Properties of a Fine-Resolution Frequency Estimator Using an Arbitrary Number of DFT Coefficients

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Abstract

This document investigates the properties of a fine-resolution frequency estimator which uses an arbitrary number of DFT coefficients. A detailed derivation of the estimator is first presented. An analysis for the mean-square-error (MSE) of the estimator is made under high signal to noise ratio (SNR) conditions. Cross-covariance between the estimates using different (discrete Fourier transform) DFT bins is also derived under high-SNR assumption. Details on the fusion rule are presented by approximating the high-SNR MSE and cross-correlation via a small $|\delta|$ assumption. The quality of the approximate analytical MSE and cross-covariance expressions is evaluated via simulations. Some results on the number of DFT bins to use in the fused estimate are given. Finally, a Matlab implementation of the estimator is provided.

Contents

1 Preliminaries **2**

2 Derivation of the Estimator **2**

3 MSE for the Estimator $\widehat{\delta}_k$ under High-SNR **3**

4 Cross-Covariance Between the Estimates $\widehat{\delta}_k$ and $\widehat{\delta}_{k'}$ $(k \neq k')$ under High-SNR **7**

5 Fusion of the Estimates **8**
5.1 MSE and Cross-Covariance Approximation 8
5.2 Fusion Rule 9
5.3 Approximate MSE of the Fused Estimate 10

6 Simulations Related to the Approximations **10**
6.1 MSE Approximations 10
6.2 Cross-Covariance Approximation 10

7 How many bins to use? **12**

8 Matlab Implementation **15**

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1 Preliminaries

We consider a noisy complex exponential signal $r[\cdot]$ of unknown amplitude, phase and frequency given as

$$r[n] = Ae^{j(2\pi fn+\phi)} + w[n], \qquad n = 0, \ldots, N-1. \tag{1}$$

The frequency variable f in (1) is the normalized frequency defined over the interval $[0, 1)$. The frequency is defined in terms of the discrete Fourier transform (DFT) bins, that is $f = (k_p+\delta)/N$ where k_p is an integer in $[0, N-1]$ and δ is a real number in $-1/2 < \delta < 1/2$. The white noise $w[\cdot]$ is circularly symmetric complex-valued Gaussian distributed with zero-mean and variance σ_w^2, that is, $w[n] \sim \mathcal{CN}(0, \sigma_w^2)$. The signal-to-noise ratio (SNR) is defined as $\text{SNR} = A^2/\sigma_w^2$.

The aim is the estimation of the frequency $f = (k_p+\delta)/N$ from the samples of the signal $r[\cdot]$. The estimation is carried out in two stages. The first stage calculates that DFT of the signal $r[\cdot]$ and finds the index of the bin with the maximum magnitude which gives an estimate of k_p, the coarse part of the frequency. In the second stage, the fine-resolution frequency estimate is calculated using the DFT samples via the formula

$$\widehat{\delta}_k = \frac{N}{\pi}\tan^{-1}\left(\tan\left(\frac{\pi}{N}k\right)\text{Re}\left\{\frac{R[k_p+k]e^{-j\frac{\pi}{N}k} - R[k_p-k]e^{j\frac{\pi}{N}k}}{R[k_p+k]e^{-j\frac{\pi}{N}k} + R[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2R[k_p]}{\cos\left(\frac{\pi}{N}k\right)}}\right\}\right) \tag{2}$$

which gives an estimate of δ using 3 DFT bins, that is, $R[k_p]$ and two bins which are k bins away from k_p. It is possible to fuse the estimates $\widehat{\delta}_k$, $k = 1, \ldots, N/2-1$ to obtain a final fused estimate $\widehat{\delta}_F$. This document investigates the properties of the fine-resolution frequency estimator $\widehat{\delta}_k$ given in (2) and the fused estimator $\widehat{\delta}_F$.

2 Derivation of the Estimator

We consider the noiseless DFT samples $R[k_p + k]$ where

$$R[k_p+k] = Ae^{j\phi}e^{j\pi\frac{N-1}{N}(\delta-k)}\frac{\sin(\pi(\delta-k))}{\sin(\pi(\delta-k)/N)} \tag{3}$$

and k_p is index of the maximum magnitude bin. Substituting $k = 0$ into (3), we get

$$R[k_p] = Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}\frac{\sin(\pi\delta)}{\sin(\pi\delta/N)}. \tag{4}$$

We now define the ratio $\gamma_k \triangleq \frac{R[k_p+k]}{R[k_p]}$ which can be given as

$$\gamma_k = e^{-j\pi\frac{N-1}{N}k}\frac{\sin(\pi(\delta-k))}{\sin(\pi(\delta-k)/N)}\frac{\sin(\pi\delta/N)}{\sin(\pi\delta)}. \tag{5}$$

Expanding the sine functions in (5), we obtain

$$\gamma_k = e^{-j\pi\frac{N-1}{N}k}\frac{\sin(\pi\delta)\cos(\pi k) - \sin(\pi k)\cos(\pi\delta)}{\sin(\pi\delta/N)\cos(\pi k/N) - \sin(\pi k/N)\cos(\pi\delta/N)}\frac{\sin(\pi\delta/N)}{\sin(\pi\delta)} \tag{6}$$

$$= e^{j\frac{\pi}{N}k}\frac{\sin(\pi\delta/N)}{\sin(\pi\delta/N)\cos(\pi k/N) - \sin(\pi k/N)\cos(\pi\delta/N)} \tag{7}$$

where we have used the facts that $\sin(\pi k) = 0$ and $\cos(\pi k) = e^{j\pi k}\ \forall k$. The expression (7) can be written as

$$\gamma_k e^{-j\frac{\pi}{N}k}\left[\sin\left(\pi\delta/N\right)\cos\left(\pi k/N\right) - \sin\left(\pi k/N\right)\cos\left(\pi\delta/N\right)\right] = \sin\left(\pi\delta/N\right). \tag{8}$$

Making the substitution $k \leftarrow -k$, we can write (8) as

$$\gamma_{-k} e^{j\frac{\pi}{N}k}\left[\sin\left(\pi\delta/N\right)\cos\left(\pi k/N\right) + \sin\left(\pi k/N\right)\cos\left(\pi\delta/N\right)\right] = \sin\left(\pi\delta/N\right). \tag{9}$$

We now sum the expressions (8) and (9) and then divide both sides by $\cos\left(\pi\delta/N\right)\cos(\pi k/N)$ to obtain

$$\tan\left(\pi\delta/N\right)\left(\gamma_k e^{-j\frac{\pi}{N}k} + \gamma_{-k} e^{j\frac{\pi}{N}k}\right) + \tan\left(\pi k/N\right)\left(-\gamma_k e^{-j\frac{\pi}{N}k} + \gamma_{-k} e^{j\frac{\pi}{N}k}\right) = 2\frac{\tan\left(\pi\delta/N\right)}{\cos\left(\pi k/N\right)} \tag{10}$$

which is equivalent to

$$\tan\left(\pi\delta/N\right)\left(\gamma_k e^{-j\frac{\pi}{N}k} + \gamma_{-k} e^{j\frac{\pi}{N}k} - \frac{2}{\cos(\pi k/N)}\right) = \left(\gamma_k e^{-j\frac{\pi}{N}k} - \gamma_{-k} e^{j\frac{\pi}{N}k}\right)\tan\left(\pi k/N\right). \tag{11}$$

