EXERCISES on SEQUENCES and SERIES

I. Determine whether the following sequences are convergent:

1) $(\frac{n}{3^n})_{n=1}^{\infty}$ 2) $(\frac{n!}{(n+1)!-n!})_{n=1}^{\infty}$ 3) $(\frac{\sin n}{n})_{n=1}^{\infty}$

4) $(\frac{1}{\sin\frac{1}{n}})_{n=1}^{\infty}$ 5) $((\sin\frac{n\pi}{3})^n)_{n=1}^{\infty}$ 6) $(\frac{(2n)!}{(n!\,2^n)^2})_{n=1}^{\infty}$

7) $(\frac{n!}{n^n})_{n=1}^{\infty}$ 8) $(\frac{n!2^{n+3}}{(n+3)+2^n})_{n=1}^{\infty}$ 9) $(n\ell n(1+\frac{2}{n}))_{n=1}^{\infty}$

10) $(\int_0^n e^{-x^2}\,dx)_{n=1}^{\infty}$ 11) $(1+\frac{1}{2}+\frac{1}{6}+\cdots+\frac{1}{n!})_{n=1}^{\infty}$ 12) $(\frac{(-n)^n}{n!+2\,n^n})_{n=1}^{\infty}$

II. Determine whether the following series are convergent:

1) $\sum_{n=0}^{\infty}\frac{2^n}{3^n+4}$ 2) $\sum_{n=0}^{\infty}\frac{3^n-2^n}{4^n}$ 3) $\sum_{n=1}^{\infty}(\frac{1}{\sqrt{n}}-\frac{1}{n})$

4) $\sum_{n=1}^{\infty}(\frac{1}{2^n}+\frac{1}{3n})$ 5) $\sum_{n=0}^{\infty}\frac{n}{n^2+10^3}$ 6) $\sum_{n=0}^{\infty}\frac{n^{1981}}{e^n}$

7) $\sum_{n=1}^{\infty}\frac{\ell nn}{n^2}$ 8) $\sum_{n=1}^{\infty}\frac{\sqrt{n+3}-\sqrt{n}}{n}$ 9) $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$

10) $\sum_{n=1}^{\infty}\frac{\ell nn}{n}$ 11) $\sum_{n=0}^{\infty}\frac{e^{100n}}{n!}$ 12) $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

13) $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ 14) $\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}$ 15) $\sum_{n=1}^{\infty}\frac{4^n n!}{n^n}$

16) $\sum_{n=1}^{\infty}\frac{\ell n(n+e^n)}{n+\ell nn}$ 17) $\sum_{n=5}^{\infty}\frac{1}{2^n-n^2}$ 18) $\sum_{n=1}^{\infty}\frac{n^{2n}}{2^n\,(2n)!}$

19) $\sum_{n=1}^{\infty}\frac{n!}{100^n}$ 20) $\sum_{n=1}^{\infty}\frac{1}{2n^2-n}$ 21) $\sum_{n=0}^{\infty}\frac{n}{2+n^3}$

22) $\sum_{n=1}^{\infty}\arcsin 2^{-n}$ 23) $\sum_{n=2}^{\infty}\frac{1}{\ell nn}$ 24) $\sum_{n=0}^{\infty}\frac{n^3}{3^n}$

25) $\sum_{n=1}^{\infty}\sin\frac{1}{n}$ 26) $\sum_{n=0}^{\infty}\sin\frac{1}{n^2}$ 27) $\sum_{n=1}^{\infty}(\frac{\pi}{2}-\arctan n^2)$

28) $\sum_{n=1}^{\infty}\frac{\sin e^n}{n^2}$ 29) $\sum_{n=0}^{\infty}e^n\sin n^2\,2^{-n}$ 30) $\sum_{n=1}^{\infty}\frac{\sin n^e}{n^e}$

31) $\sum_{n=1}^{\infty}2^{-n}\sin^2 e^n$ 32) $\sum_{n=1}^{\infty}\sin(-e)^{-n}$ 33) $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n^2+3}}$

34) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$ 35) $\sum_{n=3}^{\infty}\frac{\cos n+e^{-n}}{n^3-2n+3}$ 36) $\sum_{n=0}^{\infty}\frac{\arctan n}{\sqrt{1+n^2}}$

37) $\sum_{n=1}^{\infty}\frac{n^2}{n^4+n}$ 38) $\sum_{n=0}^{\infty}\frac{|\sin n|}{n^2+1}$ 39) $\sum_{n=0}^{\infty}\frac{n|\cos n|}{2^n+n}$

40) $\sum_{n=0}^{\infty}\frac{n^2}{n^3+n^2+1}$ 41) $\sum_{n=1}^{\infty}\frac{n}{2^{-n}}$ 42) $\sum_{n=0}^{\infty}\frac{n^{155}}{2^n}$

43) $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n^3+1}}$ 44) $\sum_{n=0}^{\infty}\frac{n+1}{\sqrt{n^4+n+1}}$ 45) $\sum_{n=0}^{\infty}\frac{n+1}{2^n}$

