Exercises:

1) Consider the curve $y=x^3-4x+1$

a) Find an equation for the line perpendicular to the tangent line to the curve at the point $(2,1)$

b) Find the eqns. for the tangent lines to the curve at all points where the slope of the tangent line is 8.

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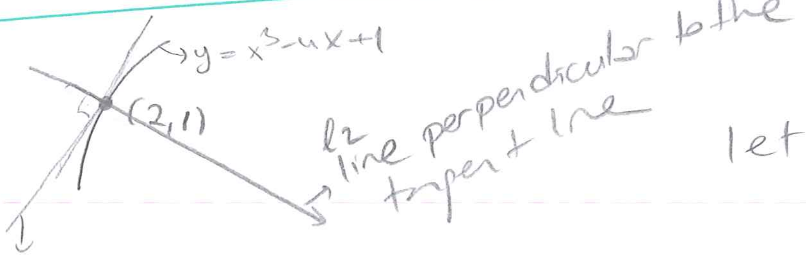
a)



let $f(x)=x^3-4x+1$

l_1: tangent line

Slope of tangent line at $x=2$ is $f'(2)$

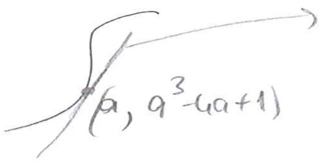
$f'(x)=3x^2-4 \qquad f'(2)=3\cdot 2^2-4=8$

$m_{l_2}\cdot m_{l_1}=-1 \Rightarrow m_{l_2}\cdot 8=-1 \Rightarrow m_{l_2}=-\frac{1}{8}$ → slope of second line

equation of l_2: $y-1=-\frac{1}{8}(x-2)$

$\Rightarrow y=-\frac{x}{8}+\frac{1}{4}+1$

b)



→ slope $\Rightarrow f'(a)=8$ solve

$f'(x)=3x^2-4$

$3x^2-4=8$

$3x^2-12=0$

$3(x^2-4)=0$

$x=\pm 2$

If $x=2$, $y=1$ $\Rightarrow y-1=8(x-2) \rightarrow y=8x-15$

If $x=-2$, $y=1$ $\Rightarrow y-1=8(x-(-2)) \rightarrow y=8x+17$

2) How many tangent lines to the curve $y = \frac{x}{x+1}$ pass through the point (1,2)?
At which points do these tangent lines touch the curve?

 $\left(a, \frac{a}{a+1}\right)$, slope $= f'(a)$ $\left(f(x) = \frac{x}{x+1}\right)$ $\quad f'(x) = \frac{1}{(x+1)^2}$

$= \frac{1}{(a+1)^2}$

eqn. of tangent : $y - \frac{a}{a+1} = \underbrace{\frac{1}{(a+1)^2}}_{\text{slope}} (x-a)$

(1,2) passes through line $\to$ substitute

$$2 - \frac{a}{a+1} = \frac{1-a}{(a+1)^2}$$

$$\frac{2a+2-a}{a+1} = \frac{1-a}{(a+1)^2}$$

$$(a+2)(a+1)^2 = (1-a)(a+1)$$

$$(a+1)\left[a^2+3a+2+a-1\right] = 0$$

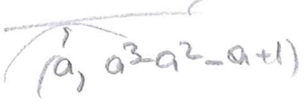
$$(a+1)\cdot(a^2+4a+1) = 0$$

$a = -1$ $\to$ not in domain

$a = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$ $\to$ 2 solns. 2 lines

at $x = \sqrt{3}-2$
and $x = -\sqrt{3}-2$

3) Find the equations of all horizontal lines are tangent to the curve $y=x^3-x^2-x+1 = f(x)$



$\longrightarrow$ horizontal $\Rightarrow$ slope $=0$

$= f'(a)$

$3a^2-2a-1=0$

$3a \quad +1$

$a \quad -1$

$(3a+1).(a-1)=0$

$a=-\frac{1}{3}$ $\quad$ $a=1$

$y=x^3-x^2-x+1$
$= -\frac{1}{27}-\frac{1}{9}+\frac{1}{3}+1 = \frac{32}{27}$

$m=0$ horizontal line

eqn of tangent;

$y-\frac{32}{27} = 0(x-(-\frac{1}{3}))$

$\boxed{y=\frac{32}{27}}$

$y=1-1-1+1=0$

$m=0$ (horizontal line)

$y-0=0(x-1)$

$\boxed{y=0}$

4) where is the function $h(x)=|x-1|+|x+2|$ differentiable Give a formula for h' and sketch the graph of h and h'!