We can rearrange the terms in (11) to get

$$\tan\left(\pi\delta/N\right) = \frac{\left(\gamma_k e^{-j\frac{\pi}{N}k} - \gamma_{-k} e^{j\frac{\pi}{N}k}\right)\tan\left(\pi k/N\right)}{\gamma_k e^{-j\frac{\pi}{N}k} + \gamma_{-k} e^{j\frac{\pi}{N}k} - \frac{2}{\cos(\pi k/N)}} \tag{12}$$

$$= \tan\left(\pi k/N\right)\text{Real}\left\{\frac{R[k_p+k]e^{-j\frac{\pi}{N}k} - R[k_p-k]e^{j\frac{\pi}{N}k}}{R[k_p+k]e^{-j\frac{\pi}{N}k} + R[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2R[k_p]}{\cos(\pi k/N)}}\right\} \tag{13}$$

where we used the definitions of γ_k, γ_{-k} with the fact that both sides of the first equation are real to obtain the second equation. Solving for δ then gives

$$\widehat{\delta}_k = \frac{N}{\pi}\tan^{-1}\left(\tan\left(\pi k/N\right)\text{Real}\left\{\frac{R[k_p+k]e^{-j\frac{\pi}{N}k} - R[k_p-k]e^{j\frac{\pi}{N}k}}{R[k_p+k]e^{-j\frac{\pi}{N}k} + R[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2R[k_p]}{\cos(\pi k/N)}}\right\}\right). \tag{14}$$

3 MSE for the Estimator $\widehat{\delta}_k$ under High-SNR

In this section, under sufficiently high SNR assumption we derive an approximate expression for the MSE of the propose estimator. The proposed estimator $\widehat{\delta}_k$ given in (2) is repeated below for convenience.

$$\widehat{\delta}_k = \frac{N}{\pi}\tan^{-1}\left(\tan\left(\pi k/N\right)\text{Real}\left\{\frac{R[k_p+k]e^{-j\frac{\pi}{N}k} - R[k_p-k]e^{j\frac{\pi}{N}k}}{R[k_p+k]e^{-j\frac{\pi}{N}k} + R[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2R[k_p]}{\cos(\pi k/N)}}\right\}\right). \tag{15}$$

Note that in a real scenario, $R[k_p+k]$, $R[k_p-k]$ and $R[k_p]$ will be noisy DFT samples and they will be related to the noiseless DFT samples as

$$R[k_p] = \overline{R}[k_p] + \eta_{k_p} \tag{16}$$

$$R[k_p+k] = \overline{R}[k_p+k] + \eta_{k_p+k} \tag{17}$$

$$R[k_p-k] = \overline{R}[k_p-k] + \eta_{k_p-k} \tag{18}$$

where η_k, $k = 0, \ldots, N/2-1$ are independent identically distributed circularly-symmetric complex Gaussian noise samples, with zero-mean and variance $N\sigma_w^2$. The quantities $\overline{R}[k_p+k]$, $\overline{R}[k_p-k]$ and $\overline{R}[k_p]$ represent the noiseless DFT samples. Substituting $R[k_p+k]$, $R[k_p-k]$ and $R[k_p]$ into the argument of the real operator in (15), we get

$$\begin{aligned}
&\frac{R[k_p+k]e^{-j\frac{\pi}{N}k} - R[k_p-k]e^{j\frac{\pi}{N}k}}{R[k_p+k]e^{-j\frac{\pi}{N}k} + R[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2R[k_p]}{\cos(\pi k/N)}} \\
\approx &\frac{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} - \overline{R}[k_p-k]e^{j\frac{\pi}{N}k}}{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} + \overline{R}[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2\overline{R}[k_p]}{\cos(\pi k/N)}} \\
&+ \frac{\eta_{k_p+k}e^{-j\frac{\pi}{N}k} - \eta_{k_p-k}e^{j\frac{\pi}{N}k}}{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} + \overline{R}[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2\overline{R}[k_p]}{\cos(\pi k/N)}} \\
&- \frac{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} - \overline{R}[k_p-k]e^{j\frac{\pi}{N}k}}{\left(\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} + \overline{R}[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2\overline{R}[k_p]}{\cos(\pi k/N)}\right)^2} \left(\eta_{k_p+k}e^{-j\frac{\pi}{N}k} + \eta_{k_p-k}e^{j\frac{\pi}{N}k} - \frac{2\eta_{k_p}}{\cos(\pi k/N)}\right)
\end{aligned} \tag{19}$$

$$\begin{aligned}
=&\frac{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} - \overline{R}[k_p-k]e^{j\frac{\pi}{N}k}}{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} + \overline{R}[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2\overline{R}[k_p]}{\cos(\pi k/N)}} \\
&+ \frac{1}{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} + \overline{R}[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2\overline{R}[k_p]}{\cos(\pi k/N)}} \Bigg(\eta_{k_p+k}e^{-j\frac{\pi}{N}k} - \eta_{k_p-k}e^{j\frac{\pi}{N}k} \\
&- \frac{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} - \overline{R}[k_p-k]e^{j\frac{\pi}{N}k}}{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} + \overline{R}[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2\overline{R}[k_p]}{\cos(\pi k/N)}} \left(\eta_{k_p+k}e^{-j\frac{\pi}{N}k} + \eta_{k_p-k}e^{j\frac{\pi}{N}k} - \frac{2\eta_{k_p}}{\cos(\pi k/N)}\right) \Bigg)
\end{aligned} \tag{20}$$

where we have made a first-order Taylor series expansion with the assumption that the noise terms are small (under high SNR). Noting that the noiseless DFT samples $\overline{R}[k_p+k]$, $\overline{R}[k_p-k]$ and $\overline{R}[k_p]$ satisfy the expression

$$\delta = \frac{N}{\pi}\tan^{-1}\left(\tan(\pi k/N)\,\text{Real}\left\{\frac{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} - \overline{R}[k_p-k]e^{j\frac{\pi}{N}k}}{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} + \overline{R}[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2\overline{R}[k_p]}{\cos(\pi k/N)}}\right\}\right) \tag{21}$$

and the argument of the real operator is always real, we can write

$$\frac{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} - \overline{R}[k_p-k]e^{j\frac{\pi}{N}k}}{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} + \overline{R}[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2\overline{R}[k_p]}{\cos(\pi k/N)}} = \frac{\tan(\pi\delta/N)}{\tan(\pi k/N)}. \tag{22}$$

