1) Convergent or Divergent?

8) $\int_{-1}^{1} \frac{2^{\sin^{-1}x}}{1-x}\,dx \to \int_{-1}^{1} \frac{2^{\arcsin x}}{1-x}\,dx$ $= f(x)$

$2^{\arcsin x} > 0$

$x < 1 \Rightarrow 1-x > 0$ $\Big\} \frac{2^{\arcsin x}}{1-x} > 0$

Apply LCT with $\int_{-1}^{1} \frac{1}{1-x}\,dx$ $\left(\frac{1}{1-x} > 0\right)$

$\lim_{x\to 1} \frac{\frac{2^{\arcsin x}}{\cancel{1-x}}}{\frac{1}{\cancel{1-x}}} = 2^{\pi/2} \to$ finite nonzero limit $\Rightarrow$ integrals $\int_{-1}^{1} f(x)\,dx$ and $\int_{-1}^{1} \frac{1}{1-x}\,dx$ both conv or diverge

consider $\int_{-1}^{1} \frac{1}{1-x}\,dx = -\int_{2}^{0} \frac{1}{t}\,dt \to$ divergent by p test

$1-x = t$
$-dx = dt$
$x = 1 \Rightarrow t = 0$
$x = -1 \Rightarrow t = 2$

So $\int_{-1}^{1} \frac{2^{\arcsin x}}{1-x}\,dx$ is divergent by limit comparison test.

9) $\int_{1}^{\infty} \frac{\ln x}{x+a}\,dx \quad a \in \mathbb{R}^+$

$x > 1 \Rightarrow \ln x > 0 \Rightarrow$ $\frac{\ln x}{x+a} > 0$ for $x > 1$

$\frac{\ln x}{x} > 0$ for $x > 1$

$\lim_{x\to\infty} \frac{\frac{\ln x}{x+a}}{\frac{\ln x}{x}} = 1$ $\begin{matrix} \neq 0 \\ \neq \infty \end{matrix}$ finite limit $\Big\} \Rightarrow \int_{1}^{\infty} \frac{\ln x}{x+a}\,dx$ & $\int_{1}^{\infty} \frac{\ln x}{x}\,dx$ both conv. or diverge

Check $\int_{1}^{\infty} \frac{\ln x}{x}\,dx = \lim_{a\to\infty} \int_{1}^{a} \frac{\ln x}{x}\,dx = \lim_{a\to\infty} \int_{0}^{\ln a} u\,du = \lim_{a\to\infty} \frac{u^2}{2}\Big|_0^{\ln a}$

$\ln x = u$
$\frac{1}{x}dx = du$

$= \lim_{a\to\infty} \left(\frac{(\ln a)^2}{2} - 0\right)$
$= \infty$

$\int_{1}^{\infty} \frac{\ln x}{x}\,dx$ is divergent (by direct computation) then by LCT $\int_{1}^{\infty} \frac{\ln x}{x+a}\,dx$ is divergent

h) $\int_0^\infty \frac{1-\cos x}{x^2}\,dx \Rightarrow \int_0^1 \frac{1-\cos x}{x^2}\,dx \qquad \int_1^\infty \frac{1-\cos x}{x^2}\,dx$

$$\lim_{x\to0}\frac{1-\cos x}{x^2}=\lim_{x\to0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}=\lim_{x\to0}\underbrace{\frac{\sin^2 x}{x^2}}_{1}\frac{1}{(1+\cos x)}=\frac{1}{2}$$

not improper (function has a bounded limit at $x=0$. so

$\int_1^\infty \frac{1-\cos x}{x^2}\,dx \to -1\le \cos x\le 1$

$-1\le -\cos x\le +1 \to 0\le 1-\cos x\le 2$

divide everywhere x^2

$\underline{\underline{0}}\le \frac{1-\cos x}{x^2}\le\frac{2}{x^2}$

$\int_1^\infty \frac{2}{x^2}\,dx \to p=2>1$ conv. by p-test

So $\int_1^\infty \frac{1-\cos x}{x^2}\,dx$ is convergent by comparison test

Since two parts are convergent $\int_0^\infty \frac{1-\cos x}{x^2}\,dx$ is conv.

---

2) Evaluate areas of finite regions bounded by the curves

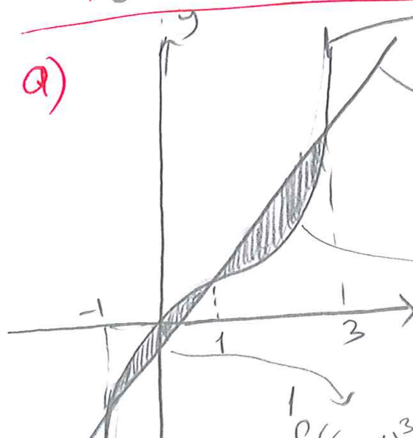
a) $y=(x-1)^3+1$ and $y=4x-3$

b) $y=(x+1)^2$, $x=\sin y$, $y=0$ and $y=1$ for $x\ge 0$

c) $y=\arcsin x$, $y=\arccos x$ and x axis.

---

a)



$y=(x-1)^3+1$

$y=4x-3$

to find intersections:

$(x-1)^3+1=4x-3$

$x^3-3x^2+3x-1+1=4x-3$

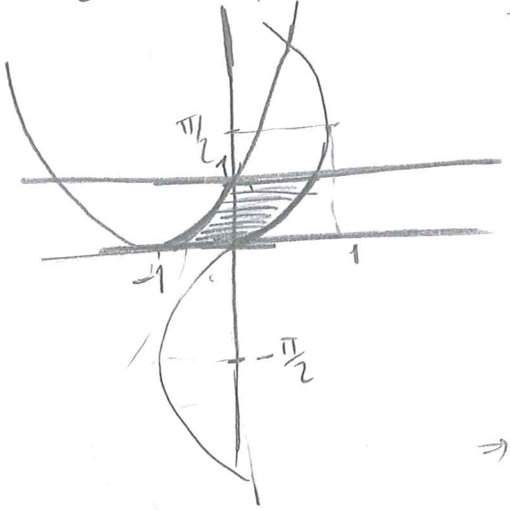
$x^3-3x^2-x+3=0 \to$ guess $x=1$ is one of the roots.

$$\begin{array}{l|l} x^3-3x^2-x+3 & x-1 \\ \cancel{\mp x^3}\pm x^2 & x^2-2x-3=(x+1)(x-3) \\ \hline -2x^2-x+3 & \quad x=-1 \quad x=3 \\ \pm 2x^2 \mp 2x & \\ \hline -3x+3 & \\ -3x+3 & \\ \hline 0 & \end{array}$$

$$\int_{-1}^{1}\left(\left((x-1)^3+1\right)-(4x-3)\right)dx+\int_1^3\left((4x-3)-\left((x-1)^3+1\right)\right)dx$$

③

b) $y=(x+1)^2$, $x=\sin y$, $y=0$ $y=1$ $x\geq 0$.

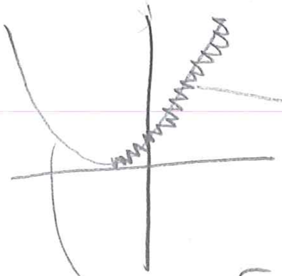


$$\int_0^1 \left(\sin y-(\sqrt{y}-1)\right)dy$$

$y=(x+1)^2$

$\Rightarrow x+1=\sqrt{y}$

$x=\sqrt{y}-1$



$\rightarrow x=\sqrt{y}-1$

$\rightarrow x=-\sqrt{y}+1.$

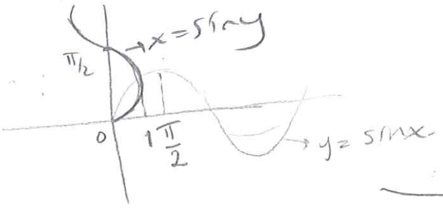
c) $y=\arcsin x$, $y=\arccos x$ & x-axis.