Soln: $x\geq 1 \Rightarrow |x-1|=x-1$ $\quad$ $x\geq -2 \Rightarrow |x+2|=x+2$

$x<1 \Rightarrow |x-1|=1-x$ $\quad$ $x<-2 \Rightarrow |x+2|=-x-2$

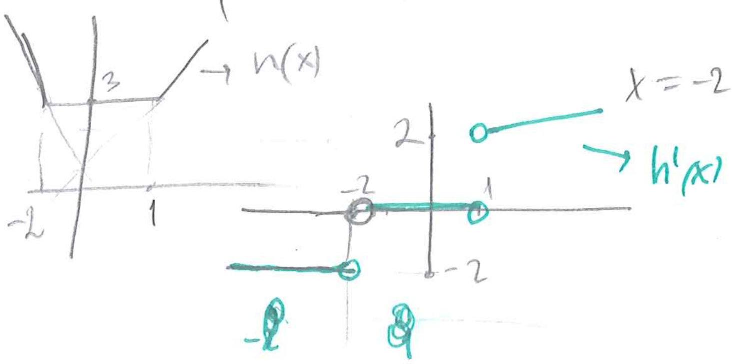
$$h(x)=\begin{cases} x-1+x+2 & x\geq 1 \\ 1-\cancel{x}+\cancel{x}+2 & -2\leq x<1 \\ 1-x-x-2 & x<-2 \end{cases} \Rightarrow h(x)=\begin{cases} 2x+1 & x\geq 1 \\ 3 & -2\leq x<1 \\ -1-2x & x<-2 \end{cases}$$

$$h'(x)=\begin{cases} 2 & x>1 \\ 0 & -2<x<1 \\ -2 & x<-2 \end{cases}$$

Check cut points $x=1$ & $x=-2$

$x=1$ $\quad \lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$ $\quad \lim_{x\to 1^+}\frac{2x+1-3}{x-1}=2$ $\quad \lim_{x\to 1^-}\frac{3-3}{x-1}=0$ $\quad \neq$ not diffble at $x=1$

$x=-2 \Rightarrow \lim_{x\to -2}\frac{f(x)-f(-2)}{x-(-2)}$ $\quad \lim_{x\to -2^+}\frac{3-3}{x+2}=0$ $\quad \lim_{x\to -2^-}\frac{-1-2x-3}{x+2}=-2$ $\quad \neq$ not diffble at $x=-2$



5) Solve the following eqns:

sol^n:

a) $\log_5 x = 2$ $\longrightarrow$ $x = 5^2 = 25$

b) $\ln 5 + \ln x = 2$ $\longrightarrow$ $\ln 5x = 2 \to 5x = e^2 \quad x = \frac{e^2}{5}$

c) $\log_2(x+1) - \log_2(x-1) = 1 \longrightarrow \log_2 \frac{x+1}{x-1} = 1 \to \frac{x+1}{x-1} = 2^1 \to x+1 = 2x-2$
$3 = x$

d) $3^{2x+1} = 5^x \longrightarrow \ln 3^{2x+1} = \ln 5^x \to$
$(2x+1)\ln 3 = x \cdot \ln 5$
$2\ln 3 \cdot x + \ln 3 = \ln 5 \cdot x \to x = \frac{\ln 3}{\ln 5 - 2\ln 3}$

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6) Find the derivative of the following functions.

a) $f(x) = e^{\sin(x^3-3)} \longrightarrow f'(x) = e^{\sin(x^3-3)} \cdot \cos(x^3-3) \cdot 3x^2$

b) $f(x) = \ln \frac{\tan x}{e^{\tan x}} \longrightarrow f(x) = \ln \tan x - \ln e^{\tan x} = \ln \tan x - \tan x$
$f'(x) = \frac{\sec^2 x}{\tan x} - \sec^2 x$

c) $f(x) = \ln(3^{x^2+1} \ln x) \to f(x) = \ln 3^{(x^2+1)} + \ln(\ln x) \to f'(x) = 2x \cdot \ln 3 + \frac{\frac{1}{x}}{\ln x}$
$= (x^2+1) \cdot \ln 3 + \ln(\ln x)$

