1. For the given functions, first determine (and justify) whether the function has absolute maximum/minimum. Then, find and classify all local and absolute extreme values of the function.

   a) $f(x) = |x^2 - x - 12|$ on $[-4, 5]$

   b) $f(x) = \dfrac{x^2}{\sqrt{9 - x^2}}$

   c) $f(x) = 2x - \arcsin(x)$

2. Sketch the graphs of the following functions:

   a) $f(x) = \dfrac{x^3}{(x+1)^2}$

   b) $f(x) = \dfrac{\ln(x)}{x^2}$

   c) $f(x) = \dfrac{|x| + 1}{x - 2}$

   d) $f(x) = e^{-x}\sin(x)$ for $x \in [-\pi, 2\pi]$
   (Also, indicate how the graph looks for $x \in (-\infty, \infty)$)

1. A one meter length wire is cut into two pieces. One piece is bent into a circle and the other pieces is bent into a square. Where should the wire be cut so that the sum of the areas of the circle and the square is a) maximum b) minimum?

2. Find the shortest distance from the point $(8, 1)$ to the curve $y = 1 + x^{\frac{3}{2}}$

3. Consider the partition $P_n = \{x_0 = 1, x_1 = 2^{\frac{1}{n}}, x_2 = 2^{\frac{2}{n}}, \ldots, x_i = 2^{\frac{i}{n}}, \ldots x_n = 2^{\frac{n}{n}} = 2\}$ of the interval $[1, 2]$. By using this partition, show that the area of the region bounded by $y = x^3, y = 0, x = 1, x = 0$ is $\dfrac{2^4 - 1}{4}$.

4. Show that the function $f(x) = 2^x$ is integrable on the interval $[0, 2]$ and find $\displaystyle\int_0^2 2^x \, dx$ by computing $L(f, P_n)$ and $U(f, P_n)$ where P_n is the partition of $[0, 2]$ into n subintervals of equal length.

5. Express the limit
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \ln\left(1 + \frac{2i}{n}\right)$$
as a definite integral.

6. Evaluate the following integrals by interpreting them as areas.

   a. $\displaystyle\int_0^2 \sqrt{2x - x^2} \, dx$

   b. $\displaystyle\int_1^2 \sqrt{4 - x^2} \, dx$

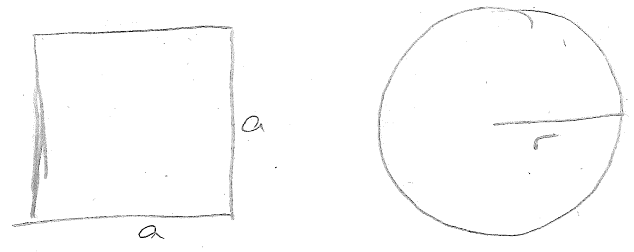
1

Q) A one meter length wire is cut into two pieces. One piece is bent into a circle and the other piece is bent into a square. Where should the wire be cut so that the sum of the areas of the circle and the square is

a) maximum b) minimum

→ 1 meter wire

Soln:



$0 \le 4a \le 1$
$0 \le a \le \frac{1}{4}$

perimeter : $4a + 2\pi r = 1 \Rightarrow r = \frac{1-4a}{2\pi}$ &

Area $= a^2 + \pi\left(\frac{1-4a}{2\pi}\right)^2 = a^2 + \frac{(1-4a)^2}{4\pi}$ → find max & min of this function

($f(x)$ is continuous & $x \in [0,\frac{1}{4}]$ ↳ interval is closed & bdd so f has abs max & min) $f(x) = x^2 + \frac{(1-4x)^2}{4\pi}$ $x \in [0,\frac{1}{4}]$

$f'(x) = 2x + \frac{2(1-4x)\cdot(-4)}{4\pi} = 2x - \frac{2}{\pi} + \frac{8x}{\pi} = 2\left(1+\frac{4}{\pi}\right)x - \frac{2}{\pi}$

$f'(x) = 0 \Rightarrow 2x + \frac{8x}{\pi} = \frac{2}{\pi} \Rightarrow \cancel{2}x\left(1+\frac{4}{\pi}\right) = \frac{\cancel{2}}{\pi}$ $x = \frac{1}{\pi+4}$ (CP)

	0		$\frac{1}{\pi+4}$		$\frac{1}{4}$
$f'(x)$		$-$		$+$	
$f(x)$	↓ local max	↘	↓ local min	↗	↓ local max

$f(0) = \frac{1}{4\pi} < \underbrace{f\left(\frac{1}{4}\right) = \frac{1}{16}}_{\text{abs max value}}$

$f\left(\frac{1}{\pi+4}\right) = \left(\frac{1}{\pi+4}\right)^2 + \left(\frac{1-\frac{4}{\pi+4}}{4\pi}\right)^2$

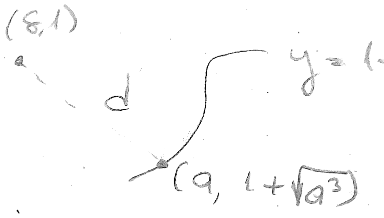
$= \frac{1}{(\pi+4)^2} + \frac{\pi^{\cancel{2}}}{(\pi+4)^2 \cdot 4\cancel{\pi}} = \frac{4+\pi}{4\cdot(\pi+4)^2}$

abs min value

2

Q) Find the shortest distance from the point $(8,1)$ to the curve $y = 1 + x^{3/2}$

Soln:



$y = 1 + x^{3/2} = 1 + \sqrt{x^3} \qquad x \geq 0$

$d = \sqrt{(a-8)^2 + (1+\sqrt{a^3} - 1)^2} = \sqrt{(a-8)^2 + a^3}$

let $f(x) = (x-8)^2 + x^3 \qquad \left((d(x))^2 = f(x) \right)$

$f(x)$ & d have abs max & min at the same point on intersection of their domain.

$f'(x) = 3x^2 + 2(x-8) = 3x^2 + 2x - 16 = (3x+8)(x-2)$

$\cancel{x = -\frac{8}{3}} \qquad x = 2$

($y = 1 + \sqrt{x^3}$ is defined for $x \geq 0$ so we can consider f on $[0, \infty)$ interval)

x	0		2	
$f'(x)$		$-$		$+$
$f(x)$		$\searrow$		$\nearrow$
			local min	

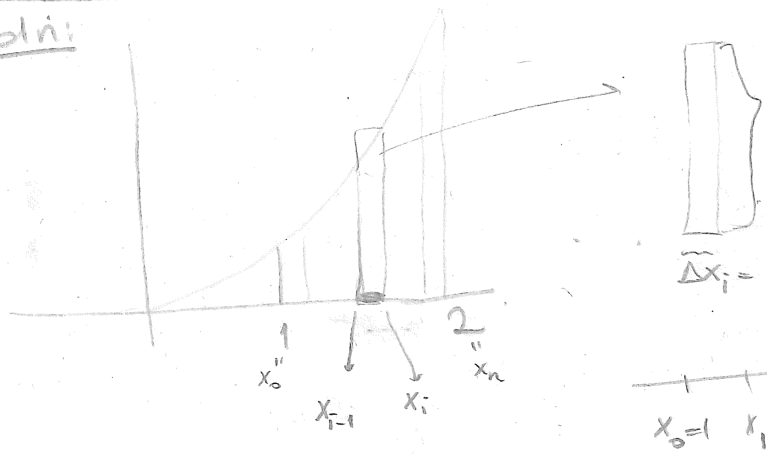
$f(0) = 64$
$\lim_{x \to \infty} f(x) = \infty$ $\Big\} \Rightarrow$ local min value is the abs min of $f(x)$

