Q4) Express the limit $\lim_{n\to\infty} S_n$ as a definite integral if

a) $S_n = \sum_{i=1}^{n} \frac{2}{n} \ln\left(1+\frac{2i}{n}\right)$

b) $S_n = \sum_{i=1}^{n} \frac{n}{n^2+i^2}$

√?

$\lim_{n\to\infty} \Delta x_i = 0$ must hold

Soln: $S_n = \sum_{i=1}^{n} \Delta x_i \cdot f(x_i^*)$ $\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx$

pick $\Delta x_i = \frac{2}{n}$ with equal length intervals partition

so $\frac{b-a}{n} = \frac{x_n - x_0}{n} = \frac{2}{n}$ & $x_i = a + \frac{i2}{n} = x_0 + \frac{2i}{n}$

$f(x_i^*) = \ln\left(1+\frac{2i}{n}\right)$ pick $x_i^* = x_i = x_0 + \frac{2i}{n}$

→ with $x_0 = 1$ then $x_n = 3$

$f\left(1+\frac{2i}{n}\right) = \ln\left(1+\frac{2i}{n}\right) \Rightarrow f(x) = \ln x$

$\int_1^3 \ln x\,dx = \ldots$

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alternative x_0 → pick $x_0 = 0$ then $x_n = 2$

then $f\left(0+\frac{2i}{n}\right) = \ln\left(1+\frac{2i}{n}\right) \Rightarrow f(x) = \ln(x+1)$

$\int_0^2 \ln(x+1)\,dx$

Alternative $\Delta x = \frac{1}{n}$

$\frac{b-a}{n} = \frac{1}{n}$

$f(x_i^*) = 2\ln\left(1+2\cdot\frac{i}{n}\right)$

$x_i^* = x_i = x_0 + \frac{i}{n}$

$x_0 = 0 \rightarrow x_1 = 1$

$f\left(\frac{i}{n}\right) = 2\ln\left(1+\frac{2i}{n}\right)$

$f(x) = 2\ln(1+2x)$

$\int_0 2\ln(1+2x)\,dx$

With $\Delta x = 1/n$

Consider $x_i = x_0 + \frac{i}{n}$ $x_i^* = x_i$

$f(x_i^*) = 2\ln\left(1+2\frac{i}{n}\right)$

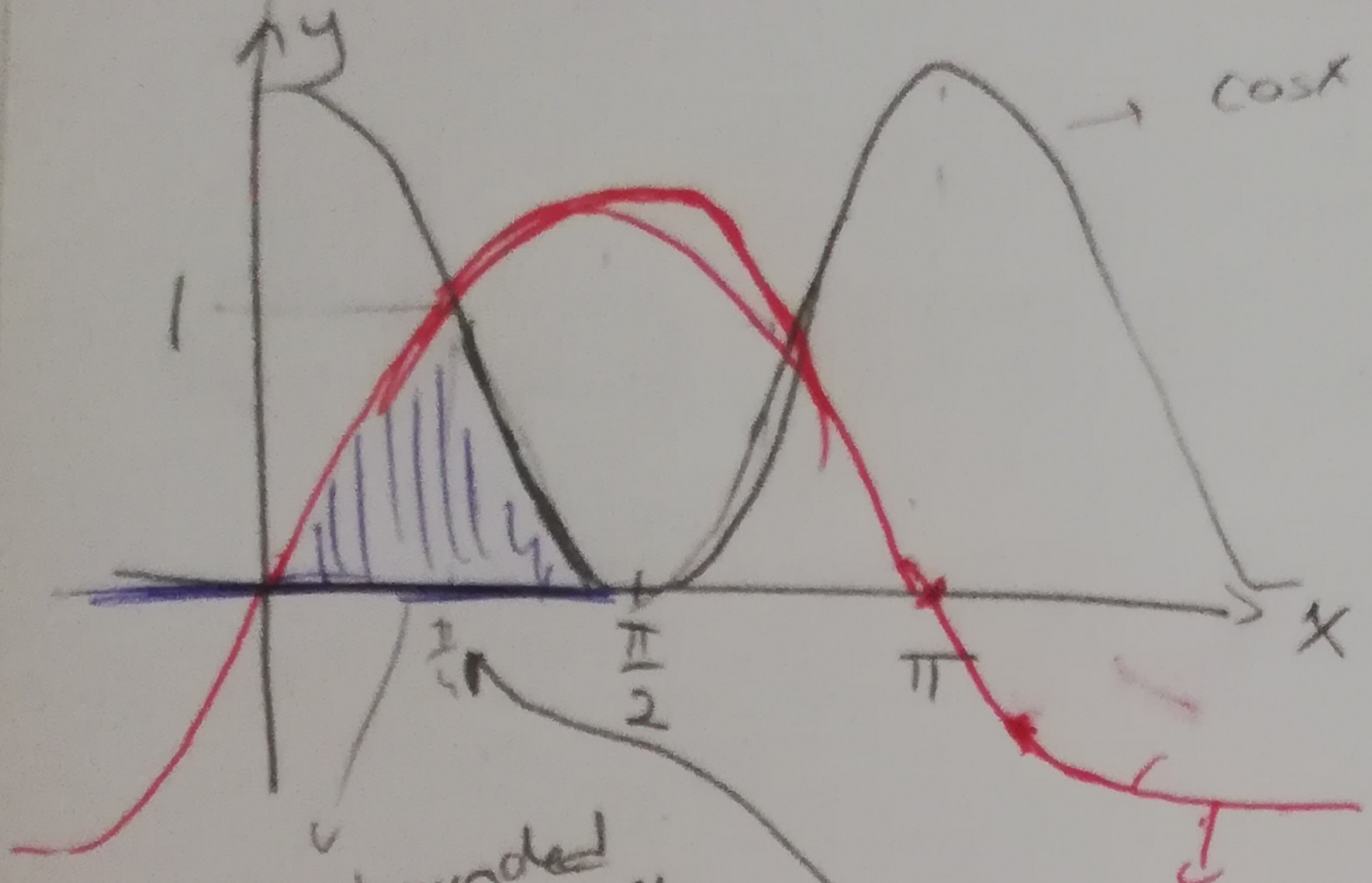
$f(x_i) = 2\ln\left(2\left(\frac{1}{2}+\frac{i}{n}\right)\right)$