d) $f(x) = x(x-1)(x-2)(x-3) \to \ln f(x) = \ln(x(x-1)(x-2)(x-3)) = \ln x + \ln(x-1) + \ln(x-2) + \ln(x$

e) $f(x) = x^{\sec x} \longrightarrow \ln f(x) = \ln x^{\sec x} \quad \ln f(x) = \sec x \ln x$
diff $\downarrow \frac{f'(x)}{f(x)} = \sec x \tan x \cdot \ln x + \sec x \cdot \frac{1}{x}$

f) $f(x) = e^{\sqrt{x}} \longrightarrow f'(x) = e^{\sqrt{x}} \frac{1}{2\sqrt{x}}$

g) $f(x) = \ln(2+\sin x) \to f'(x) = \cos x/(2+\sin x)$

h) $f(x) = (x^2+1)^{\sqrt[3]{x}} \to f(x) = e^{\ln (x^2+1)^{\sqrt[3]{x}}} \to f(x) = e^{\sqrt[3]{x} \cdot \ln(x^2+1)}$

i) $f(x) = e^{\arcsin x}$

j) $f(x) = \arctan(x^2+1)$

7) Use logarithmic differentiation to find the derivative of the following function.

$$f(x) = \frac{\sin^2 x \tan^4 x}{(x^2+1)^2}$$

$$\ln f(x) = \ln \frac{\sin^2 x \tan^4 x}{(x^2+1)^2} = \ln \sin^2 x + \ln \tan^4 x - \ln (x^2+1)^2$$

diff both sides.

$$\frac{f'(x)}{f(x)} = \frac{2\sin x \cos x}{\sin^2 x} + \frac{4\tan^3 x}{\tan^4 x} - \frac{2(x^2+1) \cdot 2x}{(x^2+1)^2}$$

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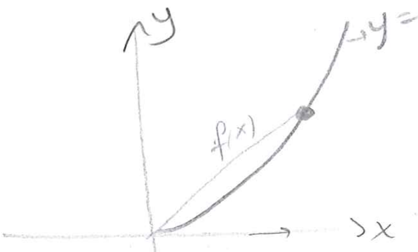
8) A particle moves along a parabola $y=x^2$ in the first quadrant in such a way that its x-coordinate increases at a steady 17 units per-second. How fast is the distance of the particle to the origin changing when $x=4$?



let distance $\longrightarrow f(t) = \sqrt{(x(t))^2 + (y(t))^2}$

$x = x(t)$
$y = y(t)$

$$= \sqrt{(x(t))^2 + \underbrace{(x(t))^4}_{y=x^2}}$$

$$\frac{dx}{dt} = 17 \text{ units/sec.}$$

$$\left.\frac{df}{dt}\right|_{x=4} = ?$$

$$\frac{d f(t)}{dt} = \frac{1}{2\sqrt{(x(t))^2 + (x(t))^4}} \cdot \left(2 \underset{4}{x(t)} \cdot \underset{17}{x'(t)} + 4 \underset{4}{(x(t))^3} \cdot \underset{17}{x'(t)}\right)$$

$$\left.\frac{d f(t)}{dt}\right|_{x=4} = \frac{2 \cdot 4 \cdot 17 + 4 \cdot 4^3 \cdot 17}{2\sqrt{4^2 + 4^4}} = \frac{\cancel{17}^{\sqrt{17}} \cdot \cancel{8} \cdot (1+32)}{\cancel{2 \cdot 4} \cdot \sqrt{\cancel{17}}} = 33\sqrt{17}$$

9) Evaluate the limits:

a) $\lim_{x\to\infty} \frac{5^{x+1} - 4^x}{3^{x+1} - 2^{x-2}}$

b) $\lim_{x\to 1} \frac{x^e - 1}{x^\pi - 1}$

c) $\lim_{x\to 0^+} (\ln \frac{1}{x})^x = \lim_{x\to 0^+} e^{\ln(\ln\frac{1}{x})^x} = e^{\lim_{x\to 0^+} \frac{\ln(\ln\frac{1}{x})}{\frac{1}{x}}}$ use LH

d) $\lim_{x\to\infty} (e^x + x)^{1/x} \to \lim_{x\to\infty} e^{\ln(e^x+x)^{1/x}}$

e) $\lim_{x\to 1} \frac{\sqrt{2x - x^4} - x^{1/3}}{1 - x^{3/4}} \to$ LH

f) $\lim_{x\to\infty} x \cdot \sin(1/x) \to$ LH OR $\lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{t\to 0} \frac{\sin t}{t} = 1$