We now use (22) to write (20) as

$$\begin{aligned}
&\frac{R[k_p+k]e^{-j\frac{\pi}{N}k} - R[k_p-k]e^{j\frac{\pi}{N}k}}{R[k_p+k]e^{-j\frac{\pi}{N}k} + R[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2R[k_p]}{\cos(\pi k/N)}} \approx \frac{\tan(\pi\delta/N)}{\tan(\pi k/N)} \\
&+ \overline{S}_k(\delta)\left(\eta_{k_p+k}e^{-j\frac{\pi}{N}k} - \eta_{k_p-k}e^{j\frac{\pi}{N}k} - \frac{\tan(\pi\delta/N)}{\tan(\pi k/N)}\left(\eta_{k_p+k}e^{-j\frac{\pi}{N}k} + \eta_{k_p-k}e^{j\frac{\pi}{N}k} - \frac{2\eta_{k_p}}{\cos(\pi k/N)}\right)\right)
\end{aligned} \tag{23}$$

where

$$\overline{S}_k(\delta) \triangleq \frac{1}{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k} + \overline{R}[k_p-k]e^{j\frac{\pi}{N}k} - \frac{2\overline{R}[k_p]}{\cos(\pi k/N)}} \tag{24}$$

is a deterministic complex-valued function of δ and k. Substituting (23) into (15), we get

$$\begin{aligned}\widehat{\delta}_k \approx & \frac{N}{\pi}\tan^{-1}\Bigg(\tan(\pi\delta/N) + \mathrm{Real}\Big\{\overline{S}_k(\delta)\Big(\tan(\pi k/N)\left(\eta_{k_p+k}e^{-j\frac{\pi}{N}k} - \eta_{k_p-k}e^{j\frac{\pi}{N}k}\right) \\ & - \tan(\pi\delta/N)\left(\eta_{k_p+k}e^{-j\frac{\pi}{N}k} + \eta_{k_p-k}e^{j\frac{\pi}{N}k} - \frac{2\eta_{k_p}}{\cos(\pi k/N)}\right)\Big)\Big\}\Bigg) \qquad (25) \\ \approx & \delta + \frac{N}{\pi}\frac{1}{1+\tan^2(\pi\delta/N)} \\ & \times \mathrm{Real}\Big\{\overline{S}_k(\delta)\Big(\tan(\pi k/N)\left(\eta_{k_p+k}e^{-j\frac{\pi}{N}k} - \eta_{k_p-k}e^{j\frac{\pi}{N}k}\right) \\ & - \tan(\pi\delta/N)\left(\eta_{k_p+k}e^{-j\frac{\pi}{N}k} + \eta_{k_p-k}e^{j\frac{\pi}{N}k} - \frac{2\eta_{k_p}}{\cos(\pi k/N)}\right)\Big)\Big\} \qquad (26)\end{aligned}$$

where we have made another first-order Taylor series expansion in the $\tan^{-1}(\cdot)$ function with the assumption that noise terms are small (under high SNR).

Hence the estimation error is given as

$$\begin{aligned}\widehat{\delta}_k - \delta = & \frac{N}{\pi}\frac{1}{1+\tan^2(\pi\delta/N)}\mathrm{Real}\Big\{\overline{S}_k(\delta)\Big(\tan(\pi k/N)\left(\eta_{k_p+k}e^{-j\frac{\pi}{N}k} - \eta_{k_p-k}e^{j\frac{\pi}{N}k}\right) \\ & - \tan(\pi\delta/N)\left(\eta_{k_p+k}e^{-j\frac{\pi}{N}k} + \eta_{k_p-k}e^{j\frac{\pi}{N}k} - \frac{2\eta_{k_p}}{\cos(\pi k/N)}\right)\Big)\Big\}. \qquad (27)\end{aligned}$$

We now rearrange the terms in (27) to get

$$\begin{aligned}\widehat{\delta}_k - \delta = & \frac{N}{\pi}\frac{1}{1+\tan^2(\pi\delta/N)}\Bigg((\tan(\pi k/N) - \tan(\pi\delta/N))\,\mathrm{Real}\Big\{\overline{S}_k(\delta)\eta_{k_p+k}e^{-j\frac{\pi}{N}k}\Big\} \\ & - (\tan(\pi k/N) + \tan(\pi\delta/N))\,\mathrm{Real}\Big\{\overline{S}_k(\delta)\eta_{k_p-k}e^{j\frac{\pi}{N}k}\Big\} + \frac{2\tan(\pi\delta/N)}{\cos(\pi k/N)}\mathrm{Real}\Big\{\overline{S}_k(\delta)\eta_{k_p}\Big\}\Bigg). \qquad (28)\end{aligned}$$

Taking the square of both sides and then taking the expected value, we obtain

$$\begin{aligned}\mathrm{MSE}_k \triangleq & E[(\widehat{\delta}_k - \delta)^2] \qquad (29) \\ = & \frac{N^2}{\pi^2}\frac{1}{(1+\tan^2(\pi\delta/N))^2}\Bigg((\tan(\pi k/N) - \tan(\pi\delta/N))^2\,E\left[\mathrm{Real}^2\Big\{\overline{S}_k(\delta)\eta_{k_p+k}e^{-j\frac{\pi}{N}k}\Big\}\right] \\ & + (\tan(\pi k/N) + \tan(\pi\delta/N))^2\,E\left[\mathrm{Real}^2\Big\{\overline{S}_k(\delta)\eta_{k_p-k}e^{j\frac{\pi}{N}k}\Big\}\right] \\ & + \frac{4\tan^2(\pi\delta/N)}{\cos^2(\pi k/N)}E\left[\mathrm{Real}^2\Big\{\overline{S}_k(\delta)\eta_{k_p}\Big\}\right]\Bigg) \qquad (30)\end{aligned}$$

where we used the independence of the noise terms η_{k_p+k}, η_{k_p-k} and η_{k_p} to get rid of the cross terms. Since the noise terms η_{k_p+k}, η_{k_p-k} and η_{k_p} are also identically distributed and circularly symmetric, the following equalities hold.

$$E\left[\mathrm{Real}^2\Big\{\overline{S}_k(\delta)\eta_{k_p+k}e^{-j\frac{\pi}{N}k}\Big\}\right] = E\left[\mathrm{Real}^2\Big\{\overline{S}_k(\delta)\eta_{k_p-k}e^{j\frac{\pi}{N}k}\Big\}\right] = E\left[\mathrm{Real}^2\Big\{\overline{S}_k(\delta)\eta_{k_p}\Big\}\right] \tag{31}$$

which gives

$$\begin{aligned}
\mathrm{MSE}_k =& \frac{N^2}{\pi^2}\frac{1}{\left(1+\tan^2(\pi\delta/N)\right)^2}E\left[\mathrm{Real}^2\left\{\overline{S}_k(\delta)\eta_{k_p}\right\}\right] \\
&\times\left(\left(\tan(\pi k/N)-\tan(\pi\delta/N)\right)^2+\left(\tan(\pi k/N)+\tan(\pi\delta/N)\right)^2+\frac{4\tan^2(\pi\delta/N)}{\cos^2(\pi k/N)}\right) \quad (32)\\
=& \frac{N^2}{\pi^2}\frac{1}{\left(1+\tan^2(\pi\delta/N)\right)^2}E\left[\mathrm{Real}^2\left\{\overline{S}_k(\delta)\eta_{k_p}\right\}\right] \\
&\times\left(2\tan^2(\pi k/N)+2\tan^2(\pi\delta/N)+\frac{4\tan^2(\pi\delta/N)}{\cos^2(\pi k/N)}\right) \quad (33)\\
=& \frac{2N^2}{\pi^2}\frac{\sin^2(\pi k/N)+\left(2+\cos^2(\pi k/N)\right)\tan^2(\pi\delta/N)}{\left(1+\tan^2(\pi\delta/N)\right)^2\cos^2(\pi k/N)}E\left[\mathrm{Real}^2\left\{\overline{S}_k(\delta)\eta_{k_p}\right\}\right] \quad (34)
\end{aligned}$$

