# Representation Theory of Finite Groups

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CHAPTER 1

# Introduction

This is class notes for the course on representation theory of finite groups taught by the author at IISER Pune to undergraduate students. We study character theory of finite groups and illustrate how to get more information about groups. The Burnside's theorem is one of the very good applications. It states that every group of order $p^a q^b$, where $p, q$ are distinct primes, is solvable. We will always consider finite groups unless stated otherwise. All vector spaces will be considered over general fields in the beginning but for the purpose of character theory we assume the field is that of complex numbers.

We assume knowledge of the basic group theory and linear algebra. The point of view I projected to the students in the class is that we have studied linear algebra hence we are familiar with the groups $GL(V)$, the general linear group or $GL_n(k)$ in the matrix notation. The idea of representation theory is to compare (via homomorphisms) finite (abstract) groups with these linear groups (some what concrete) and hope to gain better understanding of them.

The students were asked to read about "linear groups" from the book by Alperin and Bell (mentioned in the bibiliography) from the chapter with the same title. We also revised, side-by-side in the class, Sylow's Theorem, Solvable groups and motivated ourselves for the Burnside's theorem.

The aim to start with arbitrary field was to give the feeling that the theory is dependent on the base field and it gets considerably complicated if we move away from characteristic 0 algebraically closed field. This we illustrate by giving an example of higher dimensional irreducible representation of cyclic group over $\mathbb{Q}$ while all its irreducible representations are one dimensional over $\mathbb{C}$. Thus this puts things in perspective why we are doing the theory over $\mathbb{C}$ and motivates us to develop "Character Theory".

CHAPTER 2

# Representation of a Group

Let $G$ be a finite group. Let $k$ be a field. We will assume that characteristic of $k$ is 0, e.g.,$\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$ though often $char(k) \nmid |G|$ is enough.

**Definition 2.1** (Representation)**.** *A* **representation of** $G$ **over** $k$ *is a homomorphism* $\rho\colon G \to GL(V)$ *where* $V$ *is a vector space of finite dimension over field* $k$*. The vector space* $V$ *is called a* **representation space** *of* $G$ *and its dimension the* **dimension of representation***.*

Strictly speaking the pair $(\rho, V)$ is called representation of $G$ over field $k$. However if there is no confusion we simply call $\rho$ a representation or $V$ a representation of $G$. Let us fix a basis $\{v_1, v_2, \dots, v_n\}$ of $V$. Then each $\rho(g)$ can be written in a matrix form with respect to this basis. This defines a map $\tilde{\rho}\colon G \to GL_n(k)$ which is a group homomorphism.

**Definition 2.2** (Invariant Subspace)**.** *Let* $\rho$ *be a representation of* $G$ *and* $W \subset V$ *be a subspace. The space* $W$ *is called a* $G$**-invariant (or** $G$**-stable)** *subspace if* $\rho(g)(w) \in W \forall w \in W$ *and* $\forall g \in G$*.*

Notice that once we have a $G$-invariant subspace $W$ we can restrict the representation to this subspace and define another representation $\rho_W\colon G \to GL(W)$ where $\rho_W(g) = \rho(g)|_W$. Hence $W$ is also called a **subrepresentation**.

**Example 2.3** (Trivial Representation)**.** Let $G$ be a group and $k$ and field. Let $V$ be a vector space over $k$. Then $\rho(g) = 1$ for all $g \in G$ is a representation. This is called trivial representation. In this case every subspace of $V$ is an invariant subspace.

**Example 2.4.** Let $G = \mathbb{Z}/m\mathbb{Z}$ and $k = \mathbb{C}$. Let $V$ be a vector space of dimension $n$.

(1) Suppose $\dim(V) = 1$. Define $\rho_r\colon \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}^*$ by $1 \mapsto e^{\frac{2\pi i r}{m}}$ for $1 \leq r \leq m-1$.
(2) Define $\rho\colon \mathbb{Z}/m\mathbb{Z} \to GL(V)$ by $1 \mapsto T$ where $T^m = 1$. For example if $\dim(V) = 2$ once can take $T = \mathrm{diag}\{e^{\frac{2\pi i r_1}{m}}, e^{\frac{2\pi i r_2}{m}}\}$. There is a general theorem in Linear Algebra which says that any such matrix over $\mathbb{C}$ is diagonalizable.