$f(2) = 36 + 8 = 44$

min value for $d = \sqrt{f(2)} = \sqrt{44} = 2\sqrt{11}$

Q) Consider the partition $P_n = \{x_0 = 1, x_1 = 2^{\frac{1}{n}}, x_2 = 2^{\frac{2}{n}}, \dots, x_i = 2^{\frac{i}{n}}, \dots, x_n = 2^{\frac{n}{n}}\}$ of the interval $[1,2]$. By using this partition, show that the area of the region bounded by $y = x^3$, $y = 0$, $x = 1$, $x = 2$ is $\dfrac{2^4 - 1}{4}$

Soln:



$f(x_i) = x_i^3$

$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

$x_i^* \in [x_{i-1} - x_i]$

$\Delta x_i = x_i - x_{i-1} = 2^{i/n} - 2^{(i-1)/n}$

Choose $x_i^* = x_i \Rightarrow f(x_i^*) = x_i^3 = 2^{3i/n}$

$$S_n = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i = \sum_{i=1}^{n} 2^{\frac{3i}{n}} \left(2^{\frac{i}{n}} - 2^{\frac{i-1}{n}}\right) = \sum_{i=1}^{n} 2^{\frac{3i}{n}} \left(2^{\frac{i}{n}} - 2^{\frac{i}{n} - \frac{1}{n}}\right)$$

$$= \sum_{i=1}^{n} 2^{\frac{3i}{n}} \left(2^{\frac{i}{n}} - 2^{\frac{i}{n}} \cdot 2^{-\frac{1}{n}}\right) = \sum_{i=1}^{n} 2^{\frac{3i}{n}}\, 2^{\frac{i}{n}} \left(1 - 2^{-\frac{1}{n}}\right)$$

$$= \sum_{i=1}^{n} 2^{\frac{4i}{n}} \left(\frac{2^{\frac{1}{n}} - 1}{2^{1/n}}\right) = \left(\frac{2^{1/n} - 1}{2^{1/n}}\right) \sum_{i=1}^{n} \left(2^{\frac{4}{n}}\right)^i \to r$$

(not depending on i so consider as a constant)

$$\left(\sum_{i=0}^{n-1} r^n = 1 + r + r^2 + \dots + r^{n-1} = \frac{r^n - 1}{r - 1}\right)$$

$$\left(\begin{aligned} \sum_{i=1}^{n} r^n &= r + r^2 + \dots + r^n \\ &= r(1 + r + \dots + r^{n-1}) \\ &= r \sum_{i=0}^{n-1} r^n = r \cdot \frac{r^n - 1}{r - 1} \end{aligned}\right) \star$$

$$\overset{\star}{=} \frac{2^{1/n} - 1}{2^{1/n}} \cdot 2^{\frac{4}{n}} \cdot \left(\frac{\left(2^{\frac{4}{n}}\right)^n - 1}{2^{4/n} - 1}\right)$$

$$= \frac{\cancel{2^{1/n} - 1}}{2^{1/n}} \cdot 2^{4/n} \cdot \frac{2^4 - 1}{\left(2^{2/n} + 1\right)\left(2^{1/n} + 1\right)\cancel{\left(2^{1/n} - 1\right)}}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{2^{4/n}\left(2^4 - 1\right)}{\left(2^{1/n}\right)\left(2^{2/n} + 1\right)\left(2^{1/n} + 1\right)} = \frac{2^4 - 1}{4}$$

4

Q) Show that the function $f(x) = 2^x$ is integrable on the interval $[0,2]$ and find $\int_0^2 2^x dx$ by computing $L(f, P_n)$ and $U(f, P_n)$ where P_n is the partition of $[0,2]$ into n subintervals of equal length.

Soln:

5

8) Express the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \ln\left(1+\frac{2i}{n}\right) = A$$

as a $\boxed{\text{definite integral}}$

Recall: Area under $f(x)$ from a to b:

$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i$ where $\{x_0, x_1, \dots, x_n\}$ is a partition of $[a,b]$

Soln: I.

let $\frac{2}{n} = \Delta x$

$x_i = x_0 + i\frac{2}{n}$

$f(x_i^*) = \ln\left(1+\frac{2i}{n}\right)$

let $x_i^* = x_i$

So $f(x_i) = \ln\left(1+\frac{2i}{n}\right)$

i) let $x_0 = 0$ then $x_i = \frac{2i}{n}$

and if $\Delta x = \frac{x_n - x_0}{n} = \frac{x_n - 0}{n} = \frac{2}{n}$

$x_n = 2$

and $f(x_i) = \ln(1+x_i)$

So $f(x) = \ln(1+x)$

$\boxed{A = \int_0^2 \ln(1+x)}$ → soln I

(ii) let $x_0 = 1$ then $x_i = 1 + \frac{2i}{n}$

and if $\Delta x = \frac{x_n - x_0}{n} = \frac{x_n - 1}{n} = \frac{2}{n}$

$\Rightarrow x_n = 3$

and $f(x_i) = \ln x_i$

So $f(x) = \ln x$

$\boxed{A = \int_1^3 \ln x}$ → soln 2

II.

Let $\frac{1}{n} = \Delta x$

then $2\ln\left(1+\frac{2i}{n}\right) = f(x_i^*)$

$x_i = x_0 + i\Delta x = x_0 + \frac{i}{n}$

let $x_i^* = x_i$

So $f(x_i) = 2\ln\left(1+\frac{2i}{n}\right)$

iii) let $x_0 = 0$ then $x_i = \frac{i}{n}$

$f(x_i) = 2\ln\left(1+\frac{2i}{n}\right)$

$f(x_i) = 2\ln(1+2x_i)$

$\frac{x_n - x_0}{n} = \frac{1}{n}$

$\Rightarrow x_n = 1$

$\Rightarrow f(x) = 2\ln(1+2x)$

$\boxed{A = \int_0^1 2\ln(1+2x)dx}$ → soln 3

iv) let $x_0 = a$ (whatever you want to be) $\Rightarrow x_i = a + \frac{i}{n}$

$\frac{x_n - x_0}{n} = \frac{1}{n}$

$x_n = a+1$

$f(x_i) = 2\ln\left(1+2\frac{i}{n}\right)$

$f(x_i) = 2\ln\left(1+2\left(\underbrace{\frac{i}{n}+a}_{x_i}-a\right)\right)$

$f(x_i) = 2\ln(1+2(x_i - a))$

$f(x) = 2\ln(1-2a+2x)$

$\boxed{A = \int_a^{a+1} 2\ln(1-2a+2x)dx}$

Soln 4

(You should choose Δx, x_0 & w.r.t. that values find $f(x)$, & x_n to write integral $\int_{x_0}^{x_n} f(x)dx$)

Q) Evaluate the following integrals by interpreting them as areas.