In order to be able to take the expected value on the right hand side of (34), we now concentrate on the quantity $\overline{S}_k(\delta)$ which is given as

$$\overline{S}_k(\delta) \triangleq \frac{1}{\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k}+\overline{R}[k_p-k]e^{j\frac{\pi}{N}k}-\frac{2\overline{R}[k_p]}{\cos(\pi k/N)}}. \quad (35)$$

We write the denominator as

$$\begin{aligned}
&\overline{R}[k_p+k]e^{-j\frac{\pi}{N}k}+\overline{R}[k_p-k]e^{j\frac{\pi}{N}k}-\frac{2\overline{R}[k_p]}{\cos\pi k/N} \\
&=Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}e^{-j\pi\frac{N-1}{N}k}\frac{\sin(\pi(\delta-k))}{\sin\left(\pi(\delta-k)/N\right)}e^{-j\frac{\pi}{N}k} \\
&\quad+Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}e^{j\pi\frac{N-1}{N}k}\frac{\sin(\pi(\delta+k))}{\sin\left(\pi(\delta+k)/N\right)}e^{j\frac{\pi}{N}k} \\
&\quad-\frac{2}{\cos(\pi k/N)}Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}\frac{\sin(\pi\delta)}{\sin\left(\pi\delta/N\right)} \quad (36)\\
&=Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}e^{-j\pi k}\frac{\sin(\pi(\delta-k))}{\sin\left(\pi(\delta-k)/N\right)}+Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}e^{j\pi k}\frac{\sin(\pi(\delta+k))}{\sin\left(\pi(\delta+k)/N\right)} \\
&\quad-\frac{2}{\cos(\pi k/N)}Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}\frac{\sin(\pi\delta)}{\sin\left(\pi\delta/N\right)} \quad (37)\\
&=Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}\frac{\sin(\pi\delta)}{\sin\left(\pi(\delta-k)/N\right)}+Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}\frac{\sin(\pi\delta)}{\sin\left(\pi(\delta+k)/N\right)} \\
&\quad-\frac{2}{\cos(\pi k/N)}Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}\frac{\sin(\pi\delta)}{\sin\left(\pi\delta/N\right)} \quad (38)\\
&=Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}\sin(\pi\delta)\left(\frac{1}{\sin\left(\pi(\delta-k)/N\right)}+\frac{1}{\sin\left(\pi(\delta+k)/N\right)}-\frac{2}{\cos(\pi k/N)\sin(\pi\delta/N)}\right) \quad (39)\\
&=Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}\sin(\pi\delta) \\
&\quad\times\left(\frac{1}{\sin\left(\pi(\delta-k)/N\right)}+\frac{1}{\sin\left(\pi(\delta+k)/N\right)}-\frac{4}{\sin\left(\pi(\delta-k)/N\right)+\sin\left(\pi(\delta+k)/N\right)}\right) \quad (40)\\
&=\frac{Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}\sin(\pi\delta)\left(\sin\left(\pi(\delta-k)/N\right)-\sin\left(\pi(\delta+k)/N\right)\right)^2}{\sin\left(\pi(\delta-k)/N\right)\sin\left(\pi(\delta+k)/N\right)\left(\sin\left(\pi(\delta-k)/N\right)+\sin\left(\pi(\delta+k)/N\right)\right)} \quad (41)
\end{aligned}$$

$$= \frac{Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}}{2\cos(\pi k/N)}\frac{\sin(\pi\delta)}{\sin(\pi\delta/N)}\frac{\left(\sin\left(\pi(\delta-k)/N\right)-\sin\left(\pi(\delta+k)/N\right)\right)^2}{\sin\left(\pi(\delta-k)/N\right)\sin\left(\pi(\delta+k)/N\right)} \tag{42}$$

$$= \frac{Ae^{j\phi}e^{j\pi\frac{N-1}{N}\delta}}{\cos(\pi k/N)}\frac{\sin(\pi\delta)}{\sin(\pi\delta/N)}\frac{2\sin^2(\pi k/N)\cos^2(\pi\delta/N)}{\sin\left(\pi(\delta-k)/N\right)\sin\left(\pi(\delta+k)/N\right)} \tag{43}$$

where we have used the trigonometric sum formulae (for $\sin(a+b)$) and the sum-to-product formula (for the sine function) to obtain the final form. Then $\overline{S}_k(\delta)$ is given as

$$\overline{S}_k(\delta) = \frac{1}{2A}\cos(\pi k/N)e^{-j\phi}e^{-j\pi\frac{N-1}{N}\delta}\frac{\sin(\pi\delta/N)}{\sin(\pi\delta)}\frac{\sin\left(\pi(\delta-k)/N\right)\sin\left(\pi(\delta+k)/N\right)}{\sin^2(\pi k/N)\cos^2(\pi\delta/N)}. \tag{44}$$

We can now take the expected value in (34) as follows.

$$E\left[\text{Real}^2\left\{\overline{S}_k(\delta)\eta_{k_p}\right\}\right] = \frac{N}{8\text{SNR}}\cos^2(\pi k/N)\frac{\sin^2(\pi\delta/N)}{\sin^2(\pi\delta)}\frac{\sin^2\left(\pi(\delta-k)/N\right)\sin^2\left(\pi(\delta+k)/N\right)}{\sin^4(\pi k/N)\cos^4(\pi\delta/N)} \tag{45}$$

where we used the circular symmetricity of the noise term. Now, substituting the expected value (45) into (34), we get

$$\begin{aligned}\text{MSE}_k =& \frac{2N^2}{\pi^2}\frac{\sin^2(\pi k/N)+\left(2+\cos^2(\pi k/N)\right)\tan^2(\pi\delta/N)}{\left(1+\tan^2(\pi\delta/N)\right)^2\cos^2(\pi k/N)} \\ &\times \frac{N}{8\text{SNR}}\cos^2(\pi k/N)\frac{\sin^2(\pi\delta/N)}{\sin^2(\pi\delta)}\frac{\sin^2\left(\pi(\delta-k)/N\right)\sin^2\left(\pi(\delta+k)/N\right)}{\sin^4(\pi k/N)\cos^4(\pi\delta/N)} \qquad (46)\\ =& \frac{N^3}{4\pi^2\text{SNR}}\left(\sin^2(\pi k/N)+\left(2+\cos^2(\pi k/N)\right)\tan^2(\pi\delta/N)\right) \\ &\times \frac{\sin^2(\pi\delta/N)}{\sin^2(\pi\delta)}\frac{\sin^2\left(\pi(\delta-k)/N\right)\sin^2\left(\pi(\delta+k)/N\right)}{\sin^4(\pi k/N)}. \qquad (47)\end{aligned}$$