pick $x_0 = \frac{1}{2} \rightarrow x_n = \frac{3}{2}$

$f\left(\frac{1}{2}+\frac{i}{n}\right) = 2\ln\left(2\left(\frac{1}{2}+\frac{i}{n}\right)\right)$

$f(x) = 2\ln(2x) \rightarrow \int_{1/2}^{3/2} 2\ln(2x)\,dx$

Question find the area of region bdd by ($y = 1 + \cos 2x$) ($x = 0$) ($y = \sqrt{2}\sin x$) in the first quadrant



→ cosx squeezed in x direction by multiple $\frac{1}{2}$

sinx stretched in y direction by multiple $\sqrt{2}$

first quadrant ⇒ $x \in [0, \frac{\pi}{2}]$

find intersections: $1 + \cos 2x = \sqrt{2}\sin x$

$1 + 2\cos^2 x - 1 = \sqrt{2}\sin x$

$2(1 - \sin^2 x) = \sqrt{2}\sin x$

$\sin x = t$ → $2 - 2t^2 = \sqrt{2}t$

$2t^2 + \sqrt{2}t - 2 = 0$

$2t \quad -\sqrt{2}$

$t \quad +\sqrt{2}$

$(2t - \sqrt{2})(t + \sqrt{2}) = 0$

⇒ $t = \frac{\sqrt{2}}{2}$ or $t = -\sqrt{2}$

$t = \sin x$ so since $x \in [0, \frac{\pi}{2}]$

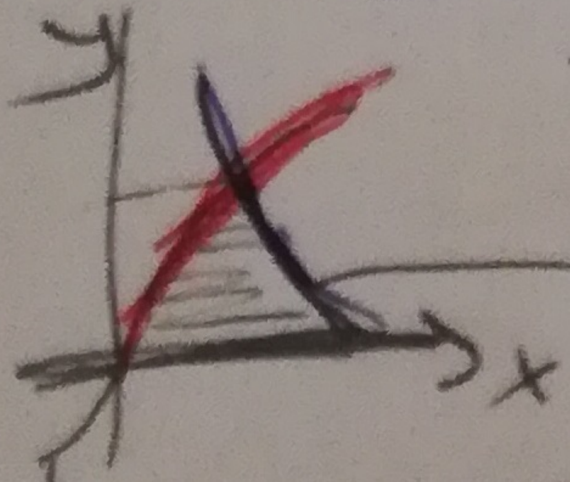
$\sin x > 0$

so $\sin x = \frac{\sqrt{2}}{2}$ $x = \frac{\pi}{4}$

2nd way

area in terms of y

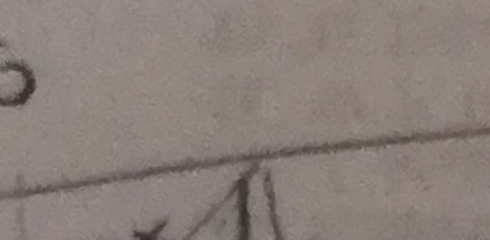
the func. on right is greater



→ $y = 1 + \cos 2x$ ⇒ $x = \frac{\arccos(y-1)}{2}$

$y = \sin x \to x = \arcsin y$

area $\int_0^1 \left(\frac{\arccos(y-1)}{2} - \arcsin y\right) dy$



area in terms of integral wrt. x:

$$\underbrace{\int_0^{\pi/4} \sin x \, dx}_{A} + \int_{\pi/4}^{\pi/2} 1 + \cos 2x \, dx = -\cos x\Big|_0^{\pi/4} + \left(x + \frac{\sin 2x}{2}\right)\Big|_{\pi/4}^{\pi/2}$$

$$= \left(-\frac{\sqrt{2}}{2} + 1\right) + \left(\frac{\pi}{2} + 0 - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right) = \ldots$$