a) $\int_0^2 \sqrt{2x-x^2}\,dx$

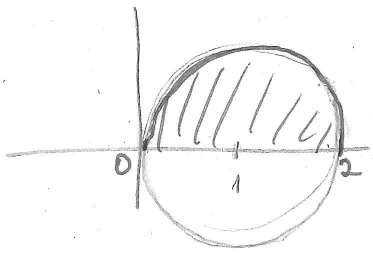
b) $\int_1^2 \sqrt{4-x^2}\,dx$

Soln: a) $\int_0^2 \sqrt{2x-x^2}\,dx$ gives the area under the curve $y=\sqrt{2x-x^2}$ and between $x=0$ & $x=2$

$y=\sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow$ $x^2-2x+y^2=0$ (add 1 to each side)

$x^2-2x+1+y^2=1$

$(x-1)^2+y^2=1 \rightarrow$ circle with center = (1,0) radius = 1

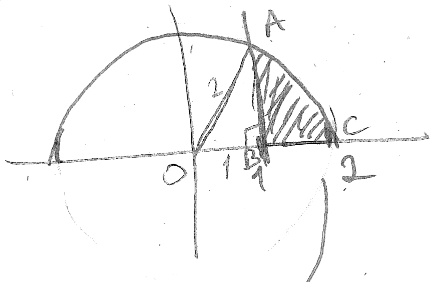


$y=\sqrt{2x-x^2}$ so $y \geq 0$

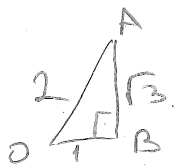
given integral is area of half circle with radius 1 so it is $\frac{\pi \cdot 1^2}{2} = \frac{\pi}{2}$

b) $\int_1^2 \sqrt{4-x^2}\,dx$ gives the area between $y=\sqrt{4-x^2}$ and x axis and bounded by $x=1$ & $x=2$

$y=\sqrt{4-x^2} \Rightarrow y^2=4-x^2 \Rightarrow x^2+y^2=4 \rightarrow$ circle centered at the origin, with radius 2



$y=\sqrt{4-x^2}$ so $y \geq 0$



Shaded area = area of $\frac{1}{6}$ circle $\underset{(\overset{\frown}{OAC})}{}$ − triangle (OAB)

$= \frac{1}{6} \cdot \pi \cdot 2^2 - \frac{1 \cdot \sqrt{3}}{2} = \frac{4\pi - 3\sqrt{3}}{6}$

Q) The implicit eqn $\sin(x+y)=xy$ defines y as a function of x say $y=f(x)$ Find a linear approximation for $f(x)$ valid about the point $P_0(\pi,0)$

$L(f,P_0) = f'(P_0).(x-x_0)+y_0$ → linearization of $f(x)$ around P_0

$\left.\frac{dy}{dx}\right|_{P_0} = ?$ $\sin(x+y)=xy$ (differentiate both sides wrt x taking $y=f(x)$ - function of x)

$\cos(x+y)\cdot(x+y)' = 1.y + x\,y'$

$\cos(x+y)(1+y') = y+xy'$

Subs. $(x,y)=(\pi,0)$ to find $\left.y'\right|_{(\pi,0)} = y'(\pi)$ →

$\underbrace{\cos(\pi+0)}_{-1}(1+y') = 0+\pi.y'$

$-1-y' = \pi\,y'$

$y'(\pi+1) = -1 \rightarrow y' = -\frac{1}{\pi+1}$

$\Rightarrow L(f,P_0) = \left(-\frac{1}{\pi+1}\right).(x-\pi)+0$

So $y \approx \left(-\frac{1}{\pi+1}\right)(x-\pi)$

(or $f(x) \approx \left(-\frac{1}{\pi+1}\right)(x-\pi)$)

here be carefull about = & ≈ signs! or you may lose point in the exam.

Q) Estimate $\sqrt[3]{127}$ with a suitable approximation.

Soln: $\sqrt[3]{125}$ is a value near $\sqrt[3]{127}$ and we can evaluate and linearize it so let $f(x)=\sqrt[3]{x}$ & $a=125$ Approximate $\sqrt[3]{127}$ $f(x)$ around 125 to ~~Approximate~~ $\sqrt[3]{127}$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3}\cdot\frac{1}{\sqrt[3]{x^2}}$ $f'(125) = \frac{1}{75}$

$f(x) \approx L(f,125) = f'(125)(x-125)+f(125)$

$= \frac{1}{75}(x-125)+5$

$f(x) \approx \frac{1}{75}(x-125)+5 \Rightarrow f(127) \approx \frac{1}{75}(127-125)+5 = \frac{2}{75}+5 = \frac{377}{75}$

1. Find the average value of $f(x) = |x|$ over $[-1, 2]$.

2. Find all functions $f(x)$ and all real numbers a such that

$$\int_x^a f(t)dt = 2 - 2\sqrt{x}, \quad for\, all\, x > 0$$

3. Evaluate the following limits

   (a) $\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{2n^2} \right)$

   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n+3i}$

   (c) $\lim_{x\to 0} \frac{1}{x} \int_{\sin x}^{x^3} \sqrt{t^2+1}dt$

4. Evaluate the definite integrals.

   (a) $\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

   (b) $\int_1^{e^{\pi/4}} \sin(\ln x)dx$

5. Evaluate the indefinite integrals.

   (a) $\int x^3\sqrt{x^2+1}dx$

   (b) $\int \frac{x^{\ln x}\ln x}{x}dx$

   (c) $\int e^{\sqrt{2x+9}}dx$

   (d) $\int 2x\arcsin(x^2)dx$

   (e) $\int \sec^4 x\tan^2 x dx$

   (f) $\int \sin^3 x\cos^3 x dx$

   (g) $\int \sqrt{1+\sin x}$

6. Additional problems

   (a) $\int_0^{\pi/2} 5(\sin x)^{3/2}\cos x\, dx$

   (b) $\int \frac{\tan(\ln x)}{}dx$

   (c) $\int \frac{dx}{\sqrt{e^{2x}-1}}$

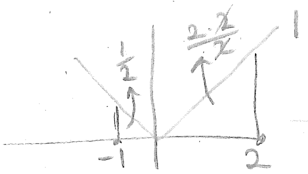
   (d) $\int \frac{1+x}{1+x^2}dx$

1

Q) Find the average value of $f(x) = |x|$ over $[-1,2]$

Soln: Recall that average value of an integrable function $f(x)$ over the interval $[a,b]$ is :

$$\frac{\int_a^b f(x)\,dx}{b-a}$$



by area property of integral

So we need to evaluate $\dfrac{\int_{-1}^{2} |x|\,dx}{2-(-1)} = \dfrac{\frac{1}{2}+2}{3} = \dfrac{5}{6}$

Q Find all functions $f(x)$ and all real numbers "a" such that

$$\int_x^a f(t)\,dt = 2 - 2\sqrt{x}, \quad \text{for all } x > 0.$$

Solution: Recall that $\dfrac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$