4 Cross-Covariance Between the Estimates $\widehat{\delta}_k$ and $\widehat{\delta}_{k'}$ $(k \neq k')$ under High-SNR

In order to calculate the cross-covariance between the estimates $\widehat{\delta}_k$ and $\widehat{\delta}_{k'}$ where $k \neq k'$, we consider the estimation error in (28) given as

$$\begin{aligned}\widehat{\delta}_k - \delta =& \frac{N}{\pi}\frac{1}{1+\tan^2(\pi\delta/N)}\Bigg(\left(\tan(\pi k/N)-\tan(\pi\delta/N)\right)\text{Real}\left\{\overline{S}_k(\delta)\eta_{k_p+k}e^{-j\frac{\pi}{N}k}\right\} \\ &-\left(\tan(\pi k/N)+\tan(\pi\delta/N)\right)\text{Real}\left\{\overline{S}_k(\delta)\eta_{k_p-k}e^{j\frac{\pi}{N}k}\right\}+\frac{2\tan(\pi\delta/N)}{\cos(\pi k/N)}\text{Real}\left\{\overline{S}_k(\delta)\eta_{k_p}\right\}\Bigg).\end{aligned} \tag{48}$$

We now write the cross-covariance $E\left[(\widehat{\delta}_k-\delta)(\widehat{\delta}_{k'}-\delta)\right]$ as

$$\begin{aligned}E\left[(\widehat{\delta}_k-\delta)(\widehat{\delta}_{k'}-\delta)\right] =& \frac{N^2}{\pi^2}\frac{1}{(1+\tan^2(\pi\delta/N))^2}\frac{4\tan^2(\pi\delta/N)}{\cos(\pi k/N)\cos(\pi k'/N)} \\ &\times E\left[\text{Real}\left\{\overline{S}_k(\delta)\eta_{k_p}\right\}\text{Real}\left\{\overline{S}_{k'}(\delta)\eta_{k_p}\right\}\right]\end{aligned} \tag{49}$$

where all of the other terms in the multiplication $(\widehat{\delta}_k - \delta)(\widehat{\delta}_{k'} - \delta)$ have zero expectations. Substituting the expression of $\overline{S}_k$ and $\overline{S}_{k'}$ into (49) and then taking the expected value, we get

$$E\left[(\widehat{\delta}_k - \delta)(\widehat{\delta}_{k'} - \delta)\right] = \frac{N^3}{2\pi^2 \mathrm{SNR}} \tan^2(\pi\delta/N) \frac{\sin^2(\pi\delta/N)}{\sin^2(\pi\delta)} \frac{\sin\left(\pi(\delta-k)/N\right)\sin\left(\pi(\delta+k)/N\right)}{\sin^2(\pi k/N)} \times \frac{\sin\left(\pi(\delta-k')/N\right)\sin\left(\pi(\delta+k')/N\right)}{\sin^2(\pi k'/N)}. \tag{50}$$

As a result, the correlation coefficient $\rho_{kk'}$ between the estimates $\widehat{\delta}_k$ and $\widehat{\delta}_{k'}$ is given as

$$\rho_{kk'} \triangleq \frac{E\left[(\widehat{\delta}_k - \delta)(\widehat{\delta}_{k'} - \delta)\right]}{\sqrt{E\left[(\widehat{\delta}_k - \delta)^2\right]}\sqrt{E\left[(\widehat{\delta}_{k'} - \delta)^2\right]}} = \frac{E\left[(\widehat{\delta}_k - \delta)(\widehat{\delta}_{k'} - \delta)\right]}{\sqrt{\mathrm{MSE}_k}\sqrt{\mathrm{MSE}_{k'}}} \tag{51}$$

$$= \frac{2\tan^2\left(\frac{\pi}{N}\delta\right)}{\sqrt{\sin^2\left(\frac{\pi}{N}k\right) + \left(2+\cos^2\left(\frac{\pi}{N}k\right)\right)\tan^2\left(\frac{\pi}{N}\delta\right)}\sqrt{\sin^2\left(\frac{\pi}{N}k'\right) + \left(2+\cos^2\left(\frac{\pi}{N}k'\right)\right)\tan^2\left(\frac{\pi}{N}\delta\right)}}. \tag{52}$$

5 Fusion of the Estimates

In this section we will define a fusion rule for the estimates $\widehat{\delta}_k$. For this purpose, we first approximate the MSE in (47) and the cross-covariance in (50) by using a small $|\delta|$ assumption.

5.1 MSE and Cross-Covariance Approximation

In the previous sections the MSE of the kth estimate $\widehat{\delta}_k$ and the cross covariance between the estimates $\widehat{\delta}_k$ and $\widehat{\delta}_{k'}$ $(k \neq k')$ were found as

$$\mathrm{MSE}_k = \frac{N^3}{4\pi^2 \mathrm{SNR}} \left(\sin^2(\pi k/N) + (2+\cos^2(\pi k/N))\tan^2(\pi\delta/N)\right) \times \frac{\sin^2(\pi\delta/N)}{\sin^2(\pi\delta)} \frac{\sin^2\left(\pi(\delta-k)/N\right)\sin^2\left(\pi(\delta+k)/N\right)}{\sin^4(\pi k/N)}. \tag{53}$$

$$E\left[(\widehat{\delta}_k - \delta)(\widehat{\delta}_{k'} - \delta)\right] = \frac{N^3}{2\pi^2 \mathrm{SNR}} \tan^2(\pi\delta/N) \frac{\sin^2(\pi\delta/N)}{\sin^2(\pi\delta)} \frac{\sin\left(\pi(\delta-k)/N\right)\sin\left(\pi(\delta+k)/N\right)}{\sin^2(\pi k/N)} \times \frac{\sin\left(\pi(\delta-k')/N\right)\sin\left(\pi(\delta+k')/N\right)}{\sin^2(\pi k'/N)}. \tag{54}$$

These MSE and correlation expressions which are valid under high-SNR are too complicated to use in a fusion rule and moreover, they depend on the unknown true value of δ (i.e., the quantity to be estimated) in a complicated manner. Therefore, in this section, we make a small $|\delta|$ assumption, which enables us to have the following approximations.

$$\tan^2(\pi\delta/N) \approx 0. \tag{55}$$

$$\frac{\sin^2\left(\pi(\delta-k)/N\right)\sin^2\left(\pi(\delta+k)/N\right)}{\sin^4(\pi k/N)} \approx 1. \tag{56}$$

Substituting these approximations into (53) and (54), we get

$$\mathrm{MSE}_k \approx \frac{1}{4N\mathrm{SNR}} \frac{\sin^2(\pi k/N)}{\pi^2/N^2} \frac{N^2 \sin^2(\pi\delta/N)}{\sin^2(\pi\delta)} \tag{57}$$

$$E\left[(\widehat{\delta}_k - \delta)(\widehat{\delta}_{k'} - \delta)\right] \approx 0 \tag{58}$$

which are approximately valid under both high-SNR and small $|\delta|$ assumptions. The expression (58) means that the estimates are approximately uncorrelated under both high-SNR and small $|\delta|$ assumptions.