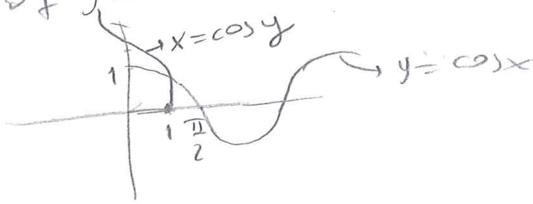
$x=\sin y$

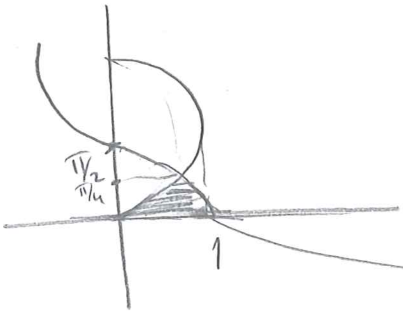
Symmetric of $y=\sin x$ w.r.t. $y=x$

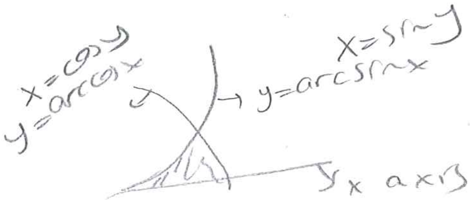
$x=\cos y$.

symmetric of $y=\cos x$ w.r.t $y=x$





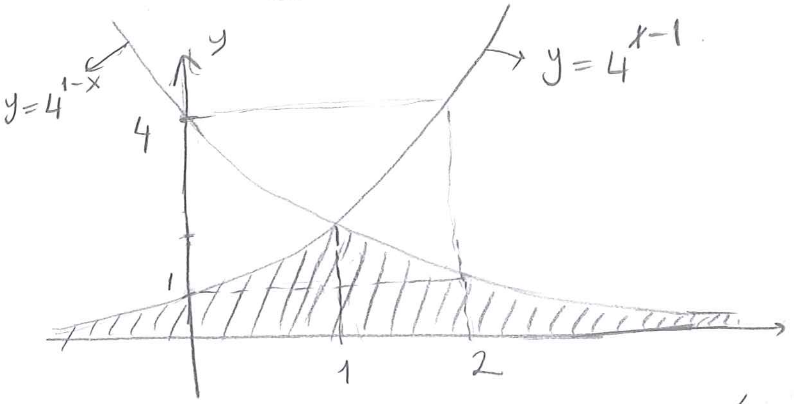




$$\int_0^{\pi/4} (\cos x-\sin y)\,dy$$

when everything is in terms of y the function on the right hand side is greater.

3.Q) Evaluate area of a region lying below the curves $y = 4^{x-1}$, $y = 4^{1-x}$ and above x-axis.



$$\int_{-\infty}^{1} 4^{x-1}\,dx \qquad\qquad \int_{1}^{\infty} 4^{1-x}\,dx = \lim_{a\to\infty}\int_{1}^{a} 4\cdot 4^{-x}\,dx = \lim_{a\to\infty} 4\left(\frac{4^{-x}}{-\ln 4}\right)\Bigg|_{1}^{a}$$

$$= \lim_{a\to\infty} 4\cdot\left(\frac{4^{-a}}{-\ln 4} - \frac{4^{-1}}{-\ln 4}\right)$$

$$= 4\cdot\frac{1}{4\ln 4} = \frac{1}{\ln 4}$$

$$= \lim_{a\to-\infty}\int_{a}^{1} 4^{x}\cdot\frac{1}{4}\,dx$$

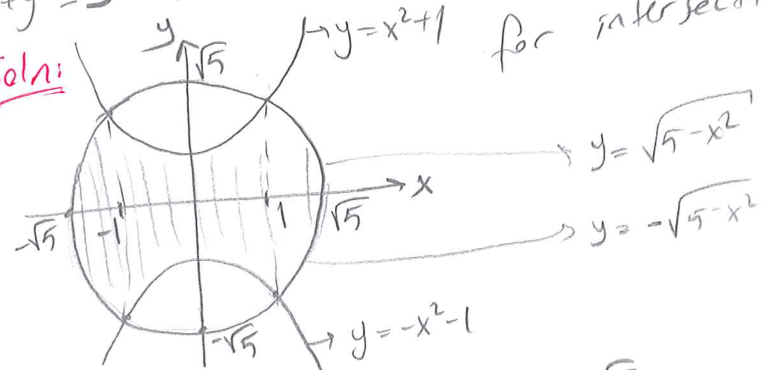
$$= \lim_{a\to-\infty}\frac{1}{4}\,4^{x}\cdot\frac{1}{\ln 4}\Bigg|_{a}^{1} = \lim_{a\to-\infty}\frac{1}{4\ln 4}\left(4^{1}-4^{a}\right), \quad 4^{a}\to 0$$

$$= \frac{4}{4\ln 4} = \frac{1}{\ln 4}$$

$$\text{area} = \frac{1}{\ln 4} + \frac{1}{\ln 4} = \frac{2}{\ln 4}$$

4Q) Parabolas $y = x^2+1$ and $y = -x^2-1$ divide the circle $x^2+y^2=5$ into three parts. Find the area of the middle part.

Soln:



for intersections put $y=x^2+1$ into $x^2+y^2=5$

$x^2 = y-1$

$y-1+y^2=5$

$y^2-y-6=0$

$(y-2)(y+3)=0$

$y=2$ $\qquad$ ~~$y=-3 \Rightarrow x^2=-4$~~

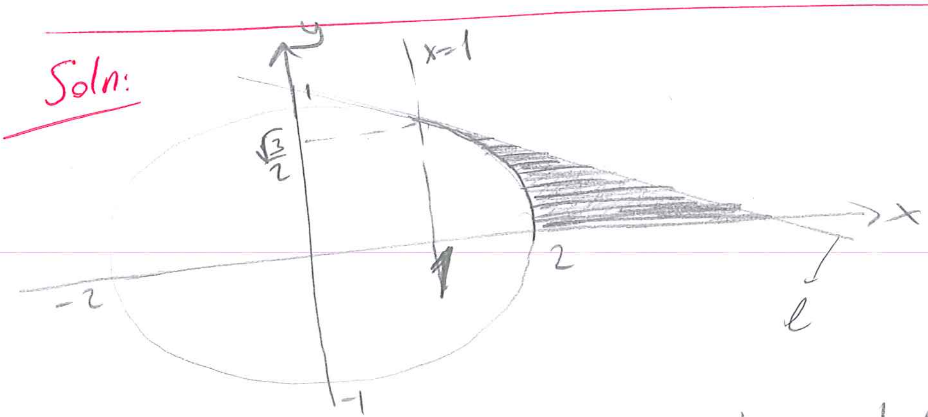
$x^2=2-1$

$x=\pm 1$

$$2\cdot\left[\int_{0}^{1}\left((x^2+1)-(-x^2-1)\right)dx + \int_{1}^{\sqrt{5}}\left(\sqrt{5-x^2}-\left(-\sqrt{5-x^2}\right)\right)dx\right]$$

5a) Let ℓ be the tangent line to the ellipse $\frac{x^2}{4}+y^2=1$ at a point $(1, \frac{\sqrt{3}}{2})$. Find the area of the region bounded by the ellipse, the tangent line, $x-2x\sqrt{3}$ and lying to the right of the line $x=1$ and above the line $y=0$.

Soln:



for slope of ℓ find $\frac{dy}{dx}$ at $(1, \frac{\sqrt{3}}{2})$

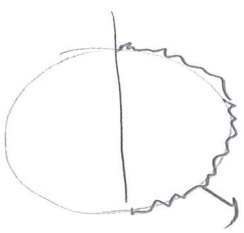
implicit differentiation of $\frac{x^2}{4}+y^2=1$

$$\frac{2x}{4}+2yy'=0 \Rightarrow y'=-\frac{x}{4y}$$

$$y'\Big|_{(1,\frac{\sqrt{3}}{2})} = -\frac{1}{4\cdot\frac{\sqrt{3}}{2}} = -\frac{1}{2\sqrt{3}}$$

Equation of ℓ: $y-\frac{\sqrt{3}}{2} = -\frac{1}{2\sqrt{3}}(x-1)$

$$-2\sqrt{3}y+3 = x-1 \Rightarrow x=-2\sqrt{3}y+4$$

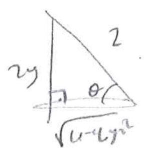


$\frac{x^2}{4}+y^2=1$

$x^2=4-4y^2$

$x=\sqrt{4-4y^2}$

$$\text{area} = \int_0^{\frac{\sqrt{3}}{2}} \Big(\underbrace{(4-2\sqrt{3}y)}_{\text{above function}} - \underbrace{\sqrt{4-4y^2}}_{\text{below function}}\Big)\,dy$$



$\sin\theta = y$

$\cos\theta\, d\theta = dy$

$\frac{\sqrt{4-4y^2}}{2} = \cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

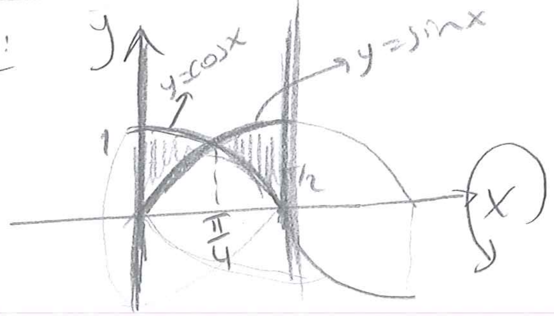
$$=\int_0^{\frac{\sqrt{3}}{2}} (4-2\sqrt{3}y)\,dy - \int_0^{\pi/3} \underbrace{2\cos\theta\cdot\cos\theta}_{\cos 2\theta+1}\,d\theta$$

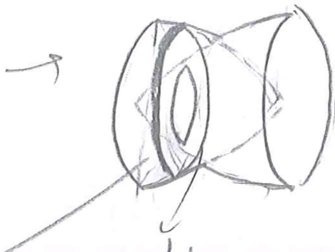
$$= 4y - \frac{2\sqrt{3}y^2}{2}\Big|_0^{\frac{\sqrt{3}}{2}} - \left(\frac{\cos 2\theta}{2}+\theta\right)\Big|_0^{\pi/3}$$

$$=\left(4\frac{\sqrt{3}}{2}-\frac{3}{4}\sqrt{3}\right)-0-\left(-\frac{1}{4}+\frac{\pi}{3}-0\right) = \frac{5\sqrt{3}}{4}+\frac{1}{4}-\frac{\pi}{3}$$

6 B) Find the volume of the solid generated by rotating the finite region bounded by the region bounded by the curves $y=\sin x$, $y=\cos x$, $x=0$ and $x=\frac{\pi}{2}$ about x-axis.