So

$$\underbrace{\frac{d}{dx}\left(\int_x^a f(t)\,dt\right)}_{\parallel} = \frac{d}{dt}\,(2-2\sqrt{x})$$

$$-f(x) = -2\cdot\frac{1}{2\sqrt{x}} \Rightarrow \boxed{f(x) = \frac{1}{\sqrt{x}}}$$ → substitute into original equation to find a

$$\int_x^a \underbrace{2\frac{1}{2\sqrt{t}}}_{\substack{\parallel \\ (\sqrt{t})'}}\,dt = 2 - 2\sqrt{x}$$

(By Fundamental Thm of calculus (FTC))

$$2\sqrt{t}\Big|_x^a = 2 - 2\sqrt{x}$$

$$\parallel$$

$$2\sqrt{a} - 2\sqrt{x} = 2 - 2\sqrt{x}$$

$$\boxed{a = 1}$$

Recall

FTC: let $F'(x) = f(x)$
Part 2
then
$$\int_a^b f(x)\,dx = F'(b) - F'(a)$$

9) Evaluate the following limits

a) $\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\dots+\frac{n}{2n^2}\right)=A$

b) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n+3i}=B$

c) $\lim_{x\to 0}\frac{1}{x}\int_{\sin x}^{x^3}\sqrt{t^2+1}\,dt$

2nd way for (c) let $F(x)=\int_{\sin x}^{x^3}\sqrt{t^2+1}\,dt$

$\lim_{x\to 0}\frac{F(x)-F(0)}{x-0}=F'(0)$ so given limit is $F'(0)$

(a) $A=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2\left(1+\left(\frac{i}{n}\right)^2\right)}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n\cdot\left(1+\left(\frac{i}{n}\right)^2\right)}$

(take denominator into n^2 parenthesis) (because we need Δx & $f(x)$ ($\lim_{n\to\infty}\Delta x=0$))

let $\frac{1}{n}=\Delta x$ & $f(x_i)=\frac{1}{1+\left(\frac{i}{n}\right)^2}$ with $x_i^*=x_i$ & $x_i=x_0+i\Delta x$
$=x_0+i\cdot\frac{1}{n}$

also let $\boxed{x_0=0}$ then $\frac{x_n-x_0}{n}=\frac{1}{n}\Rightarrow\boxed{x_n=1}$

$\Rightarrow f(x_i)=\frac{1}{1+(x_i)^2}\Rightarrow\boxed{f(x)=\frac{1}{1+x^2}}$

then $A=\int_0^1\frac{1}{1+x^2}=\arctan x\Big|_0^1=\arctan 1-\arctan 0$

$(\arctan x)'=\frac{1}{1+x^2}$

$=\frac{\pi}{4}-0=\frac{\pi}{4}$ //

(b) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n+3i}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n(1+3\frac{i}{n})}$

(we need Δx)

let $\Delta x=\frac{3}{n}$ $f(x_i)=\frac{1}{(1+3\frac{i}{n})}$

$x_i=x_0+i.\Delta x$ let $\boxed{x_0=1}$
$=x_0+i.\frac{3}{n}$

then $\frac{x_n-x_0}{n}=\frac{3}{n}\Rightarrow\boxed{x_n=4}$

$f(x_i)=\frac{1}{x_i}$ $\boxed{f(x)=\frac{1}{x}}$

$B=\int_1^4\frac{1}{x}dx=\ln x\Big|_1^4=\ln 4-\ln 1=\underline{\ln 4}$

$(\ln x)'=\frac{1}{x}$

(c) $\lim_{x\to 0}\frac{\int_{\sin x}^{x^3}\sqrt{t^2+1}\,dt}{x}\to\frac{0}{0}$ form

(as $x\to 0$ $\sin x$ & $x^3\to 0$
& $\int_a^a f(x)dx=0$)

$\overset{LH}{=}\lim_{x\to 0}\frac{\sqrt{(x^3)^2+1}\cdot(x^3)'-\sqrt{\sin^2x+1}\,(\sin x)'}{1}$

$=\lim_{x\to 0}\sqrt{x^6+1}\,(3x^2)-\sqrt{\sin^2+1}\,.\cos x$

$=-1$ //

Remember

$\frac{d}{dx}\int_{g(x)}^{h(x)}f(t)dt=f(h(x))\cdot h'(x)-f(g(x))\cdot g'(x)$

9) Evaluate the definite integrals

a) $\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$ b) $\int_1^{e^{\pi/4}} \sin(\ln x)\,dx$

Soln:

a) $\int_0^1 \frac{dx}{(1+\sqrt{x})^4} = \int_1^2 \frac{2(u-1)}{u^4}du = 2\int_1^2 \frac{u}{u^4} - \frac{1}{u^4}\,du$

let $1+\sqrt{x}=u$

$\Rightarrow \frac{1}{2\sqrt{x}}dx = du$

$dx = 2\sqrt{x}\,du$

$\sqrt{x} \to u-1$

$dx = 2(u-1)du$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=2$

$= 2\int_1^2 (u^{-3} - u^{-4})\,du$

$= 2\left(\frac{u^{-3+1}}{-2} + \frac{u^{-4+1}}{-3}\right)\Big|_1^2$

$= 2\left[\left(\frac{2^{-2}}{-2} + \frac{2^{-3}}{3}\right) - \left(-\frac{1}{2} + \frac{1}{3}\right)\right]$

$= 2\left[\left(-\frac{1}{8} + \frac{1}{24}\right) - \left(-\frac{1}{6}\right)\right] = -\frac{1}{6} + \frac{2}{6} = \frac{1}{6}$

$-\frac{1}{12}$

b) $\int_1^{e^{\pi/4}} \sin(\ln x)\,dx = \underbrace{\int_0^{\pi/4} \sin t\, e^t dt}_{\text{call } I} = \sin t \cdot e^t\Big|_0^{\pi/4} - \int_0^{\pi/4} e^t \cos t\,dt$

$\int u \cdot dv = u \cdot v - \int v\,du$

substitution

let $\ln x = t \Rightarrow x = e^t$

$\Rightarrow \frac{1}{x}dx = dt$

$dx = x\,dt = e^t dt$

$x=1 \Rightarrow t=0$

$x=e^{\pi/4} \Rightarrow t=\frac{\pi}{4}$

integration by parts

$u=\sin t \Rightarrow du = \cos t\,dt$

$dv = e^t dt \Rightarrow v = e^t$

$= (\sin\frac{\pi}{4} \cdot e^{\pi/4} - \underbrace{\sin 0\, e^0}_{0}) - \int_0^{\pi/4} \underbrace{\cos t}_{u}\, \underbrace{e^t dt}_{dv}$

$= u \cdot v - \int v\,du$

integration by parts again

$u = \cos t \Rightarrow du = -\sin t\,dt$

$dv = e^t dt \Rightarrow v = e^t$

$= \sin\frac{\pi}{4} e^{\pi/4} - \left(\cos t\, e^t\Big|_0^{\pi/4} - \int_0^{\pi/4} e^t(-\sin t)dt\right)$

$= \sin\frac{\pi}{4} e^{\pi/4} - \left(\cos\frac{\pi}{4} e^{\pi/4} - \cos 0\, e^0 + I\right)$

So $I = \sin\frac{\pi}{4} e^{\pi/4} - \cos\frac{\pi}{4} e^{\pi/4} + 1 - I$

$\Rightarrow 2I = \cancel{\frac{\sqrt{2}}{2} e^{\pi/4}} - \cancel{\frac{\sqrt{2}}{2} e^{\pi/4}} + 1 \Rightarrow 2I = 1 \Rightarrow I = \frac{1}{2}$