46) $\sum_{n=0}^{\infty}\frac{1+\cos(\frac{n\pi}{2})}{e^n}$ 47) $\sum_{n=1}^{\infty}\frac{n^5}{e^{n^2}}$ 48) $\sum_{n=1}^{\infty}\frac{n^n}{(n+1)!}$

49) $\sum_{n=1}^{\infty}\frac{\sin(\frac{n\pi}{3})}{n}$ 50) $\sum_{n=0}^{\infty}\frac{n+(-1)^n}{3^n}$ 51) $\sum_{n=2}^{\infty}e^{\sin n}$

52) $\sum_{n=0}^{\infty}\frac{n+1}{n!(n+2)}$ 53) $\sum_{n=2}^{\infty}\frac{1}{n\ell nn}$ 54) $\sum_{n=1}^{\infty}\frac{(\ell nn)^3}{n^2}$

55) $\sum_{n=0}^{\infty}\frac{1+\cos n\pi}{n!}$ 56) $\sum_{n=2}^{\infty}\frac{1}{n\ell n^2 n}$ 57) $\sum_{n=1}^{\infty}\frac{1}{n(2n-\frac{1}{n})}$

58) $\sum_{n=1}^{\infty}\frac{3\cdot5\cdot7\cdots(2n+1)}{1\cdot4\cdot7\cdots(3n-2)}$ 59) $\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{1}{n}}}$ 60) $\sum_{n=2}^{\infty}\frac{1}{2^{\ell nn}}$

61) $\sum_{n=2}^{\infty}\frac{1}{(\ell nn)^{\ell nn}}$ 62) $\sum_{n=0}^{\infty}n^2\ell n(1+\frac{1}{2^n})$ 63) $\sum_{n=0}^{\infty}\frac{n2^n+(n!)^2}{2^{n^2}}$

III. Determine whether the following series are convergent:

1) $\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+n)}{4n^2+5n-1}$
2) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ell nn}$
3) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$
4) $\sum_{n=1}^{\infty} (-1)^{n+1} \ell n(1+\frac{1}{n})$
5) $\sum_{n=0}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{2n+1}$
6) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{3n-1}$
7) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+\cos n\pi)}{n}$
8) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$
9) $\sum_{n=0}^{\infty} (-1)^n \frac{n^2}{n^3+1}$
10) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+2}{n^3+n-1}$
11) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}-1}$

IV. Determine the values of the parameters "x" for which the following series converge.

1) $\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}x^n$
2) $\sum_{n=0}^{\infty} \frac{n!}{3^n}x^n$
3) $\sum_{n=1}^{\infty} \frac{n^2}{x^n}$
4) $\sum_{n=1}^{\infty} \frac{e^{nx}}{n}$
5) $\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{n^2 3^n}$
6) $\sum_{n=1}^{\infty} n\, e^{-nx}$
7) $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\sqrt{n+1}}$
8) $\sum_{n=0}^{\infty} n^3 x^{3n}$
9) $\sum_{n=0}^{\infty} \frac{n^5}{5^n}(x-5)^n$
10) $\sum_{n=1}^{\infty} \frac{(x-4)^n}{\sqrt{3n}}$
11) $\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{3^n+n}$
12) $\sum_{n=0}^{\infty} 3(x+1)^n$
13) $\sum_{n=0}^{\infty} \frac{\sqrt{n}}{(n+1)(2x+3)^n}$
14) $\sum_{n=0}^{\infty} e^{\sin n} x^n$
15) $\sum_{n=0}^{\infty} \frac{(x-2)^{4n}}{2^n}$
16) $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}x^n$
17) $\sum_{n=0}^{\infty} n!(x-3)^n$
18) $\sum_{n=2}^{\infty} \frac{(x+6)^n}{\ell nn}$

V. Determine the **exact values** of the following sums:

1) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$
2) $\sum_{n=0}^{\infty} \frac{x^n}{e^{nx}}$
3) $\sum_{n=1}^{\infty} 3(x+1)^n\, 2^n$
4) $\sum_{n=1}^{\infty} (\sin \frac{n\pi}{2})^n$
5) $\sum_{n=3}^{\infty} \frac{4n-3}{n(n^2-4)}$ (ans: $\frac{167}{96}$)
6) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)}$ (ans: $\frac{1}{12}$)
7) $\sum_{n=1}^{\infty} \ell n(\frac{n(n+2)}{(n+1)^2})$ (ans: $\ell n2$)

VI. For the following functions find the Taylor series expansion about the given point and determine the interval of convergence.

1) $f(x) = e^x, \quad x_0 = a$
2) $f(x) = \ell nx, \quad x_0 = 2$
3) $f(x) = \sin x, \quad x_0 = \frac{\pi}{6}$
4) $f(x) = \frac{1}{x}, \quad x_0 = 1$
5) $f(x) = \frac{1}{1-36x^2}, \quad x_0 = 0$

VII. A function satisfies the initial value problem: $f'(x) + 2x\, f(x) = 0$, $f(0) = 1$. Find $f(x)$ in the form of a power series $\sum_{n=0}^{\infty} a_n\, x^n$ and identify this function. In what interval does the series represent the function?