Soln:





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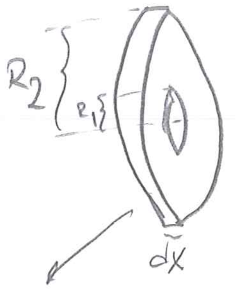
Strip



discs perpendicular to x axis so we have everything in terms of x.

$$dV=(\pi R_2^2-\pi R_1^2)\,dx$$
$$=\pi(R_2^2-R_1^2)\,dx$$

from $x=0$ to $x=\frac{\pi}{4}$: $R_2=\cos x$, $R_1=\sin x$

from $x=\frac{\pi}{4}$ to $x=\frac{\pi}{2}$: $R_2=\sin x$, $R_1=\cos x$

$$\text{Volume}=\int_0^{\frac{\pi}{4}}\pi\cdot(\underbrace{\cos^2x-\sin^2x}_{\cos 2x})\,dx+\int_{\frac{\pi}{4}}^{\pi/2}\pi\cdot(\underbrace{\sin^2x-\cos^2x}_{-\cos 2x})\,dx$$

$$=\pi\cdot\frac{\sin 2x}{2}\Big|_0^{\pi/4}+\pi\cdot\frac{-\sin 2x}{2}\Big|_{\pi/4}^{\pi/2}$$

$$=\pi\cdot\left(\frac{1}{2}-0\right)+\pi\cdot\left(-0-\left(-\frac{1}{2}\right)\right)=\frac{\pi}{2}+\frac{\pi}{2}=\pi.$$

78) Find the volume of the solid generated by rotating the finite region bounded by the curves

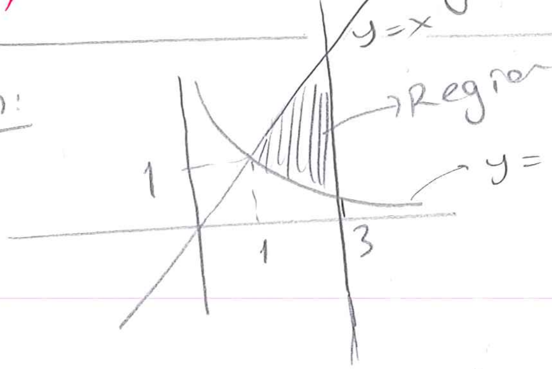
$y = x \quad y = x^{-1} \quad x = 3$

a) about the x-axis

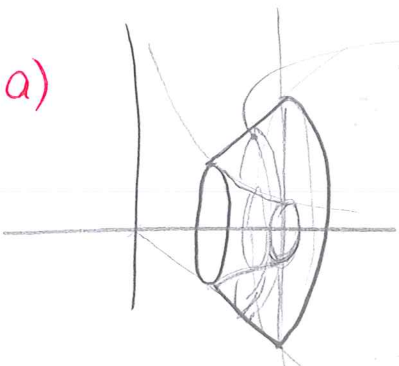
b) about the line $y = 1$

c) about the y axis.

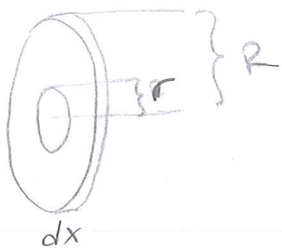
Soln:



a)



slice



Slices are perpendicular to x axis
write R & r in terms of x.

$R = x$

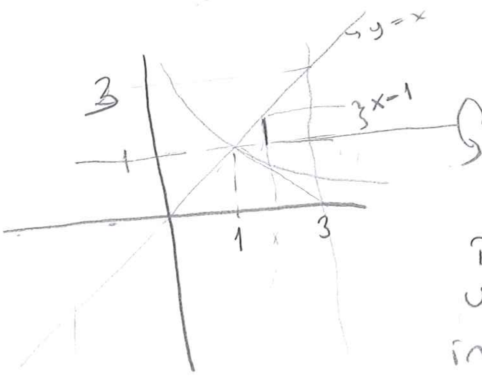
$r = \frac{1}{x}$

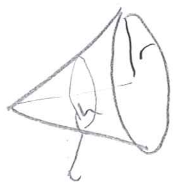
$$\underbrace{dV = (\pi R^2 - \pi r^2)dx}$$

$$\text{volume} = \int_1^3 \pi \cdot \left(x^2 - \frac{1}{x^2}\right) dx$$

$$= \pi \cdot \left(\frac{x^3}{3} + \frac{1}{x}\right)\Big|_1^3 = \pi \cdot \left(\frac{28}{3} - \frac{4}{3}\right) = \pi \cdot 8$$

b)





Cone volume $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \cdot 2^2 \cdot 2$

$= \frac{8\pi}{3}$

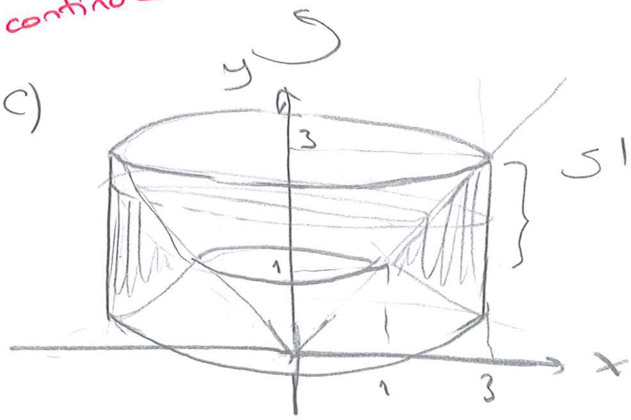
By using integral

$\to \pi r^2 \cdot dx$

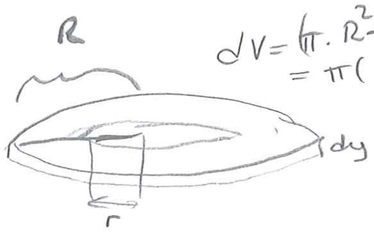
$$\int_1^3 \pi \cdot (x-1)^2 dx$$

$$= \pi \frac{(x-1)^3}{3}\Big|_1^3 = \frac{\pi \cdot 8}{3} - 0 \;//$$

c)

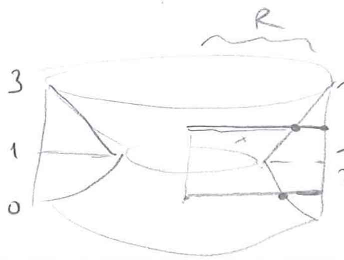


With
slicing / Washer
method



$dV = (\pi \cdot R^2 - \pi r^2)\, dy$
$= \pi (R^2 - r^2)\, dy$

Slices perp. to y axis
so R, r will be in terms of y



R = 3 all time
r ends at $y=x$ all time $\to x = y$ or $r = y$ } $1 \le y \le 3$

$\to$ R = 3 all time
r ends at $y = x^{-1}$, $x = y^{-1}$ } $r = y^{-1}$ } $0 \le y \le 1$

empty inside

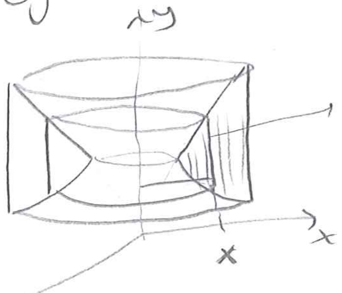
$$\text{Volume} = \int_0^1 \pi (3^2 - y^{-2})\, dy + \int_1^3 \pi (3^2 - y^2)\, dy$$

$$= \pi \cdot \left(9y - \frac{y^{-1}}{-1}\right)\Big|_0^1 + \pi \left(9y - \frac{y^3}{3}\right)\Big|_1^3$$