Q1) Evaluate the indefinite integrals:

a) $\int x^3\sqrt{x^2+1}\,dx$; b) $\int \frac{x^{\ln x}\cdot \ln x}{x}\,dx$ c) $\int e^{\sqrt{2x+9}}\,dx$

d) $\int 2x\arcsin(x^2)\,dx$ e) $\int \sec^4 x\tan^2 x\,dx$ f) $\int \sin^3 x\cos^3 x\,dx$

g) $\int \sqrt{1+\sin x}\,dx$

---

a) $\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\,x\,dx = \int (u-1)(\sqrt{u})\,du$

$= \int (u^{3/2} - u^{1/2})\,du$

$= \frac{u^{\frac{3}{2}+1}}{\frac{5}{2}} - \frac{u^{\frac{1}{2}+1}}{\frac{3}{2}} + C$

don't forget

(Substitute
$x^2+1=u \Rightarrow x^2=u-1$
$\Rightarrow 2x\,dx=du$)

(backsubstitution)
$u=x^2+1$

! Don't forget

$= \frac{2}{5}(x^2+1)^{5/2} - \frac{2}{3}(x^2+1)^{3/2} + C$

---

b) $\int \frac{x^{\ln x}\,\ln x}{x}\,dx = \int u\cdot\frac{1}{2u}\,du = \frac{1}{2}\int du = \frac{1}{2}u + C$

(Subs. $x^{\ln x} = u$
take $\ln$
$\ln x^{\ln x} = \ln u$
$(\ln x)^2 = \ln u$
differential $2\ln x\,\frac{1}{x}\,dx = \frac{1}{u}\,du$
$\frac{\ln x}{x}\,dx = \frac{1}{2}\frac{du}{u}$)

$= \frac{1}{2}x^{\ln x} + C$

(backsubstitution
$u = x^{\ln x}$)

$\int u\,dv = uv - \int v\,du$

c) $\int e^{\sqrt{2x+9}}\,dx = \int e^t\,t\,dt = te^t - \int e^t\,dt = te^t - e^t + C$

$=(t-1)e^t+C$

(Subs. $\sqrt{2x+9} = t$
$\Rightarrow 2x+9 = t^2$
diff. $2dx = 2t\,dt$
$dx = t\,dt$)

($u = t \Rightarrow du = dt$
$dv = e^t dt \Rightarrow v = e^t$
integration by parts)

$= (\sqrt{2x+9}-1)e^{\sqrt{2x+9}} + C$

backsubst.
$t = \sqrt{2x+9}$!

5

d) $\int 2x \arcsin(x^2)\,dx = \int \underbrace{\arcsin(x^2)}_{\arcsin u} \cdot \underbrace{2x\,dx}_{du} = \int \underbrace{\arcsin t}_{u} \cdot \underbrace{dt}_{dv}$

(let $x^2 = t$
then $2x\,dx = dt$)

$\left(\begin{array}{ll} u = \arcsin t & du = \dfrac{1}{\sqrt{1-t^2}} \\ dv = dt \Rightarrow & v = t \end{array} \right)$

IBP (integration by parts)

$= (\arcsin t) \cdot t - \int \dfrac{t}{\sqrt{1-t^2}}\,dt$

$\left((\sqrt{1-t^2})' = \dfrac{-2t}{2\sqrt{1-t^2}} \right)$

$= \arcsin t \cdot t + \sqrt{1-t^2} + C$

$= \arcsin x^2 \cdot x^2 + \sqrt{1-x^4} + C$

(back subs $t = x^2$)

---

e) $\int \sec^4 x \tan^2 x\,dx = \int \overbrace{\tan^2 x}^{u^2}\, \overbrace{\sec^2 x}^{1+u^2} \cdot \overbrace{\underbrace{\sec^2 x\,dx}_{(\tan x)'}}^{du}$

subs $\tan x = u$ (also $\sec^2 x = 1 + \tan^2 x$)
diff $\sec^2 x\,dx = du$

$= \int u^2 (1+u^2)\,du = \int (u^2 + u^4)\,du$

$= \dfrac{u^3}{3} + \dfrac{u^5}{5} + C$

$u = \tan x$

$= \dfrac{\tan^3 x}{3} + \dfrac{\tan^5 x}{5} + C$

---

f) $\int \sin^3 x \cos^3 x\,dx = \int \overbrace{\sin^3 x}^{u^3} \cdot \overbrace{\underbrace{\cos^2 x}_{(1-\sin^2 x)}}^{1-u^2} \cdot \overbrace{\cos x\,dx}^{du} = \int u^3 \cdot (1-u^2)\,du$

(subs $\sin x = u$
$\Rightarrow \cos x\,dx = du$)

$= \int u^3 - u^5\,du = \dfrac{u^4}{4} - \dfrac{u^6}{6} + C$

$(u = \sin x)$

$= \dfrac{\sin^4 x}{4} - \dfrac{\sin^6 x}{6} + C$

6

9) $\int \sqrt{1+\sin x}\ dx$ $\left(\sin \frac{2x}{2} = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}\right)$

$= \int \sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}\ dx$ $\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}=1\right)$

$= \int \sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}\ dx$

$= \int \left|\sin\frac{x}{2}+\cos\frac{x}{2}\right| dx$

$\left(\text{if } \sin\frac{x}{2}+\cos\frac{x}{2} \geq 0\right)$

$= \int \sin\frac{x}{2}+\cos\frac{x}{2}\ dx$

$= 2\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+C$

if $\sin\frac{x}{2}+\cos\frac{x}{2} < 0$

$\int -\sin\frac{x}{2}-\cos\frac{x}{2}\ dx$

$= 2\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)+C$

(OR multiply & divide by $\sqrt{1-\sin x}$ ...... try yourself)

Evaluate the following integrals

1. $\displaystyle\int \frac{x^2}{1-x^4}dx$
2. $\displaystyle\int \frac{dx}{x-\sqrt[3]{x}}$
3. $\displaystyle\int \frac{\ln(x^2+x+1)}{x^2}dx$ (IBP and partial fraction)
4. $\displaystyle\int \frac{x^3}{x^2+2x+5}dx$
5. $\displaystyle\int \frac{x-3}{(x^2+2x+4)^2}dx$
6. $\displaystyle\int \frac{tdt}{\sqrt{9-4t^2}}$
7. $\displaystyle\int \frac{5dx}{\sqrt{25x^2-9}}, \quad x>3/5$
8. $\displaystyle\int \frac{dx}{x(1+x^2)^{3/2}}$
9. $\displaystyle\int \sqrt{1+x^2}dx$
10. $\displaystyle\int \frac{d\theta}{2+\sin\theta}$

Rec week 11.