$t = \frac{1}{x} \Rightarrow x\to\infty,\ t\to 0$

g) $\lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x} \cdot x \sin\frac{1}{x} = 0$

$\frac{x}{\sin x} \to 1$, $x\sin\frac{1}{x} \to 0$ by squeeze

$-1 \leq \sin\frac{1}{x} \leq 1$

$-|x| \leq x\sin\frac{1}{x} \leq |x|$

$-|x| \to 0$, $|x| \to 0$

equal

so $\lim_{x\to 0} x\sin\frac{1}{x} = 0$

h) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

$\downarrow$

$\left(\lim_{x\to 0^+} \frac{\sin x - x}{x \sin x} \overset{LH}{=} \right.$ 2 L Hospitals $\left.\right)$

10) Find limit $\lim_{x\to 0} \frac{\sinh x}{x}$ without using l'Hospital's rule.

$$\lim_{x\to 0} \frac{\sinh x - 0}{x - 0} \underset{\downarrow}{=} f'(0) = 1.$$

Let $f(x) = \sinh(x)$
$f'(x) = \cosh(x)$
$f'(0) = 1$

11) Use the Squeeze Thm to show that
$\lim_{x\to 0^+} \sqrt{x}\,(1+\sin^2(2/x)) = 0$

$-1 \leq \sin\frac{2}{x} \leq 1$ for all $x \in \mathbb{R}$

$0 \leq \sin^2\frac{2}{x} \leq 1 \Rightarrow$ (add 1 to everywhere) $1 \leq 1+\sin^2\frac{2}{x} \leq 2$

mult with $\sqrt{x}$

$$\sqrt{x} \leq \sqrt{x}\left(1+\sin^2\frac{2}{x}\right) \leq 2\sqrt{x}$$

$\sqrt{x} \to 0$ as $x\to 0^+$, $2\sqrt{x} \to 0$ as $x \to 0^+$

equal so by squeeze thm $\lim_{x\to 0^+} \sqrt{x}\left(1+\sin^2\frac{2}{x}\right)$

12) Find the value of k so that the function

$$f(x)=\begin{cases} x^2+kx+5 & \text{if } x\le 1 \\ (x-k)^2+(9-k)\,x & \text{if } x>1 \end{cases}$$

Is continuous at $x=1$.

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for continuity we need $\lim_{x\to 1} f(x) = f(1)$

for limit check left & right limit

$f(1) = 1+k+5 = 6+k$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2+kx+5) = 6+k$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} ((x-k)^2+(9-k)\,x) = (1-k)^2+9-k$

must be equal for existence of limit

$k^2-2k+1+9-k = 6+k$

$\Rightarrow k^2-4k+4=0$

$(k-2)^2=0 \Rightarrow k=2$

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13) Prove that $\frac{x}{x^2+1} < \arctan x < x \quad \forall x>0$

by using MVT.

a) showing $\arctan x < x$

let $f(x)=\arctan x$
cont on $[0,x]$
diff'ble on $(0,x)$

So by MVT $\exists c\in(0,x)$

st $f'(c) = \frac{f(x)-f(0)}{x-0}$

$\frac{1}{1+c^2} = \frac{\arctan x}{x} \Rightarrow \frac{\arctan x}{x} < 1$

<1

$\Rightarrow \arctan x < x$

b) showing $\frac{x}{x^2+1} < \arctan x$

let $f(x)=\arctan x$ cont on $[0,x]$
diff'ble on $[0,x]$

So by MVT $\exists c\in[0,x)$ s.t.

$f'(c) = \frac{f(x)-f(0)}{x-0} \Rightarrow \frac{1}{1+c^2} = \frac{\arctan x}{x}$

$c\in(0,x) \Rightarrow 0<c<x \Rightarrow c^2<x^2$

$1+c^2<1+x^2$

$\frac{1}{1+c^2} > \frac{1}{1+x^2}$

$\frac{\arctan x}{x}$

$\frac{\arctan x}{x} > \frac{1}{1+x^2} \Rightarrow \arctan x > \frac{x}{1+x^2}$