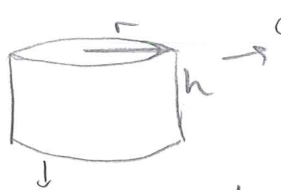
$$= \pi (9 + 1 - 0) + \pi \cdot \left(27 - \frac{27}{3} - \left(9 - \frac{1}{3}\right)\right)$$

$$= 10\pi +$$

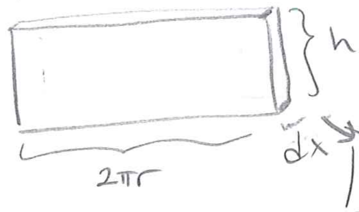
Cylindrical Shell method



Shells perpendicular to x axis so everything in terms of x



cut open :)



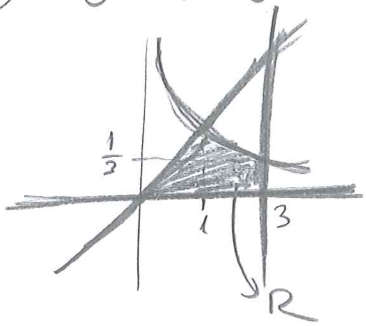
$\boxed{dV = 2\pi r h\, dx}$

h touches $y = x^{-1}$ at the bottom & $y = x$ at the top all the time, so $h = x - x^{-1}$. x changes from 1 to 3
"distance of h to y axis" $= x = r$ (when we take 1 we take -1 so we don't write -1 to -3 etc. Just take one side in the integral)

$$\text{Volume} = \int_1^3 dV = \int_1^3 2\pi \cdot x \cdot (x - x^{-1})\, dx = \int_1^3 2\pi \cdot (x^2 - 1)\, dx$$

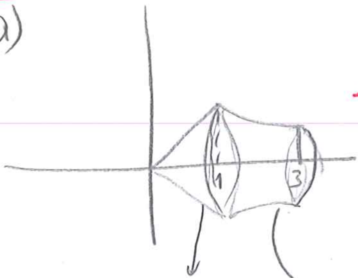
$$= 2\pi \cdot \left(\frac{x^3}{3} - x\right)\Big|_1^3 =$$

8 (a) Same as previous question but region is bounded by $y=x$, $y=x^{-1}$ $x=3$ and x-axis



a) rotate about x-axis
b) rotate about the line $y=1$
c) rotate about the y axis.

a)



$\boxed{\text{Disc method}}$

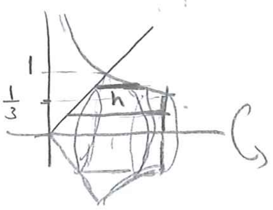
$\to \pi r^2 dx$ (discs perp to x axis write r in terms of "x")

for $x \in [0,1]$ $r=x$
$x \in [1,3]$ $r=x^{-1}$

$$\int_0^1 \pi \cdot x^2 dx + \int_1^3 \pi \cdot x^{-2} dx = \frac{\pi x^3}{3}\Big|_0^1 + \frac{\pi \cdot x^{-1}}{-1}\Big|_1^3$$

$$= \frac{\pi}{3} - 0 + \left(-\frac{\pi}{3} + \pi\right) = \pi \,//$$

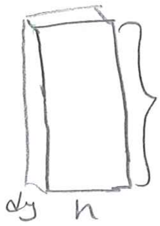
If we use $\boxed{\text{shell method}}$



cylinders perpendicular to y axis
write r, h in terms of y.

cut open

$\to dV = 2\pi r h\, dy$



$r=y$ for $y \in [0,1]$

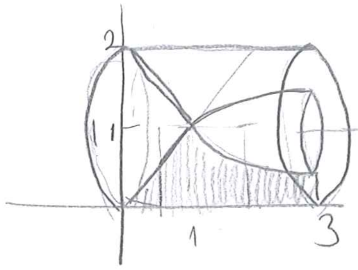
$h = $ $y \in [0, 1/3] \Rightarrow$ h cuts $x=y$ and $x=3$ so $h = 3-y$
$y \in [1/3, 1] \Rightarrow$ h cuts $x=y$ and $x=y^{-1}$ so $h = \underbrace{y^{-1}-y}$ ($y^{-1} > y$ there)

$$\Rightarrow \text{Volume} = \int_0^{1/3} 2\pi \cdot y \cdot (3-y)\, dy + \int_{\frac{1}{3}}^{1} 2\pi \cdot y\,(y^{-1}-y)\,dy$$

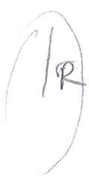
$$= \left(3\pi y^2 - \frac{2\pi y^3}{3}\right)\Big|_0^{1/3} + \left(2\pi y - \frac{2\pi y^3}{3}\right)\Big|_{1/3}^{1}$$

$$= \frac{3\pi}{9} - \frac{2\pi}{81} - 0 + 2\pi - \frac{2\pi}{3} - \frac{2\pi}{3} + \frac{2\pi}{81} = \pi \,//$$

b)



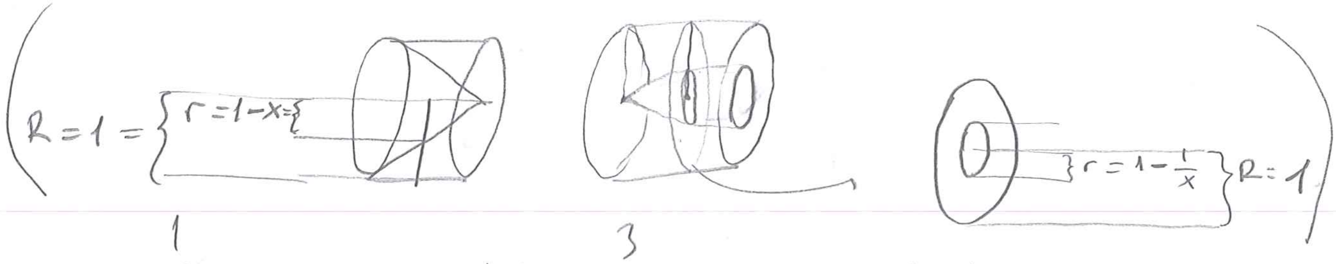
rotate about $y = 1$



$\pi R^2 dx$

$\pi (R^2 - r^2) dx$

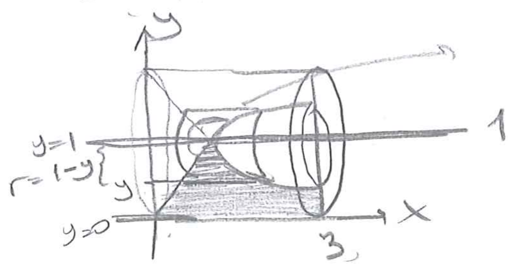
Disc method: $\int_0^1 \pi\cdot(1^2-(1-x)^2)dx + \int_1^3 \pi\cdot\left(1^2-\left(1-\frac{1}{x}\right)^2\right)dx$

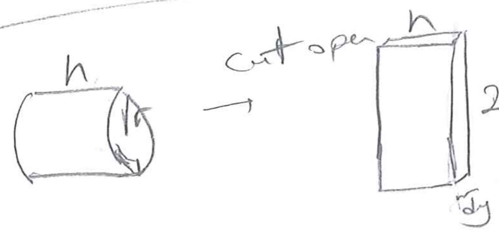


$= \pi\int_0^1 (\cancel{1} - x^2+2x-\cancel{1})dx + \pi\int_1^3 \left(\cancel{1} - \frac{1}{x^2} + \frac{2}{x} - \cancel{1}\right)dx$

$= \pi\left(\frac{x^3}{3}+x^2\right)\Big|_0^1 + \pi\cdot\left(\frac{1}{x} - 2\ln x\right)\Big|_1^3 =$

Shell method





$dV = 2\pi r h dy$

$r = 1 - y$

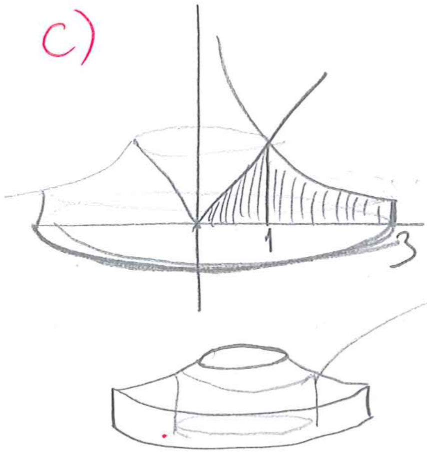
for $y \in [0, 1/3]$ $\quad h = 3 - y$

for $y \in [1/3, 1]$ $\quad h = y^{-1} - y \quad (y^{-1} > y$ there$)$

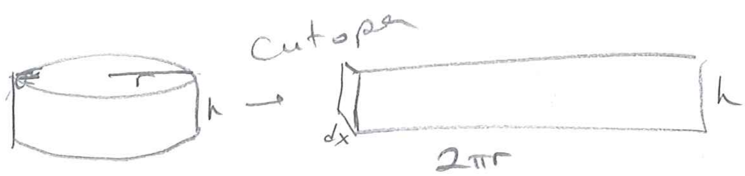
Volume $= \int_0^{1/3} 2\pi\cdot\underbrace{(1-y)\cdot(3-y)}_{3-4y+y^2}dy + \int_{1/3}^1 2\pi\cdot\underbrace{(1-y)\cdot(y^{-1}-y)}_{y^{-1}-y-1+y^2}dy$

