This is a contradiction. Therefore the degree of at least two vertices must be the same.

6. (16 pts) Let G be the graph defined as $V = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} : 1 \leqslant a, b \leqslant 5\}$ and two vertices are joined by an edge if and only if they are 1 unit apart.

(a) Draw G.



(b) Show that G is bipartite. Write V_1 and V_2 explicitly.

$V_1 = \{(a,b) \in V : a+b \text{ even}\}$

$V_2 = \{(a,b) \in V : a+b \text{ odd}\}$

There is an edge $\{v_1, v_2\}$ in G only if $v_1 \in V_1$ and $v_2 \in V_2$.

(c) Find a Hamilton path on G from $(1, 5)$ to $(5, 1)$.

$(1,5) \rightarrow \cdots \rightarrow (5,5)$

$\downarrow$

$(1,4) \leftarrow \cdots \leftarrow (5,4)$

$\downarrow$

$(1,3) \rightarrow \cdots \rightarrow (5,3)$

$\downarrow$

$(1,2) \leftarrow \cdots \leftarrow (5,2)$

$\downarrow$

$(1,1) \rightarrow \cdots \rightarrow (5,1)$

(d) Show that G does not have a Hamilton cycle.

G is bipartite with 25 vertices. Note that $|V_1| = 13$ and $|V_2| = 12$. G does not have a Hamilton cycle since $13 \neq 12$.