1) $\int \frac{x^2}{1-x^4}\,dx =$

partial fraction (degree of $x^2 <$ deg $(1-x^4)$)

$$\frac{x^2}{1-x^4} = \frac{x^2}{(1+x^2)(1-x^2)} = \frac{x^2}{(1+x^2)(1-x)(1+x)} = \frac{ax+b}{1+x^2} + \frac{c}{1-x} + \frac{d}{1+x}$$

$$\qquad\qquad (1-x)(1+x) \qquad (1+x^2)(1+x) \qquad (1-x)(1+x^2)$$

$$x^2 = (ax+b)\underbrace{(1-x)(1+x)} + c(1+x^2)(1+x) + d(1-x)(1+x^2)$$

$$x \to 1 \Rightarrow 1 = 0 + c \cdot 2 \cdot 2 + 0 \Rightarrow 4c = 1$$
$$c = \frac{1}{4}$$

$$x \to -1 \Rightarrow 1 = 0 + 0 + d(2)(2) \Rightarrow 4d = 1$$
$$d = \frac{1}{4}$$

$$x^2 = (ax+b)(1-x^2) + \frac{1}{4}(1+x^2)(1+\cancel{x}+1-\cancel{x})$$

$x = 0 \Rightarrow$

$$0 = b + \frac{1}{\cancel{4}_2} \cdot \cancel{2} \Rightarrow b = -\frac{1}{2}$$

$$x^2 = \left(ax - \frac{1}{2}\right)(1-x^2) + \frac{1}{\cancel{4}2}(1+x^2)$$

$$\underline{\underline{\cancel{x^2}}} = ax - ax^3 - \cancel{\frac{1}{2}} + \cancel{\frac{x^2}{2}} + \cancel{\frac{1}{2}} + \cancel{\frac{x^2}{2}} \qquad \forall x$$

$$\Rightarrow a = 0$$

$$\frac{x^2}{1-x^4} = \frac{1}{2}\frac{1}{1+x^2} - \frac{1}{4}\frac{1}{x-1} + \frac{1}{4}\frac{1}{x+1}$$

$$\int \frac{x^2}{1-x^4}\,dx = \int \left(\frac{1}{2}\frac{1}{1+x^2} - \frac{1}{4}\frac{1}{x-1} + \frac{1}{4}\frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}\arctan x - \frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x+1| + c$$

Week 11

2) $\int \frac{dx}{x-\sqrt[3]{x}} = \int \frac{3t^2\,dt}{t^3-t}$

$\left(\begin{array}{l} \sqrt[3]{x}=t \\ \Rightarrow x=t^3 \\ dx=3t^2dt \end{array} \right)$

degree$(3t^2)$ > degree(t^3-t)

use partial fraction

$$\frac{3t^2}{t(t^2-1)} = \frac{3t^2}{t(t-1)(t+1)} = \frac{a}{t}_{(t-1)(t+1)} + \frac{b}{t-1}_{t(t+1)} + \frac{c}{t+1}_{t(t-1)}$$

$$3t^2 = a.(t-1)(t+1) + b\,t(t+1) + c\,t(t-1)$$

equal polynomials

$t\to 0 \Rightarrow \quad 0 = a.(-1) + 0 + 0 \Rightarrow a=0$

$t\to 1 \Rightarrow \quad 3 = 0 + b.2 + 0 \Rightarrow 3=2b \quad b=\frac{3}{2}$

$t\to -1 \Rightarrow \quad 3 = 0 + 0 + c.(-1)(-2) \Rightarrow 3=2c \Rightarrow c=\frac{3}{2}$

$$\frac{3t^2}{t^3-t} = \frac{3}{2}\cdot\frac{1}{t-1} + \frac{3}{2}\,\frac{1}{t+1}$$

$$\int \frac{3t^2}{t^3-t}\,dt = \int \left(\frac{3}{2}\,\frac{1}{t-1} + \frac{3}{2}\,\frac{1}{t+1}\right)dt$$

$$= \frac{3}{2}(\ln |t-1| + |t+1|) + C$$

$$= \frac{3}{2}\ln|t^2-1| + C$$

$t=\sqrt[3]{x}$ back substitution $\Rightarrow$

$$= \frac{3}{2}\ln|\sqrt[3]{x^2}-1| + C$$

3) $\int \frac{\ln(x^2+x+1)}{x^2}\,dx$

Integration by parts:

$u = \ln x^2+x+1 \qquad dv = \frac{1}{x^2}\,dx$

$\rightarrow du = \frac{2x+1}{x^2+x+1}\,dx \qquad \rightarrow v = -\frac{1}{x}$

(Since it is difficult to find integral of logarithm and we get easier fraction when we differentiate we will use integration by parts with $u = \ln(x^2+x+1)$)

$$= \ln(x^2+x+1)\left(-\frac{1}{x}\right) - \int -\frac{1}{x}\cdot\frac{2x+1}{x^2+x+1}\,dx$$

$$= -\frac{\ln(x^2+x+1)}{x} + \int \underbrace{\frac{2x+1}{x(x^2+x+1)}}\,dx$$

degree $(2x+1) <$ deg $(x(x^2+x+1))$
we can directly use partial fraction

let $\frac{2x+1}{x(x^2+x+1)} = \frac{a}{x} + \frac{bx+c}{x^2+x+1}$ → (since x^2+x+1 is irreducible we cannot factor out)

$2x+1 = a(x^2+x+1) + (bx+c)x$ (RHS polynomial is equal to LHS)

$2x+1 = (a+b)x^2 + (a+c)x + a$

$a = 1 \qquad a+c = 2 \Rightarrow c = 1$

$a+b = 0 \Rightarrow b = -a \quad b = -1$

$$\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$\int \frac{2x+1}{x(x^2+x+1)}\,dx = \int\left(\frac{1}{x} + \frac{-x+1}{x^2+x+1}\right)dx$$

$$\frac{-x+1-\frac{1}{2}+\frac{1}{2}}{x^2+x+1} = \frac{2x+1}{2(x^2+x+1)} + \frac{3}{2(x^2+x+1)}$$

$$= -\frac{\ln(x^2+x+1)}{x} + \int\left(\frac{1}{x} + \frac{1}{2}\frac{2x+1}{x^2+x+1} + \frac{3}{2}\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\right)dx$$

$$= -\frac{\ln(x^2+x+1)}{x} + \ln x + \frac{1}{2}\ln(x^2+x+1) + \frac{3}{2}\cdot\frac{\arctan\frac{(x+\frac{1}{2})}{\frac{\sqrt{3}}{2}}}{\frac{\sqrt{3}}{2}} + C$$

$$\left(\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}\right)$$

4) $\int \frac{x^3}{x^2+2x+5} dx$

$\deg x^3 > \deg(x^2+2x+5)$

first polynomial division

$$\begin{array}{r|l} x^3 & x^2+2x+5 \\ \mp x^3 \mp 2x^2 \mp 5x & x-2 \\ \hline -2x^2-5x & \\ \pm 2x^2 \pm 4x \pm 10 & \\ \hline -5x-10 & \end{array}$$

$\frac{x^3}{x^2+2x+5} = x-2-\frac{5}{2}\,\frac{2x+2}{x^2+2x+5}$

$= x-2-\frac{5}{2}\cdot\frac{2x+2}{x^2+2x+5}-\frac{5}{\cancel{2}}\,\frac{\cancel{2}}{x^2+2x+5}$

$\int \frac{x^3}{x^2+2x+5} = \int\left(x-2-\frac{5}{2}\,\frac{2x+2}{x^2+2x+5}-5\,\frac{1}{(x+1)^2+4}\right)dx$