$= 2\pi\left(\left(3y - 2y^2 + \frac{y^3}{3}\right)\Big|_0^{1/3} + \left(\ln y - \frac{y^2}{2} - y + \frac{y^3}{3}\right)\Big|_{1/3}^1\right)$

c)



Shell method



$dV = 2\pi r h \underline{dx}$ everything in terms of x

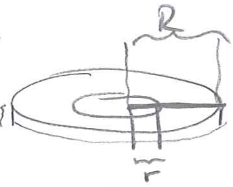
$r = x$

$\int dV = \text{Volume}$
- for $x \in [0,1]$ h between $y=0$ & $y=x$, so $h = x-0$
- for $x \in [1,3]$ h between $y=0$ and $y=x^{-1}$, so $h = x^{-1}-0$

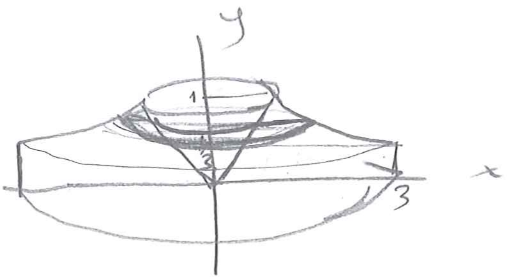
$$\int_0^1 2\pi \cdot x \cdot x \, dx + \int_1^3 2\pi \cdot x \cdot (x^{-1}-0)\,dx$$

Disc method:

Cut perp to y axis, we will get slices:



$dV = \pi \cdot (R^2 - r^2)\,dy$

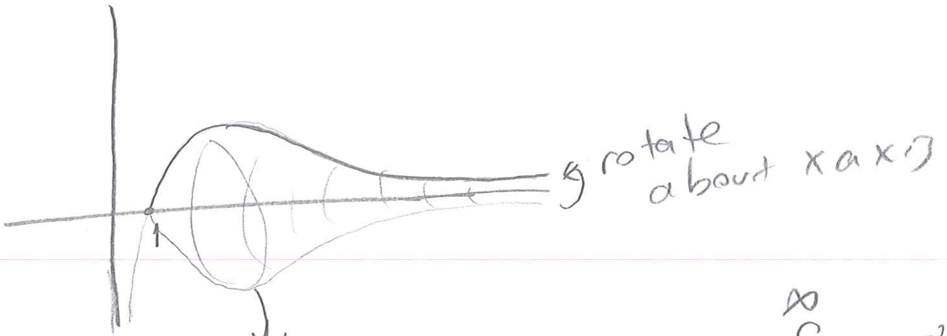


for $y \in [0, 1/3]$ $R = 3$ $r = y$ (from $y=x \equiv x=y$)

$y \in [1/3, 1] \rightarrow R = y^{-1}$ $r = y$ (from $y = x^{-1}$, $x = y^{-1}$)

$$\text{Volume} = \int_0^{1/3} \pi \cdot (9 - y^2)\,dy + \int_{1/3}^{1} \pi \cdot ((y^{-1})^2 - y^2)\,dy$$

9Θ) Determine if the infinite solid generated by rotating about x-axis the region bounded by the curves $y=e^{-x}\ln x$, $y=0$ and lying to the right of the line $x=1$ has a finite volume.



Disc method

$\rightarrow dV = \pi r^2 dx.$

dx

$r = e^{-x}\ln x$

$$V = \int_1^\infty \pi \cdot e^{-2x} (\ln x)^2 dx$$

$0 < \ln x \leq x$ for $x > 1$

$0 < (\ln x)^2 \leq x^2$

$\bigstar \quad 0 < e^{-2x} \cdot (\ln x)^2 \leq \boxed{e^{-2x} \cdot x^2}$

$e^{-2x} \cdot x^2 \geq 0$, $e^{-x} \geq 0$ we can use LCT.

consider $\int_1^\infty e^{-2x} \cdot x^2 dx$ use LCT with e^{-x}

$$\int_1^\infty e^{-x} dx = \lim_{a\to\infty} \int_1^a e^{-x} dx$$

$$= \lim_{a\to\infty} \left. \frac{e^{-x}}{-1} \right|_1^a$$

$$= \lim_{a\to\infty} e - \frac{1}{e^a} = e$$

($\frac{1}{e^a} \to 0$)

So $\int_1^\infty e^{-x} dx$ is conv.

$$\lim_{x\to\infty} \frac{e^{-2x} \cdot x^2}{e^{-x}} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{\frac{\infty}{\infty},\, LH}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\frac{\infty}{\infty},\, LH}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

then $e^{-2x} \cdot x^2 < < e^{-x}$

so $\int_1^\infty e^{-2x} x^2 dx$ is conv. by LCT.

also with $\bigstar$ and comparison test $\int_1^\infty e^{-2x} (\ln x)^2 dx$ is convergent // i.e. volume is finite

10 Q) The region $R = \{(x,y) : y \le 8 - x^2, y \ge 0, y \ge 2x\}$ is rotated

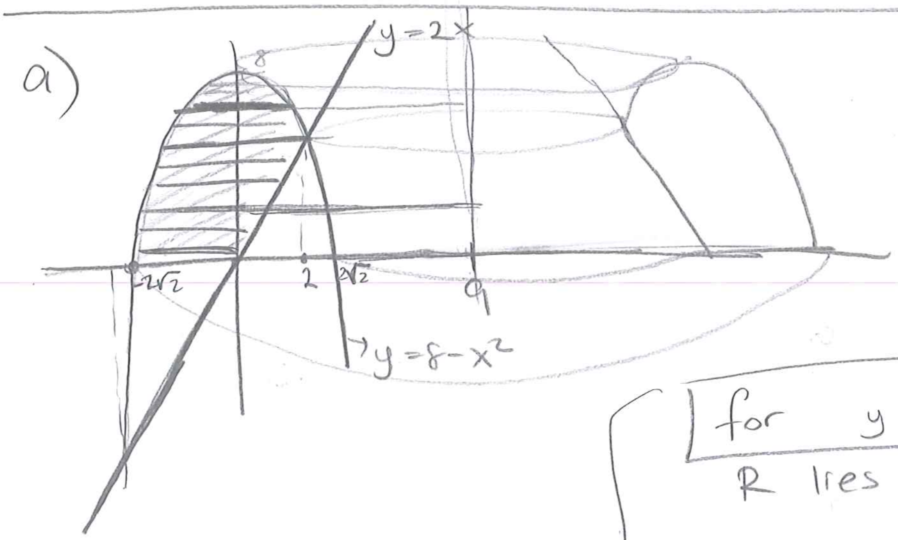
a) about the line $x = 9$

b) about the line $y = 9$

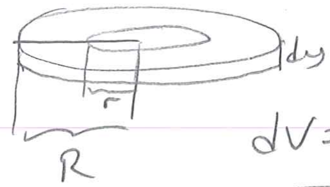
Find the volume of resulting solid.

---

a)



Cut perp to y axis we get discs



$dV = \pi(R^2 - r^2)\,dy$

$\boxed{\text{for } y \text{ from } 0 \text{ to } 4}$

R lies from left arm of $y = 8 - x^2$ to $x = 9$

write $y = 8 - x^2$ in terms of y

$x = \pm\sqrt{8-y}$ take $x = -\sqrt{8-y}$

So $R = 9 - (-\sqrt{8-y}) = 9 + \sqrt{8-y}$

r lies from $\underbrace{y = 2x}_{x = y/2}$ to $x = 9$.

