METU Mathematics Department
MATH 112: Exercise set IV

1. If eight fair dice are rolled, what is the probability that
   (a) all six numbers appear?
   (b) the sum is 12?

2. What is the probability that none of the letters is in its original position when we rearrange the letters in LAPTOP?

3. Let $x_1, x_2 \ldots, x_n$ be arbitrary integers. Show that
$$T(i,k) = x_i + x_{i+1} + \cdots + x_k$$
is divisible by n for some i and k, $1 \leq i \leq k \leq n$.

4. Let S be a set of $n+1$ distinct positive integers less than or equal to $2n$. Show that there exist two distinct elements $x, y \in S$ such that $\gcd(x, y) = 1$.

5. For an arbitrary integer $n > 0$, show that there exists a number divisible by n that contains only the digits 7 and 0.

6. (a) Show that if n pigeonholes are occupied by less than $n(n-1)/2$ pigeons then there exist two holes with the same number of pigeons (possibly empty).
   (b) Determine if it is possible to distribute 70 coins between 12 children so that each child gets a different number of coins provided that
      i. a child may receive no coins.
      ii. each child must receive a coin.

7. What is the smallest value of n such that whenever $S \subset \mathbf{Z}^+$ and $|S| = n$, then there exist three distinct elements $x, y, z \in S$, each of which has the same remainder upon division by some fixed integer $m > 0$.

8. If 10 integers are selected from $\{2, 3, 4, \ldots, 1000\}$, prove that there are at least two, say x and y, such that $0 < |\log_2(x) - \log_2(y)| < 1$.

9. During the first six weeks of her freshman year in METU, Pelin sends out one letter each day but no more than 60 letters in total. Show that there is a period of consecutive days during which she sends out exactly 23 letters.

10. Construct a sequence of 16 distinct positive integers that has no increasing or decreasing subsequence of length 5. Can you construct such a sequence with 17 integers?

11. Let S be a set of five positive integers, the maximum of which is at most 9. Prove that the sum of the elements in all the nonempty subsets of S cannot be all distinct.

12. Let p be an odd prime. Show that there exists a positive integer n such that $2^n - 1$ is divisible by p.