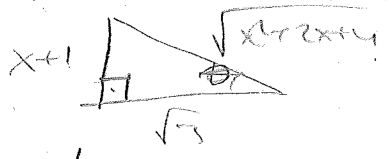
$= \frac{x^2}{2}-2x-\frac{5}{2}\ln(x^2+2x+5)-5\int\frac{1}{2\left(\left(\frac{x+1}{2}\right)^2+1\right)}dx$

$= \frac{x^2}{2}-2x-\frac{5}{2}\ln(x^2+2x+5)-\frac{5}{2}\arctan\left(\frac{x+1}{2}\right)+C$

5) $\int\frac{x-3}{(x^2+2x+4)^2}dx = \frac{1}{2}\int\frac{(2x+2)}{(x^2+2x+4)^2} - \int\frac{4}{((x+1)^2+(\sqrt{3})^2)^2}dx$

$\left(\frac{1}{x^2+2x+4}\right)' = -\frac{2x+2}{(x^2+2x+4)^2}$

$(x+1)$, $\sqrt{3}$, $\sqrt{x^2+2x+4}$



$\tan\theta = \frac{x+1}{\sqrt{3}}$

$\sqrt{3}\tan\theta = x+1$

$\sqrt{3}\sec^2\theta\, d\theta = dx$

$\frac{\sqrt{x^2+2x+4}}{\sqrt{3}} = \sec\theta$

$\sqrt{x^2+2x+4} = \sqrt{3}\sec\theta$

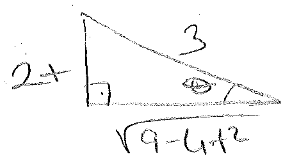
$= -\frac{1}{2}\,\frac{1}{(x^2+2x+4)} - 4\int\frac{\sqrt{3}\sec^2\theta\, d\theta}{9\sec^4\theta}$

$= -\frac{1}{2}\,\frac{1}{x^2+2x+4} - \frac{4\sqrt{3}}{9}\int\cos^2\theta\, d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

6) $\int \frac{t\,dt}{\sqrt{9-4t^2}}$



$3,\ 2t,\ \sqrt{9-4t^2}$

$\sin\theta = \frac{2t}{3}$ $\qquad$ $\cos\theta = \frac{\sqrt{9-4t^2}}{3}$

$t = \frac{3}{2}\sin\theta$ $\qquad$ $3\cos\theta = \sqrt{9-4t^2}$

$dt = \frac{3}{2}\cos\theta\, d\theta$

$$= \int \frac{\frac{3}{2}\sin\theta \cdot \frac{3}{2}\cancel{\cos\theta}\, d\theta}{3\cancel{\cos\theta}} = -\frac{9}{4}\cos\theta + C$$

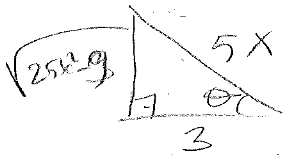
$$= -\frac{9}{4}\,\frac{\sqrt{9-4t^2}}{3} + C$$

$$= -\frac{3}{4}\sqrt{9-4t^2} + C$$

$9-4t^2 = u \Rightarrow -8t\,dt = du \Rightarrow t\,dt = -\frac{1}{8}du$

$$-\frac{1}{8}\int \frac{du}{2\sqrt{u}} = -\frac{1}{4}\sqrt{u} + C = -\frac{1}{4}\sqrt{9-4t^2} + C$$

7) $\int \frac{5\,dx}{\sqrt{25x^2-9}}$ $\quad x > 3/5$



$5X \quad 3 \quad \sqrt{25x^2-9}$

$\tan\theta = \frac{\sqrt{25x^2-9}}{3}$

$\sec\theta = \frac{5x}{3}$ $\qquad$ $\sqrt{25x^2-9} = 3\tan\theta$

$x = \frac{3}{5}\sec\theta$

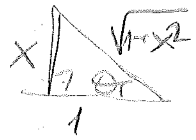
$dx = \frac{3}{5}\sec\theta\tan\theta\, d\theta$

$$= \int \frac{\cancel{5}\cdot\frac{\cancel{3}}{\cancel{5}}\sec\theta\cdot\cancel{\tan\theta}\, d\theta}{\cancel{3\tan\theta}} = \int \frac{\sec\theta\,(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$$

8) $\int \frac{dx}{x(1+x^2)^{3/2}} = \int \frac{dx}{x\left(\sqrt{(1+x^2)}\right)^3}$



$1,\ x,\ \sqrt{1+x^2}$

$$\left(\begin{array}{l} \tan\theta = x \\ \sec^2\theta\, d\theta = dx \\ \frac{1}{\sqrt{1+x^2}} = \cos\theta \\ \sqrt{1+x^2} = \sec\theta \end{array}\right)$$

$$= \int \frac{\cancel{\sec^2\theta}\, d\theta}{\underset{\frac{\sin\theta}{\cos\theta}}{\tan\theta} \cdot \underset{\frac{1}{\cos\theta}}{\sec^3\theta}} = \int \frac{\cos^2\theta}{\sin\theta}\, d\theta$$

$$= \int \frac{\cos^2\theta \cdot \sin\theta\, d\theta}{1-\cos^2\theta}$$

$\cos\theta = u \Rightarrow -\sin\theta\, d\theta = du$

$$= -\int \frac{u^2\, du}{1-u^2} = \int \frac{u^2\, du}{u^2-1}$$

$\deg u^2 \not> \deg(u^2-1)$

polynomial division $\quad \frac{u^2}{u^2-1} = 1 + \frac{1}{u^2-1}$ partial fraction

$$\begin{array}{r|l} u^2 & u^2-1 \\ -u^2 \pm 1 & 1 \\ \hline 1 & \end{array}$$

$(u^2-1)\left(\frac{1}{u^2-1}\right) = \left(\frac{a}{u+1} + \frac{b}{u-1}\right)(u-1)(u+1)$

$u \to 1 \Rightarrow 1 = 2b \Rightarrow b = \frac{1}{2}$

$u \to -1 \Rightarrow 1 = -2a \Rightarrow a = -\frac{1}{2}$

$$= \int \left(1 - \frac{1}{2}\frac{1}{u+1} + \frac{1}{2}\frac{1}{u-1}\right) du = u - \frac{1}{2}\ln|u+1| + \frac{1}{2}\ln|u-1| + C$$

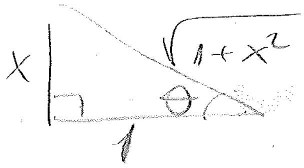
$$= u + \ln\sqrt{\frac{u-1}{u+1}} + C$$

$u = \cos\theta = \frac{1}{\sqrt{1+x^2}}$

$$= \frac{1}{\sqrt{1+x^2}} + \ln\sqrt{\frac{\frac{1}{\sqrt{1+x^2}} - 1}{\frac{1}{\sqrt{1+x^2}} + 1}} + C$$

$$= \frac{1}{\sqrt{1+x^2}} + \ln\left(\frac{1-\sqrt{1+x^2}}{1+\sqrt{1+x^2}}\right)^{1/2} + C$$