$r = 9 - \frac{y}{2}$

$\boxed{\text{for } y \text{ from } 4 \text{ to } 8}$

R lies from $\underbrace{\text{left arm of } y = 8 - x^2}_{x = -\sqrt{8-y}}$ to $x = 9$

$R = 9 - (-\sqrt{8-y})$

r lies from $\underbrace{\text{right arm of } y = 8 - x^2}_{x = \sqrt{8-y}}$ to $x = 9$

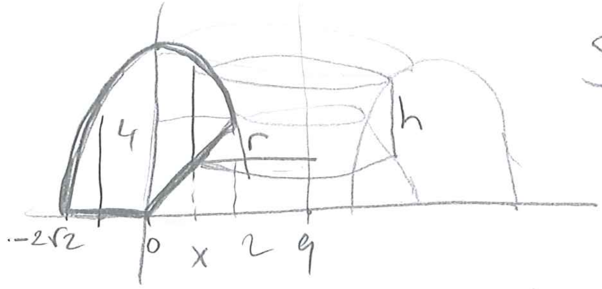
$r = 9 - \sqrt{8-y}$

So

Volume

$$= \int_0^4 \pi\left(\left(9 + \sqrt{8-y}\right)^2 - \left(9 - \frac{y}{2}\right)^2\right) dy + \int_4^8 \pi\left(\left(9 + \sqrt{8-y}\right)^2 - \left(9 - \sqrt{8-y}\right)^2\right) dy$$

evaluate integral



Shell method

$dV = 2\pi r h\,dx \rightarrow$ shells perpendicular to x axis.

$r = 9 - x$ all the time

for $x \in (-2\sqrt{2}, 0)$ h lies between $y=0$ & $y = 8 - x^2$

so $h = 8 - x^2 - 0$

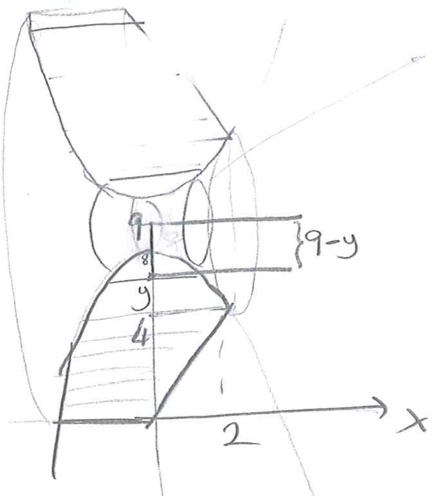
for $x \in (0, 2)$ h lies between $y = 2x$ and $y = 8 - x^2$

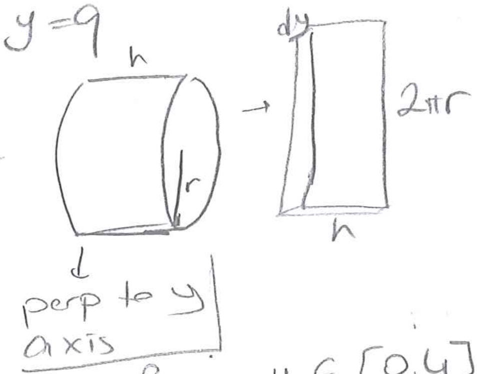
so $h = 8 - x^2 - 2x$

$$\text{Volume} = \int_{-2\sqrt{2}}^{0} 2\pi \cdot (9-x) \cdot (8-x^2)\,dx + \int_{0}^{2} 2\pi(9-x) \cdot (8-x^2-2x)\,dx$$

---- evaluate integral

b) Rotate about $y = 9$





$dV = 2\pi r h\,dy$

$r = 9 - y$

- for $y \in [0, 4]$ h is between Left arm of $y = 8 - x^2$ and $y = 2x$

$x = -\sqrt{8-y}$

$\Rightarrow x = \frac{y}{2}$

So $h = \frac{y}{2} - (-\sqrt{8-y})$

- for $y \in [4, 8]$ h is between right arm of $y = 8 - x^2$ and left arm of $y = 8 - x^2$

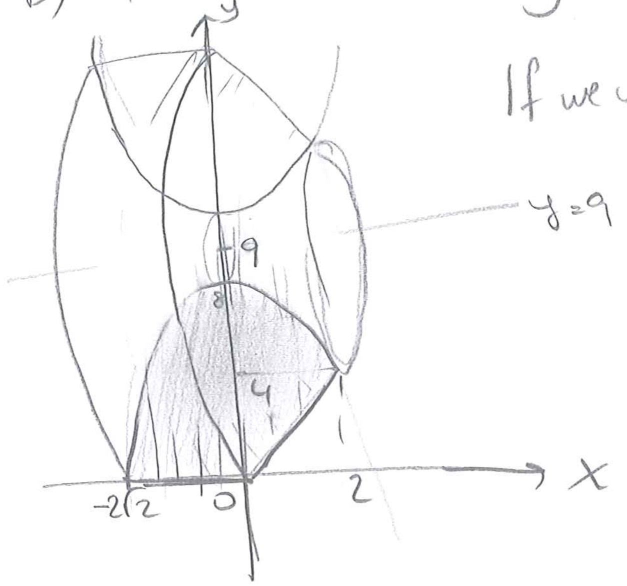
$x = \sqrt{8-y}$

$x = -\sqrt{8-y}$

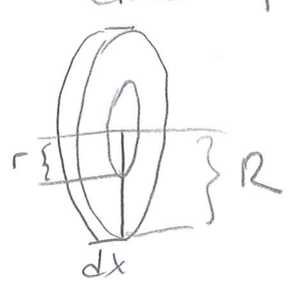
$h = \sqrt{8-y} - (-\sqrt{8-y}) = 2\sqrt{8-y}$

$$\text{Volume} = \int_{0}^{4} 2\pi \cdot (9-y) \cdot \left(\frac{y}{2} + \sqrt{8-y}\right) dy + \int_{4}^{8} 2\pi \cdot (9-y) \cdot (2\sqrt{8-y})\,dy$$

b) Rotate about $y=9$ (by disc method)



If we use Disc method we take discs perpendicular to x axis



$dV = \pi \cdot (R^2 - r^2) dx$

for $x \in [-2\sqrt{2}, 0]$ R lies between $y=9$ and $y=0$

so $R = 9 - 0 = 9$

r lies between $y=9$ & $y=8-x^2$

$r = 9-(8-x^2) = 1+x^2$

for $x \in [0, 2]$ R lies between $y=9$ and $y=2x$

so $R = 9-2x$

r lies between $y=9$ and $y=8-x^2$

$r = 9-(8-x^2) = 1+x^2$

$$\text{Volume} = \int_{-2\sqrt{2}}^{0} \pi \left(9^2 - (1+x^2)^2\right) dx + \int_{0}^{2} \pi \cdot \left((9-2x)^2 - (1+x^2)^2\right) dx$$

→ evaluate integral

11) Let R be the region in the first quadrant bounded by the curves $y=\frac{4}{\pi}\arctan x$, $x+y=2$ and $x=0$.

a) Use the cylindrical shell method to set up (but do not evaluate) a integral for the volume of the solid obtained by rotating the region R

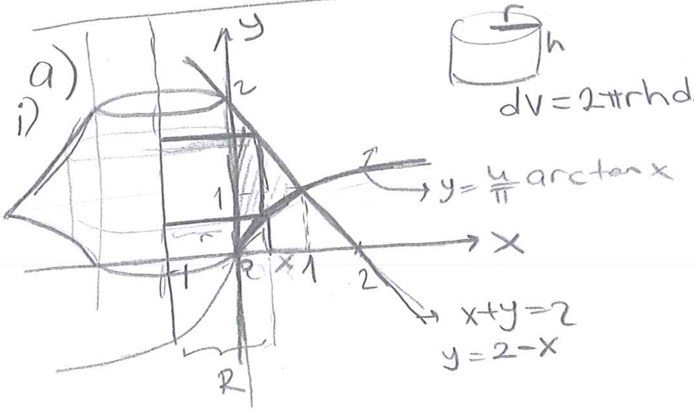
i) about the line $x=-1$;

ii) about x-axis

b) Same as a but use disc method i) about $x=-1$ ii) about x-axis ($y=0$)

---

a) i)



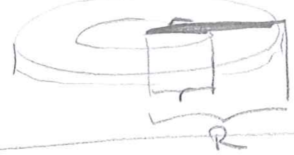
$dV = 2\pi r h\,dx$

$r = x-(-1) = x+1$

$h = (2-x) - \frac{4}{\pi}\arctan x$

$$\text{Volume} = \int_0^1 2\pi\cdot(x+1)\left(2-x-\frac{4}{\pi}\arctan x\right)dx$$

b) i)



$\to dV = \pi\cdot(R^2-r^2)\,dy$

$y\in[0,1] \Rightarrow R = \tan\frac{\pi y}{4} - (-1)$

$y=\frac{4}{\pi}\arctan x \Rightarrow x = \tan\frac{\pi y}{4}$

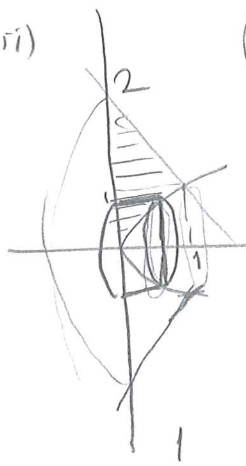
$r = 0-(-1) = 1$

$y\in[1,2] \Rightarrow R = (2-y)-(-1) = 3-y$

$r = 0-(-1) = 1$

$$\text{Volume} = \int_0^1 \pi\left(\left(\tan\frac{\pi y}{4}+1\right)^2 - 1^2\right)dy + \int_1^2 \pi\left((3-y)^2 - 1^2\right)dy$$

a) ii)