5.2 Fusion Rule

The proposed fused estimate is the best linear unbiased estimator (BLUE) which is given as (See [1, p.139])

$$\widehat{\delta}_F = \frac{\sum_{k=1}^{N/2-1} \frac{1}{(\mathrm{MSE})_k} \widehat{\delta}_k}{\sum_{k=1}^{N/2-1} \frac{1}{(\mathrm{MSE})_k}}. \tag{59}$$

Note also that any constant multiplier(s) of the weights $\mu_k \triangleq \frac{1}{(\mathrm{MSE})_k}$ which does not depend on k (although it may depend on δ) can be removed from the weights thanks to the normalization term. We know that

$$\mu_k = 4N\mathrm{SNR} \frac{\pi^2/N^2}{\sin^2(\pi k/N)} \frac{\sin^2(\pi\delta)}{N^2 \sin^2(\pi\delta/N)} \tag{60}$$

where we used the MSE approximation provided in (57). Note that all the terms except $\sin^2(\pi k/N)$ are constants with respect to k on the right hand side of (60). Therefore, these terms can be removed thanks to the normalization operation on the weights, which gives

$$\mu_k \propto \frac{1}{\sin^2(\pi k/N)}. \tag{61}$$

Hence, the BLUE $\widehat{\delta}_F$ can be obtained using the formula

$$\widehat{\delta}_F = \frac{\sum_{k=1}^{N/2-1} \frac{1}{\sin^2(\pi k/N)} \widehat{\delta}_k}{\sum_{k=1}^{N/2-1} \frac{1}{\sin^2(\pi k/N)}}. \tag{62}$$

There is an analytical formula for the summation in the denominator which is given as [2]

$$\sum_{k=1}^{N/2-1} \frac{1}{\sin^2(\pi k/N)} = \sum_{k=1}^{N/2-1} \csc^2(\pi k/N) = \frac{1}{6}(N+2)(N-2) \tag{63}$$

for even N values.

5.3 Approximate MSE of the Fused Estimate

Considering the fact that the MSE of the BLUE in (59) can be calculated as (See [1, p.139])

$$(\text{MSE})_F = \frac{1}{\sum_{k=1}^{N/2-1} \frac{1}{(\text{MSE})_k}}, \tag{64}$$

the MSE of $\widehat{\delta}_F$ in (62) is approximately given as

$$\text{MSE}_F \approx \frac{6N}{(2\pi)^2 \text{SNR}(N-2)(N+2)} \frac{N^2 \sin^2(\pi\delta/N)}{\sin^2(\pi\delta)}, \tag{65}$$

where we used (57) and (63), which converges to the CRLB for the problem [3] given as

$$\text{CRLB} \triangleq \frac{6}{(2\pi)^2 N \text{SNR}} \tag{66}$$

when N is sufficiently large and $|\delta|$ is sufficiently small.

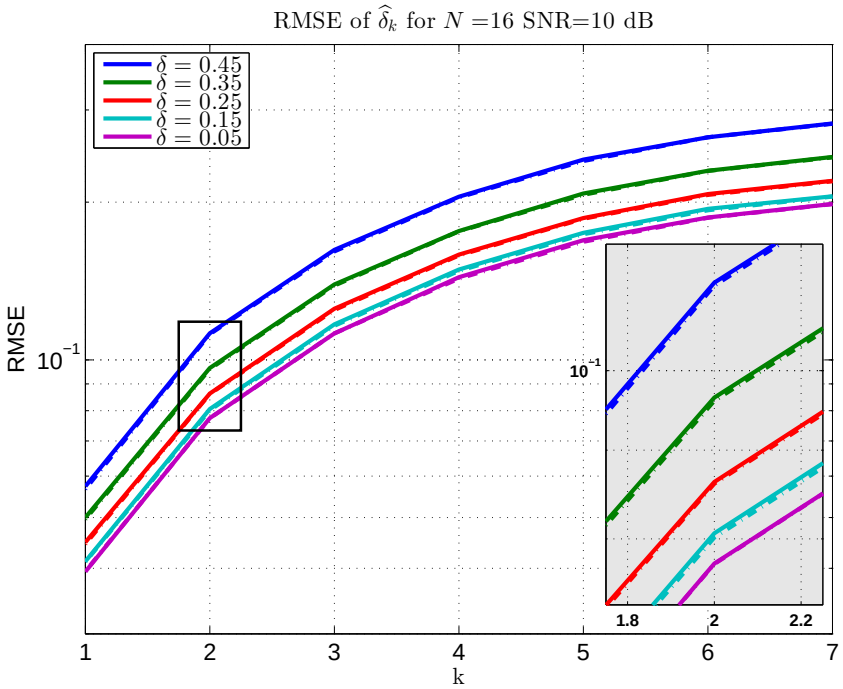
6 Simulations Related to the Approximations

6.1 MSE Approximations

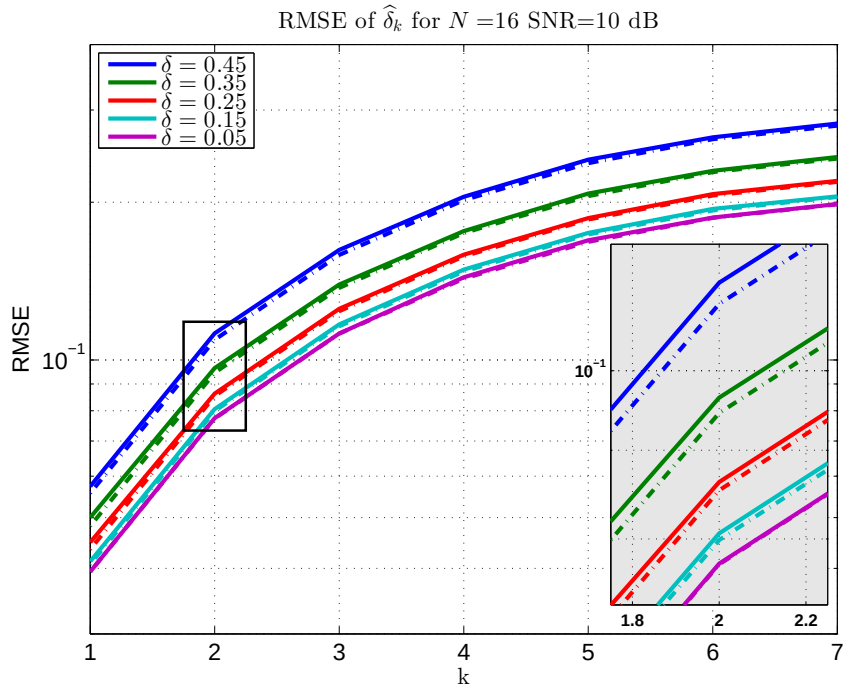
In this sub-section, we compare the simulated root mean square errors (RMSE) of the estimators $\widehat{\delta}_k$ with the RMSE approximations given by (47) and (57). We consider the complex exponential signal of length $N = 16$ and frequency $f = (2+\delta)/N$, i.e., the maximum bin $k_p = 2$. The RMSE performance of estimators $\widehat{\delta}_k$, $k = 1, \ldots, 7$ are evaluated on 100000 Monte Carlo (MC) runs for different values of δ under the condition SNR= 10 dB. In each MC-run different noise $w[\cdot]$ and phase ($\phi \sim$ Uniform$[0, 2\pi]$) realizations are used. Figures 1(a) and 1(b) show the simulated RMSE of the estimators $\widehat{\delta}_k$, $k = 1, \ldots, 7$ with respect to k along with the analytical approximations given by (47) and (57), respectively, for different true δ-values. It is seen that the RMSE values increase as true δ and k increases, which is expected. The approximations (47) and (57) both follow the simulated RMSE values quite closely while the latter being only slightly worse than the former.