**Example 2.5.** Let $G = \mathbb{Z}/m\mathbb{Z}$ and $k = \mathbb{R}$. Let $V = \mathbb{R}^2$ with basis $\{e_1, e_2\}$. Then we have representations of $\mathbb{Z}/m\mathbb{Z}$:

$$\rho_r\colon 1 \mapsto \begin{bmatrix} \cos\frac{2\pi r}{m} & -\sin\frac{2\pi r}{m} \\ \sin\frac{2\pi r}{m} & \cos\frac{2\pi r}{m} \end{bmatrix}$$

where $1 \leq r \leq m-1$. Notice that we have $m$ distinct representations.

**Example 2.6.** Let $\phi\colon G \to H$ be a group homomorphism. Let $\rho$ be a representation of $H$. Then $\rho \circ \phi$ is a representation of $G$.

**Example 2.7.** Let $G = D_m = \langle a, b \mid a^m = 1 = b^2, ab = ba^{m-1}\rangle$ the dihedral group with $2m$ elements. We have representations $\rho_r$ defined by :

$$a \mapsto \begin{bmatrix} \cos\frac{2\pi r}{m} & -\sin\frac{2\pi r}{m} \\ \sin\frac{2\pi r}{m} & \cos\frac{2\pi r}{m} \end{bmatrix}, b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Notice that we make use of the representation of $\mathbb{Z}/m\mathbb{Z}$ to construct this.

**Example 2.8** (Permutation Representation of $S_n$)**.** Let $S_n$ be the symmetric group on $n$ symbols and $k$ any field. Let $V = k^n$ with standard basis $\{e_1, \ldots, e_n\}$. We define a representation of $S_n$ as follows: $\sigma(e_i) = e_{\sigma(i)}$ for $\sigma \in S_n$. Notice that while defining this representation we don't need to specify any field.

**Example 2.9** (Group Action)**.** Let $G$ be a group and $k$ be a field. Let $G$ be acting on a finite set $X$, i.e., we have $G \times X \to X$. We denote $k[X] = \{f \mid f\colon X \to k\}$, set of all maps. Clearly $k[X]$ is a vector space of dimension $|X|$. The elements $e_x\colon X \to k$ defined by $e_x(x) = 1$ and $e_x(y) = 0$ if $x \neq y$ form a basis of $k[X]$. The action gives rise to a representation of $G$ on the space $k[X]$ as follows: $\rho\colon G \to GL(k[X])$ given by $(\rho(g)(f))(x) = f(g^{-1}x)$ for $x \in X$. In fact one can make $k[X]$ an algebra by the following multiplication:

$$(f * f')(t) = \sum_x f(x) f'(x^{-1}t).$$

Note that this is convolution multiplication not the usual point wise multiplication. If we take $G = S_n$ and $X = \{1, 2, \ldots, n\}$ we get back above example.

**Example 2.10** (Regular Representation)**.** Let $G$ be a group of order $n$ and $k$ a field. Let $V = k[G]$ be an $n$-dimensional vector space with basis as elements of the group itself. We define $L\colon G \to GL(k[G])$ by $L(g)(h) = gh$, called the left regular representation. Also $R(g)(h) = hg^{-1}$, defines right regular representation of $G$. Prove that these representations are injective. Also these representations are obtained by the action of $G$ on the set $X = G$ by left multiplication or right multiplication.