9) $\int \sqrt{1+x^2}\, dx$



$1,\ x,\ \sqrt{1+x^2}$

$$\left(\begin{array}{l} \tan\theta = x \\ \Rightarrow \sec^2\theta\, d\theta = \cancel{dx} \end{array} \qquad \sqrt{1+x^2} = \sec\theta \right)$$

$$= \int \underbrace{\sec\theta}_{u} \cdot \underbrace{\sec^2\theta\, d\theta}_{dV} \qquad \begin{array}{ll} u = \sec\theta & dV = \sec^2\theta\, d\theta \\ du = \sec\theta\tan\theta\, d\theta & V = \tan\theta \end{array}$$

$$= \sec\theta\tan\theta + \int \tan\theta \cdot \sec\theta\tan\theta\, d\theta$$

$\sec^2\theta = 1 + \tan^2\theta$
$\sec^2\theta - 1 = \tan^2\theta$

$$= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

$$I = \sec\theta\tan\theta + \ln(\sec\theta + \tan\theta) \cancel{+} I$$

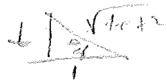
$$2I = \sec\theta\tan\theta + \ln(\sec\theta + \tan\theta) + C$$

$$I = \frac{(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta))}{2} + C$$

$$= \frac{\sqrt{1+x^2}\, x + \ln(\sqrt{1+x^2} + x)}{2} + C$$

10) $\int \frac{d\theta}{2+\sin\theta}$

$\tan\frac{\theta}{2} = t \Rightarrow \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$



$$= 2 \cdot \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

$\frac{1}{2}\sec^2\frac{\theta}{2} \cdot d\theta = dt$

$d\theta = 2\cos^2\frac{\theta}{2}\, dt = 2 \cdot \frac{1}{1+t^2}\, dt$

$$= \int \frac{1}{2 + \frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$$

$$= \int \frac{1}{1+t^2+t}\, dt = \int \frac{1}{(t+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\, dt = \int \frac{1}{(\frac{\sqrt{3}}{2})^2 \left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2 + 1}\, dt$$

$$= \arctan\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C = \arctan\frac{\tan\frac{\theta}{2} + \frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

Quiz 4 (Solutions)

Let $f(x) = \frac{x^3}{(x+1)^2}$

a) Find domain of $f(x)$, find x, y intercepts

b) Find asymptotes (if any)

c) Find $f'(x)$ and the intervals where $f(x)$ is increasing and decreasing
also find local extrema

d) Find $f''(x)$ and the intervals where $f(x)$ is concave up and down
Find inflection points (if any)

e) Draw table for f, f', f''

f) Draw graph of $f(x)$

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a) Domain of $f(x) = \mathbb{R} - \{-1\}$

x intercept: if $y=0$ $\frac{x^3}{(x+1)^2} = 0$ $x=0$

y intercept: if $x=0$ $\frac{0^3}{(0+1)^3} = 0 \Rightarrow y=0$

} Just origin.

b) $x=-1$ makes denominator zero so it is candidate for vertical asymptote.

Check $\lim\limits_{x\to(-1)} f(x) = \lim\limits_{x\to -1} \frac{x^{3} \to -1}{(x+1)^2 \to 0^+} = -\infty$ so $\boxed{x=-1 \text{ VA}}$

Checking Horizontal asymptote:

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{x^3}{(x+1)^2} = \lim\limits_{x\to\infty} \frac{x^3}{x^2(1+\frac{1}{x})^2} = \infty$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} \frac{\cancel{x^3} \to -\infty}{\cancel{x^2}(\cancel{1}+(\frac{1}{x})^2} = -\infty$

} no H.A.

Oblique asymptote: (If for $g(x) = ax+b$ $\lim\limits_{\substack{x\to\infty \\ \text{or} \\ x\to-\infty}} (f(x)-g(x)) = 0$) $y=g(x) \to$ oblique asymp.

$$\begin{array}{r|l} x^3 & x^2+2x+1 \\ -x^3 \mp 2x^2 \mp x & x-2 \\ \hline -2x^2 - x & \\ \pm 2x^2 \pm 4x \pm 2 & \\ \hline 3x+2 & \end{array}$$

$\frac{x^3}{x^2+2x+1} = \boxed{x-2} + \frac{3x+2}{(x+1)^2}$ $g(x) = x-2$

$\lim\limits_{\substack{x\to\infty \\ (x\to-\infty)}} f(x) - (x-2) = \lim\limits_{\substack{x\to\infty \\ (x\to-\infty)}} \frac{3x+2}{(x+1)^2} = \lim\limits_{\substack{x\to\infty \\ (x\to-\infty)}} \frac{\cancel{x^2}(\frac{3}{x}+\frac{2}{x^2})}{\cancel{x^2}(1+(\frac{1}{x}))^2} = 0$

So $y = x-2 \to$ oblique asymp.

c) $f(x) = \frac{x^3}{(x+1)^2} \to f'(x) = \frac{3x^2.(x+1)^2 - x^3.2(x+1)}{((x+1)^2)^2} = \frac{x^2(x+1)(3(x+1)-2x)}{(x+1)^4}$

$= \frac{x^2(x+3)}{(x+1)^3}$

$f'(x)=0 \Rightarrow x=0$ or $x=-3$
(at $x=-1$ $f'(x)$ changes its sign)

	$-\infty$ … -3	-3 … -1	-1 … 0	0 … ∞
$f'(x)$	+	−	+	+
$f(x)$	↗	↘	↗	↗

local max

f(x) not cont here (so ~~NOT~~ local min)

$f(x)$ is increasing on: $(-\infty,-3), (-1,0), (0,\infty)$
(don't take union here)

$f(x)$ is decreasing on: $(-3,-1)$
$f(-3)$ is local minimum, it has no local max
no abs max ($\lim_{x\to\infty} f(x) = \infty$)

no abs min ($\lim_{x\to\infty} f(x) = -\infty$)

d) $f''(x) = \frac{(2x.(x+3)+x^2.1).(x+1)^3 - x^2(x+3).3(x+1)^2}{((x+1)^3)^2}$

$= \frac{\cancel{(x+1)^2}.((2x^2+6x+x^2)(x+1) - 3(x^3+3x^2))}{\cancel{((x+1)^6)}\ (x+1)^4}$

$= \frac{\cancel{3x^3} + 6x^2 + 3x^2 + 6x - \cancel{3x^3} - 9x^2}{(x+1)^4} = \frac{6x}{(x+1)^4}$

$f''(x) = 0 \Rightarrow x = 0$ ~~and x=0~~

	$-\infty$ … -1	-1 … 0	0 … ∞
$f''(x)$	−	−	+
$f(x)$	∩	∩	∪

inflection pt

$(0, f(0)) = \boxed{(0,0) \text{ inflection pt.}}$

$f(x)$ is concave ~~down~~ : $(-\infty,-1)$ & $(-1,0)$
$f(x)$ is concave up ~~[illegible]~~ at : $(0,\infty)$

e)

	-3		-1		0	
$f'(x)$	$+$		$-$		$+$	$+$
$f''(x)$	$-$		$-$		$-$	$+$
$f(x)$						

f) local max at $x=-3$ $f(-3) = \dfrac{-27}{4}$

intercept : origin

put asymptotes, $\lim\limits_{x \to -1^+} f(x) = -\infty$