6.2 Cross-Covariance Approximation

In this sub-section, we evaluate the correlation coefficients given in (52) for $\delta = 0.05$ and $\delta = 0.45$. The experiment parameters are the same as those in the previous sub-section. Table 1 and Table 2 give the values of $\rho_{kk'}$ in (52) for $\delta = 0.05$ and $\delta = 0.45$, respectively, for $k, k' = 1, \ldots, 7$. Note that the expression (52) is valid only in the case $k \neq k'$. The elements of the tables for the cases $k = k'$ are naturally set to unity. It is seen that when $\delta = 0.05$, the estimates are almost uncorrelated confirming the approximation made in (58) under high-SNR and small $|\delta|$ assumptions. When $\delta = 0.45$, on the other hand, the correlation coefficients can be as high as 0.15. Most of the correlation coefficient values for $\delta = 0.45$ are still less than 0.1 and therefore the quality of the approximation (58) degrades a little with increasing $|\delta|$ but still remains at a reasonable level.



(a) MSE approximation given by (47).



(b) MSE approximation given by (57).

Figure 1: Analytical (approximate) (dash-dotted lines) and simulated RMSE (solid lines) performances of the estimators $\widehat{\delta}_k$, $k = 1, \ldots, 7$ for different true δ-values.

Table 1: Correlation coefficients $\rho_{kk'}$ for $k, k' = 1, \ldots, 7$ when $\delta = 0.05$.

k \ k'	1	2	3	4	5	6	7
1	1	0.0026	0.0018	0.0014	0.0012	0.0011	0.0010
2	0.0026	1	0.0009	0.0007	0.0006	0.0005	0.0005
3	0.0018	0.0009	1	0.0005	0.0004	0.0004	0.0004
4	0.0014	0.0007	0.0005	1	0.0003	0.0003	0.0003
5	0.0012	0.0006	0.0004	0.0003	1	0.0003	0.0002
6	0.0011	0.0005	0.0004	0.0003	0.0003	1	0.0002
7	0.0010	0.0005	0.0004	0.0003	0.0002	0.0002	1

Table 2: Correlation coefficients $\rho_{kk'}$ for $k, k' = 1, \ldots, 7$ when $\delta = 0.45$.

k \ k'	1	2	3	4	5	6	7
1	1	0.1543	0.1104	0.0879	0.0753	0.0679	0.0641
2	0.1543	1	0.0665	0.0530	0.0453	0.0409	0.0386
3	0.1104	0.0665	1	0.0379	0.0324	0.0293	0.0276
4	0.0879	0.0530	0.0379	1	0.0259	0.0233	0.0220
5	0.0753	0.0453	0.0324	0.0259	1	0.0200	0.0188
6	0.0679	0.0409	0.0293	0.0233	0.0200	1	0.0170
7	0.0641	0.0386	0.0276	0.0220	0.0188	0.0170	1

7 How many bins to use?

Although, in general, it can be said that the fusion of the estimates will improve the performance no matter how many bins are used in the fused estimate, it is difficult to make a theoretical analysis on how to choose the number of bins to use. However, some practical recommendations can still be made as follows. Consider the estimator

$$\widehat{\delta}_F^K = \frac{\sum_{k=1}^K \frac{1}{\sin^2(\pi k/N)} \widehat{\delta}_k}{\sum_{k=1}^K \frac{1}{\sin^2(\pi k/N)}} \tag{67}$$

where $K \leq N/2 - 1$ is the number of bins used to form the fused estimate. In order for the performance of $\widehat{\delta}_F^K$ to be close to the fused estimate $\widehat{\delta}_F = \widehat{\delta}_F^{N/2-1}$ which uses all of the available DFT bins, one would need that a significant mass of all weights is contained in the summation $\sum_{k=1}^K \frac{1}{\sin^2(\pi k/N)}$. A reasonable rule might be to choose K as the smallest integer such that the inequality below is satisfied.

$$\sum_{k=1}^{K} \frac{1}{\sin^2(\pi k/N)} \geq \alpha \sum_{k=1}^{N/2-1} \frac{1}{\sin^2(\pi k/N)} = \frac{\alpha}{6}(N+2)(N-2) \tag{68}$$

where $0 < \alpha < 1$ determines the confidence level of the K value selected. Noting the MSE expression for the fused estimate given in (64), the condition (68) ensures that the fused estimator

with K bins has at most $1/\alpha$ times the MSE of the fused estimator with all bins (i.e., $K = N/2-1$). The inequality (68) can be written equivalently as

$$\sum_{k=1}^{K} \frac{\frac{1}{\sin^2(\pi k/N)}}{\sum_{\ell=1}^{N/2-1} \frac{1}{\sin^2(\pi \ell/N)}} \geq \alpha. \tag{69}$$

Hence, considering the normalized weights on the left hand side of (69) to form a probability mass function, choosing the value $\alpha = 0.95$ will ensure that the used weights ($k = 1, \ldots, K$) will contain at least the 95th percentile of the available normalized weights ($k = 1, \ldots, N/2-1$).

It is interesting to see what the value of K would be for some value of α when N is very large. In order to investigate this, we can take the following limit.

$$\lim_{N\to\infty} \frac{\frac{1}{\sin^2(\pi k/N)}}{\sum_{\ell=1}^{N/2-1} \frac{1}{\sin^2(\pi \ell/N)}} = \lim_{N\to\infty} \frac{6}{(N+2)(N-2)} \frac{1}{\sin^2(\pi k/N)} \tag{70}$$

$$= \lim_{N\to\infty} \frac{6N^2}{\pi^2 k^2 (N+2)(N-2)} \tag{71}$$

$$= \frac{6}{\pi^2} \frac{1}{k^2}. \tag{72}$$

Substituting the limit into (69), we get

$$\frac{6}{\pi^2} \sum_{k=1}^{K} \frac{1}{k^2} \geq \alpha. \tag{73}$$

Noting that, as $K \to \infty$, the summation approaches from 1 to $\pi^2/6$, we see that the left hand side of (73) approaches from $6/\pi^2 \approx 0.61$ to 1 (Hence, if $\alpha \leq 6/\pi^2$, then $K = 1$.). Using the definition of the generalized harmonic numbers [4, p. 277] given as

$$H_{n,m} \triangleq \sum_{k=1}^{n} \frac{1}{k^m}, \tag{74}$$

we can write (73) as

$$H_{K,2} \geq \frac{\alpha \pi^2}{6}. \tag{75}$$

Hence, we see that, as N grows, the number of bins to use converges to the smallest integer K for which the generalized harmonic number $H_{K,2}$ is larger than $\frac{\alpha\pi^2}{6}$.