**Example 2.11.** Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and $k = \mathbb{C}$. We define a 2-dimensional representation of $Q_8$ by:

$$i \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, j \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

**Example 2.12** (Galois Theory)**.** Let $K = \mathbb{Q}(\theta)$ be a finite extension of $\mathbb{Q}$. Let $G = \mathrm{Gal}(K/Q)$. We take $V = K$, a finite dimensional vector space over $\mathbb{Q}$. We have natural representation of $G$ as follows: $\rho\colon G \to GL(K)$ defined by $\rho(g)(x) = g(x)$. Take $\theta = \zeta$, some $n$th root of unity and show that the cyclic groups $\mathbb{Z}/m\mathbb{Z}$ have representation over field $\mathbb{Q}$ of possibly dimension more than 2. This is a reinterpretation of the statement of the Kronecker-Weber theorem.

**Definition 2.13** (Equivalence of Representations)**.** *Let $(\rho, V)$ and $(\rho', V')$ be two representations of $G$. The representations $(\rho, V)$ and $(\rho', V')$ are called* **$G$-equivalent (or equivalent)** *if there exists a linear isomorphism $T\colon V \to V'$ such that $\rho'(g) = T\rho(g)T^{-1}$ for all $g \in G$.*

Let $\rho$ be a representation. Fix a basis of $V$, say $\{e_1, \dots, e_n\}$. Then $\rho$ gives rise to a map $G \to GL_n(k)$ which is a group homomorphism. Notice that if we change the basis of $V$ then we get a different map for the same $\rho$. However they are equivalent as representation, i.e. differ by conjugation with respect to a fix matrix (the base change matrix).

**Example 2.14.** The trivial representation is irreducible if and only if it is one dimensional.

**Example 2.15.** In the case of Permutation representation the subspace $W =< (1, 1, \dots, 1) >$ and $W' = \{(x_1, \dots, x_n) \mid \sum x_i = 0\}$ are two irreducible $S_n$ invariant subspaces. In fact this representation is direct sum of these two and hence completely reducible.

**Example 2.16.** One dimensional representation is always irreducible. If $|G| \geq 2$ then the regular representation is not irreducible.

**Exercise 2.17.** Let $G$ be a finite group. In the definition of a representation, let us not assume that the vector space $V$ is finite dimensional. Prove that there exists a finite dimensional $G$-invariant subspace of $V$.

Hint : Fix $v \in V$ and take $W$ the subspace generated by $\rho(g)(v)\ \forall g \in G$.

**Exercise 2.18.** A representation of dimension 1 is a map $\rho\colon G \to k^*$. There are exactly two one dimensional representations of $S_n$ over $\mathbb{C}$. There are exactly $n$ one dimensional representations of cyclic group $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{C}$.

**Exercise 2.19.** Prove that every finite group can be embedded inside symmetric group $S_n$ fro some $n$ as well as linear groups $GL_m$ for some $m$.

Hint: Make use of the regular representation. This representation is also called "God given" representation. Later in the course we will see why its so.

**Exercise 2.20.** Is the above exercise true if we replace $S_n$ by $A_n$ and $GL_m$ by $SL_m$?

**Exercise 2.21.** Prove that the cyclic group $\mathbb{Z}/p\mathbb{Z}$ has a representation of dimension $p - 1$ over $\mathbb{Q}$.

Hint: Make use of the cyclotomic field extension $\mathbb{Q}(\zeta_p)$ and consider the map left multiplication by $\zeta_p$. This exercise shows that representation theory is deeply connected to the Galois Theory of field extensions.

### 2.1. Commutator Subgroup and One dimensional representations

Let $G$ be a finite group. Consider the set of elements $\{xyx^{-1}y^{-1} \mid x, y \in G\}$ and $G'$ the subgroup generated by this subset. This subgroup is called the **commutator subgroup of** $G$. We list some of the properties of this subgroup as an exercise here.

**Exercise 2.22.**
(1) $G'$ is a normal subgroup.
(2) $G/G'$ is Abelian.
(3) $G'$ is smallest subgroup of $G$ such that $G/G'$ is Abelian.
(4) $G' = 1$ if and only if $G$ is Abelian.
(5) For $G = S_n$, $G' = A_n$; $G = D_n =< r, s \mid r^n = 1 = s^2, srs = r^{-1} >$ we have $G' =< r >\cong \mathbb{Z}/n\mathbb{Z}$ and $Q_8' = \mathcal{Z}(Q_8)$.