$\to 2\pi r h\,dy \quad r=y$

$y\in(0,1) \Rightarrow$

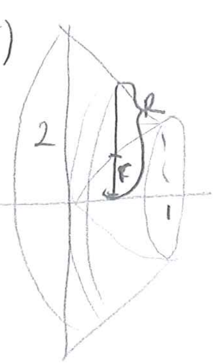
$h = \tan\frac{\pi y}{4}$

$y\in[1,2] \Rightarrow$

$h = 2-y$

$$\text{Volume} = \int_0^1 2\pi y\cdot\tan\frac{\pi y}{4}\,dy + \int_1^2 2\pi y(2-y)\,dy$$

b) ii)



$\to dV = \pi(R^2-r^2)\,dx$

$R = 2-x$

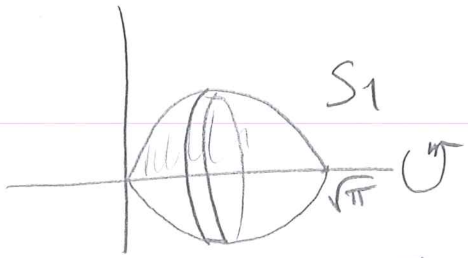
$r = \frac{4}{\pi}\arctan x$

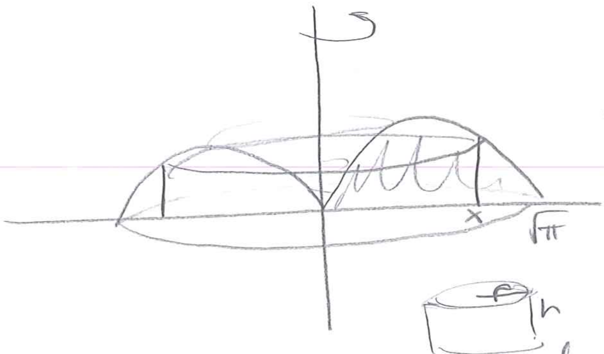
$$\text{Volume} = \int_0^1 \pi\left((2-x)^2 - \left(\frac{4}{\pi}\arctan x\right)^2\right)dx$$

12 a) Consider the region

$R := \{(x,y) : 0 \le x \le \sqrt{\pi},\ 0 \le y \le \sin(x^2)\}$,

Suppose that the solid S_1 is obtained by rotating R about x-axis and the solid S_2 is obtained by rotating R about y-axis. Which of the solids has larger volume? (Hint: $2x > x^2$ on $[0, \sqrt{\pi}]$)





↓ disc method

$r = \sin(x^2)$

$dV = \pi r^2 dx$

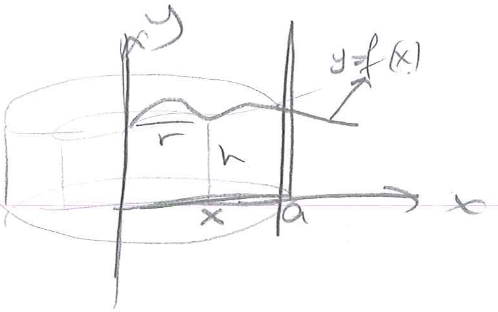
$$\text{volume} = \int_0^{\sqrt{\pi}} \pi \cdot (\sin(x^2))^2 dx$$

Shell method

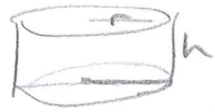
$dV = 2\pi r h\, dx$

$$\text{Volume} = \int_0^{\sqrt{\pi}} 2 \cdot \pi \cdot x \cdot \sin(x^2)\, dx$$

13 Q) Suppose f is a positive continuous function on $[0,\infty)$ and R is the region bounded by the curves $y=f(x)$, $y=0$, $x=0$, $x=a$ $(a>0)$. If the volume of the solid obtained by rotating R about y-axis is a^2e^{2a} for any $a>0$ find the function f.



By cylindrical shell method;



$2\pi r h\, dx = dV$

$h=f(x) \quad r=x$

$$\text{Volume} = \int_0^a 2\pi \cdot x \cdot f(x)\, dx$$

$$\int_0^a 2\pi x \cdot f(x)\, dx = a^2 e^{2a}$$

diff w.r.t. a

$$\frac{d}{da}\left(\int_0^a 2\pi x\, f(x)\, dx\right) = \frac{d}{da}\left(a^2e^{2a}\right)$$

By FTC.

$$2\pi a f(a) = 2ae^{2a} + a^2 \cdot 2e^{2a}$$

$$f(a) = \frac{\cancel{2}\cancel{a}e^{2a}(1+a)}{\cancel{2}\pi \cancel{a}} \Rightarrow f(a) = \frac{e^{2a}\cdot(1+a)}{\pi}$$

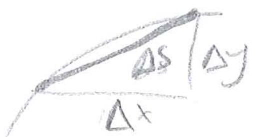
$$f(x) = \frac{e^{2x}\cdot(1+x)}{\pi}$$

Q) Find the length of the curve

14 a) $y=\frac{x^2}{8}-\ln x$ where $1\le x\le 3$

b) $y^2=x^3$ from point $(1,1)$ to point $(2,2\sqrt{2})$

a)



$\rightarrow \Delta s=\sqrt{(\Delta x)^2+(\Delta y)^2} \Rightarrow \Delta s=\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$

$$ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\frac{dy}{dx}=\frac{2x}{8}-\frac{1}{x}=\frac{x}{4}-\frac{1}{x}$$

$$\text{length}=\int_1^3\sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\,dx=\int_1^3\sqrt{1+\frac{x^2}{16}-2\cdot\frac{\cancel{x}}{4}\cdot\frac{1}{\cancel{x}}+\frac{1}{x^2}}\,dx$$

$$=\int_1^3\sqrt{\frac{x^2}{16}+\underbrace{\frac{1}{2}}_{2\cdot\frac{x}{4}\cdot\frac{1}{x}}+\frac{1}{x}}\,dx=\int_1^3\sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2}\,dx$$

$$=\int_1^3\frac{x}{4}+\frac{1}{x}\,dx$$

$$=\left(\frac{x^2}{8}+\ln x\right)\Big|_1^3$$

$$=\frac{9}{8}+\ln 3-\left(\frac{1}{8}+\ln 1\right)$$

$$=1+\ln 3 \;//$$

b) $y^2=x^3$ $(1,1)$ to $(2,2\sqrt{2})$

1st way: let $x=t^2$, $y=t^3$, $t\in[1,2]$ } same curve

$$\Delta s=\sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$=\sqrt{\Delta t^2\left(\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2\right)}$$

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$\text{length}=\int_1^2\sqrt{(2t)^2+(3t^2)^2}\,dt=\int_1^2\sqrt{\left(3t^2-\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^2}\,dt$$

2nd way: find $\frac{dy}{dx}$

$2yy'=3x^2 \Rightarrow y'=\frac{3x^2}{2y}=\frac{3x^2}{2\cdot\sqrt{x^3}}$, $y=\sqrt{x^3}$

from a) $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$\int_1^2\sqrt{1+\left(\frac{3x^2}{2\sqrt{x^3}}\right)^2}\,dx$$

evaluate ...

15 Q) Suppose f is an increasing, twice continuously differentiable function on $[0,\infty)$ and the curve $y=f(x)$ is concave up. If $F(a)$ is the length of the curve $y=f(x)$ where $0\le x\le a$ show that F is an increasing twice continuously differential function on $[0,\infty)$ and the curve $y=F(x)$ is concave up.

---

length of $f(x)$ from $x=0$ to $x=a$;

$$ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+(f'(x))^2}\,dx$$

$$F(a)=\int_0^a \underbrace{\sqrt{1+(\underbrace{f'(x)}_{\text{cont}})^2}}_{\text{cont}}\,dx$$

(f is continuously diffble means $f'(x)$ is cont)

So by FTC $F'(a)=\sqrt{\underbrace{1+(f'(a))^2}_{\ge 1}}>0$ obviously

$F'(x)>0 \Rightarrow F(x)$ is increasing function

$f(x)$ is twice cont. diffble (i.e.) $f'(x)$ is cont. diffble

so $\underbrace{\sqrt{1+(f'(x))^2}}_{\underset{F'(x)}{\shortparallel}}$ is continuously diffble

$\Rightarrow F(x)$ is twice continuously differentiable

and $F''(x)=(F'(x))'=\left(\sqrt{1+(f'(x))^2}\right)'$

$$=\frac{1}{\underbrace{2\sqrt{\underset{\ge 1}{1+(f'(x))^2}}}_{>0}}\cdot 2\underbrace{f'(x)}_{\substack{>0\\(f\text{ is increasing}\\ \text{given})}}\cdot\underbrace{f''(x)}_{\substack{>0\\(f\text{ is concave}\\ \text{up given})}}$$

So $F''(x)>0$ so $F(x)$ is concave up.