For finite N case, we argue as follows. It is known that $\sin(x) < x$ for $x > 0$. Hence we have,

$$\sin(\pi k/N) < \pi k/N \tag{76}$$

for all $k > 0$. Therefore we can write

$$\frac{1}{\sin^2(\pi k/N)} > \frac{N^2}{\pi^2 k^2}. \tag{77}$$

Multiplying both sides by the positive number $\frac{6}{(N+2)(N-2)}$ (assuming $N > 2$), we get

$$\frac{6}{(N+2)(N-2)} \frac{1}{\sin^2(\pi k/N)} > \frac{6N^2}{(N^2-4)\pi^2 k^2}. \tag{78}$$

Table 3: K values obtained for α-values in $[0.9, 0.99]$.

α	0.9	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
K	6	7	8	9	10	12	15	20	30	61

Table 4: K values obtained for several α-values close to unity.

α	0.9	0.99	0.999	0.9999	0.99999	0.999999	0.9999999	0.99999999
K	6	61	608	6079	60793	607927	6079276	61418896

Noting that $\frac{N^2}{N^2-4} = 1/(1-4/N^2)$ is strictly decreasing with increasing N, the inequality (78) is preserved when we take the limit of the right hand side of (78) as N goes to infinity. Hence we can write

$$\frac{6}{(N+2)(N-2)} \frac{1}{\sin^2(\pi k/N)} > \frac{6}{\pi^2 k^2}. \tag{79}$$

Now summing both sides of (79) from $k=1$ to $k=K$, we obtain

$$\frac{6}{(N+2)(N-2)} \sum_{k=1}^{K} \frac{1}{\sin^2(\pi k/N)} > \frac{6}{\pi^2} \sum_{k=1}^{K} \frac{1}{k^2} = \frac{6}{\pi^2} H_{K,2} \tag{80}$$

Hence we see that, the sum of the first K normalized weights is always larger than the number $\frac{6}{\pi^2} H_{K,2}$. Hence if we ensure that the number $\frac{6}{\pi^2} H_{K,2}$ is larger than α, the sum of the first K normalized weights would be larger than α. As a result, we can suggest that, for any N value, if we choose K as the smallest integer such that the generalized harmonic number $H_{K,2}$ is larger than $\frac{\alpha\pi^2}{6}$, we always satisfy the condition (69). Note that these K values do not depend on N and they depend only on α. In Table 3, K values obtained for some α values are listed. The values in Table 3 say that, if one wants to use at least 90 percent of the weights (or if one want to work at most 10/9 times the MSE of $\widehat{\delta}_F$ (i.e., MSE_F)), then it is sufficient to use only 6 bins, independent of the value of N (of course as long as we have at least 6 bins, i.e., $N \geq 14$). Similarly 99 percent of the weights (or at most 100/99 times MSE_F) would require 61 bins.

Having seen the previous values for K, we calculated K values for some more α values in Table 4. Noting that the α values in Table 4 can be written as $\alpha = 1 - 0.1^{\ell}$ for $\ell = 1, \ldots, 8$, we can write the interesting approximate empirical formula for K given as

$$K \approx \text{round}\left(\frac{0.607927}{1-\alpha}\right), \tag{81}$$

which has been seen to hold for all α values in our trials except the last two columns of Table 4. Hence the empirical formula (81) might not be correct for very close values of α to unity. Note that the empirical constant 0.607927 shares some digits with $6/\pi^2 = 0.607927101854027$ but it is not the same.

8 Matlab Implementation

```
function outfreq = fine_freq_est(data,maxorder);
% function outfreq = fine_freq_est(data,maxorder);
% Generates frequency estimates via fusing maxorder estimates given by
%
% \Δhat_k = N/pi atan(tan(pi k/N) Real { Ratio } )
%
% where
% Ratio = (R[kp+k]e^{−jk pi/N} − R[kp − k]e^{+jk pi/N}) /
%         (R[kp+k]e^{−jk pi/N} + R[kp − k]e^{+jk pi/N} − 2R[kp]/cos(pi k/N))
%
% data :  N x Mcnum matrix ,(N: number of samples, Mcnum: number of vectors)
% maxorder : Number of estimates to be fused (maxorder < N/2 )
% outfreq : 1 x Mcnum vector
%
% March 2014
%

[N,MCnum] = size(data);
if exist('maxorder')==0, maxorder = N/2 − 1; end;

umutall = zeros(maxorder,MCnum);
%%%First Estimate
morder = 1; %do the first estimation (\Δhat_1)
[outfreq_dum,outf,outmaxind] = est_sub(data,morder);
umutall(morder,:) = outfreq_dum;
clear data;

%%%Remaining Estimates
for morder=2:maxorder,  %Do the rest, if maxorder>1
    outfreq_dum = est_sub([],morder,outf,outmaxind);
    umutall(morder,:) = outfreq_dum;
end;

%Fusion
dum = 1:maxorder; fusw = 1./(sin(pi*dum/N)).^2; fusw = fusw/sum(fusw);
outfreq = fusw*umutall;


%%%%%%%%%%%%%%%
function [outfreq,outf,outmaxind] = est_sub(data,morder,outf,outmaxind);

if exist('outf')==0, %If outf is provided at the input, do not repeat
                     %                                FFT calculation
    [N,MCnum] = size(data);
    outf=fft(data,[],1); clear data;
    out = real(outf).*real(outf) + imag(outf).*imag(outf);
    [outmaxval,outmaxind] = max(out,[],1);
else
    [N,MCnum] = size(outf);
end;

dumvec = (0:length(outmaxind)−1)*N;
vec = outmaxind −   1;  % 0 ≤ vec elements ≤ N−1
vecp = mod(vec + morder, N); % 0 ≤ vec elements ≤ N−1
vecm = mod(vec − morder, N); % 0 ≤ vec elements ≤ N−1
```

```
56
57 vec = vec + dumvec + 1;
58 vecp = vecp + dumvec + 1;
59 vecm = vecm + dumvec + 1;
60 index = [vecm;vec;vecp];
61 out  = outf(index);
62
63 %
64 %Constants
65 expp=exp(j*pi*morder/N); expm=exp(-j*pi*morder/N);
66 mycos=cos(pi*morder/N); mytan=tan(pi*morder/N);
67
68 %
69 out = [-expp 0 expm; expp -2/mycos expm]*out;
70 outfreq = N/pi*atan(real(mytan*out(1,:)./out(2,:)));
71 outfreq = outfreq + outmaxind - 1;
```

References

[1] S. M. Kay, *Fundamentals of Statistical Signal Processing, Volume 1: Estimation Theory.* Prentice Hall, 1993.

[2] N. Gauthier and P. S. Bruckman, "Sums of the even integral powers of the cosecant and secant," *Fibonacci Quart.*, vol. 44, no. 3, pp. 264–273, 2006.

[3] D. Rife and R. Boorstyn, "Single tone parameter estimation from discrete-time observations," *IEEE Trans. Inf. Theory*, vol. 20, no. 5, pp. 591–598, Sep 1974.

[4] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd ed. Addison-Wesley, 1994.