16) a) Find a polar equation for the curve represented by the given Cartesian equation:

$$(x^2+y^2)^2 = 4x^2y^2$$

b) Find a Cartesian equation for the curve represented by the given polar equation $r = 5\cos\theta$

16 a) $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \quad x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$

$$(x^2+y^2)^2 = 4x^2y^2$$

$$(r^2)^2 = 4r^2\cos^2\theta \cdot r^2\sin^2\theta$$

$$\cancel{r^4} = \cancel{r^4}\underbrace{(2\sin\theta\cdot\cos\theta)}_{\sin 2\theta}{}^2$$

$$(\sin 2\theta)^2 = 1$$

$$\sin 2\theta = 1$$

OR

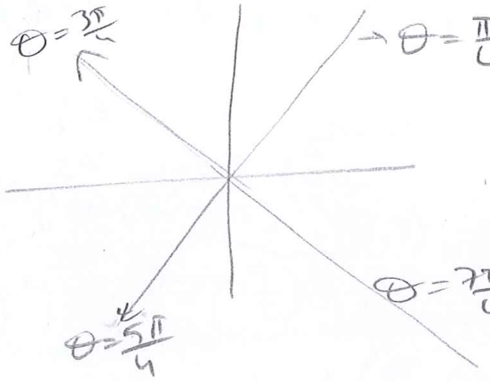
$$\sin 2\theta = -1$$

$\sin 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{2}$ or $2\theta = \frac{5\pi}{2}$

$\theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$

$\sin 2\theta = -1 \Rightarrow 2\theta = \frac{3\pi}{2}$ or $2\theta = \frac{7\pi}{2}$

$\theta = \frac{3\pi}{4}$ or $\theta = \frac{7\pi}{4}$



b) $r = 5\cos\theta$

$\left(\begin{array}{l} r\cos\theta = x \\ r\sin\theta = y \end{array}\right)$

multiply with r to get $r\cos\theta$

$$r^2 = 5\underline{r\cos\theta}$$

$$x^2+y^2 = 5\cdot x \Rightarrow x^2-5x+y^2 = 0$$

$$\underbrace{x^2-5x+\frac{25}{4}}+y^2 = \frac{25}{4}$$

$$\left(x-\frac{5}{2}\right)^2+y^2 = \left(\frac{5}{2}\right)^2$$

17 Q) Draw the polar curves

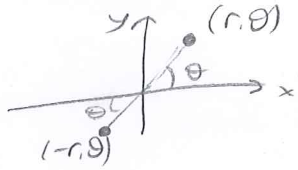
a) $r = 2 + \sin 3\theta$

b) $r = \cos\frac{\theta}{3}$

a) Check symmetries first;

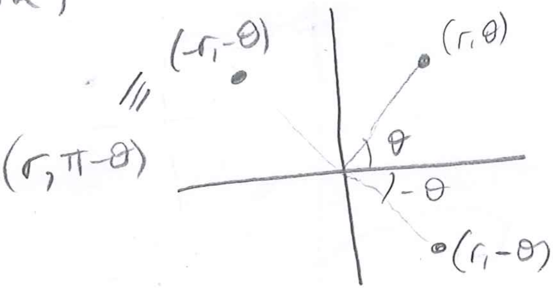
If $(r,\theta), (r,-\theta) \in$ graph, graph is symmetric wrt polar axis (x-axis) $\boxed{(r(\theta) = r(-\theta))}$

If $\overset{(r,\theta)}{(r,\theta)}, \overset{(-r,\theta)}{(r, \pi+\theta)} \in$ graph, the graph is symmetric w.r.t the pole (origin) or $\boxed{r(\theta) = r(\theta+\pi)}$



If (r,θ), $(-r,-\theta)$ OR ||| $(r,\theta), (r, \pi-\theta)$ $\in$ graph $\Big\}$ the graph is symmetric w.r.t y-axis.

$r(\theta) = r(\pi - \theta)$



a) $r = 2 + \sin 3\theta$, $r(-\theta) = 2 + \sin(-3\theta) \neq -r$, $\neq r$ $\}$ not sym wrt x-axis

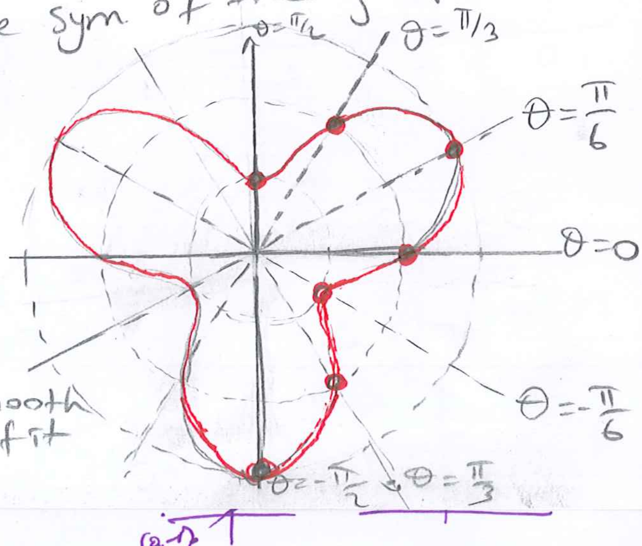
$r(\pi+\theta) = 2 + \sin(3\pi + 3\theta) = 2 + \sin(\pi + 3\theta)$
$= 2 - \sin(3\theta) \neq r(\theta)$, $\neq -r(\theta)$ $\Big\}$ not sym. wrt origin

$r(\pi - \theta) = 2 + \sin(3\pi - 3\theta) = 2 + \sin(\pi - 3\theta)$
$= 2 + \sin 3\theta = r\theta$ $\Big\}$ symmetric wrt. $y$axis (sym. wrt. $\theta = \frac{\pi}{2}$)

We have symmetry wrt $y$axis so if we draw for $[-\frac{\pi}{2}, \frac{\pi}{2}]$ take sym. of the graph we get the whole graph.

3θ	$-3\pi/2$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$2+\sin 3\theta$	3	2	1	2	3	2	1

plot these points, Join points with smooth curve and take symmetric of it



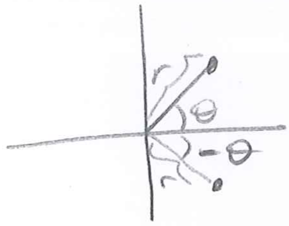
b) $r = \cos\frac{\theta}{3}$

Check symmetries first

$r(-\theta) = \cos\left(-\frac{\theta}{3}\right) = \cos\frac{\theta}{3} = r(\theta) \quad \forall \theta.$

(cosine is even)

So (r, θ) & $(r, -\theta) \in$ graph at the same time so graph is sym wrt x-axis.



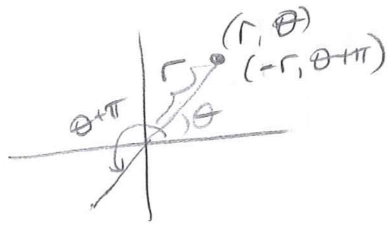
$r(\pi+\theta) = \cos\left(\frac{\pi+\theta}{3}\right) = \cos\left(\frac{\pi}{3}+\frac{\theta}{3}\right) = \cos\frac{\pi}{3}\cdot\cos\frac{\theta}{3} - \sin\frac{\pi}{3}\cdot\sin\frac{\theta}{3} \neq \cos\frac{\theta}{3}$

$\neq -\cos\frac{\theta}{3}$

we have not symmetry wrt origin (pole)

$r(\pi-\theta) = \cos\left(\frac{\pi-\theta}{3}\right) = \cos\left(\frac{\pi}{3}-\frac{\theta}{3}\right)$ similarly we don't have symmetry wrt y axis. ($\theta = \frac{\pi}{2}$ line)

$r(\theta+3\pi) = \cos\left(\frac{\theta+3\pi}{3}\right) = \cos\left(\frac{\theta}{3}+\pi\right) = -\cos\frac{\theta}{3}$



(r, θ) & $(-r, (\theta+3\pi))$ gives same point so enough to draw graph for $\theta \in [0, 3\pi]$ also graph was symmetric wrt x axis so draw half of it and then take symmetric of it wrt x axis. (polar axis)

use $\theta \in [0, \frac{3\pi}{2}]$

$\frac{\theta}{3}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
θ	0	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$
$r=\cos\frac{\theta}{3}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

use these points plot them on x-y then join the points smoothly } red curve

green curve { then draw symmetric of it w.r.t. polar-axis (x-axis)

(for $\theta \in [0, \frac{3\pi}{2}]$ $\cos\frac{\theta}{3} \geq 0$ & smooth)