Let $\widehat{G}$ be the set of all one-dimensional representations of $G$ over $\mathbb{C}$, i.e., the set of all group homomorphisms from $G$ to $\mathbb{C}^*$. For $\chi_1, \chi_2 \in \widehat{G}$ we define multiplication by:

$$(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g).$$

**Exercise 2.23.** Prove that $\widehat{G}$ is an Abelian group.

We observe that for a $\chi \in \widehat{G}$ we have $G' \subset ker(\chi)$. Hence we can prove,

**Exercise 2.24.** Show that $\widehat{G} \cong G/G'$.

**Exercise 2.25.** Calculate directly $\widehat{G}$ for $G = \mathbb{Z}/n\mathbb{Z}, S_n$ and $D_n$.

Let $G$ be a group. The group $G$ is called **simple** if $G$ has no proper normal subgroup.

**Exercise 2.26.**
(1) Let $G$ be an Abelian simple group. Prove that $G$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ where $p$ is a prime.
(2) Let $G$ be a simple non-Abelian group. Then $G = G'$.

CHAPTER 3

# Maschke's Theorem

In the last chapter we saw a representation can have possibly a subrepresentation. This motivates us to define:

**Definition 3.1** (Irreducible Representation)**.** *A representation $(\rho, V)$ of $G$ is called* **irreducible** *if it has no proper invariant subspace, i.e., only invariant subspaces are* $0$ *and* $V$*.*

Let $(\rho, V)$ and $(\rho', V')$ be two representations of $G$ over field $k$. We can define **direct sum** of these two representations $(\rho \oplus \rho', V \oplus V')$ as follows: $\rho \oplus \rho' \colon G \to GL(V \oplus V')$ such that $(\rho \oplus \rho')(g)(v, v') = (\rho(g)(v), \rho'(g)(v'))$. In the matrix notation if we have two representations $\rho \colon G \to GL_n(k)$ and $\rho' \colon G \to GL_m(k)$ then $\rho \oplus \rho'$ is given by

$$g \mapsto \begin{bmatrix} \rho(g) & 0 \\ 0 & \rho'(g) \end{bmatrix}.$$

This motivates us to look at those nice representations which can be obtained by taking direct sum of irreducible ones.

**Definition 3.2** (Completely Reducible)**.** *A representation $(\rho, V)$ is called completely reducible if it is a direct sum of irreducible ones. Equivalently if $V = W_1 \oplus \ldots \oplus W_r$, where each $W_i$ is $G$-invariant irreducible representations.*

This brings us to the following questions:

(1) Is it true that every representation is direct sum of irreducible ones?
(2) How many irreducible representations are there for $G$ over $k$?

The answer to the first question is affirmative in the case $G$ is a finite group and $char(k) \nmid |G|$ which is the Maschake's theorem proved below. (It is also true for Compact Groups where it is called Peter-Weyl Theorem). The other exceptional case comes under the subject 'Modular Representation Theory'. We will answer the second question over the field of complex numbers (and possible over $\mathbb{R}$) which is the character theory. For the theory over $\mathbb{Q}$ the subject is called 'rationality questions' (refer to the book by Serre).

**Theorem 3.3** (Maschke's Theorem)**.** *Let $k$ be a field and $G$ be a finite group. Suppose $char(k) \nmid |G|$, i.e. $|G|$ is invertible in the field $k$. Let $(\rho, V)$ be a finite dimensional representation of $G$. Let $W$ be a $G$-invariant subspace of $V$. Then there exists $W'$ a $G$-invariant* *subspace such that $V = W \oplus W'$. Conversely, if $char(k) \mid |G|$ then there exists a representation, namely the regular representation, and a proper $G$-invariant subspace which does not have a $G$-invariant compliment.*

**Proposition 3.4** (Complete Reducibility)**.** *Let $k$ be a field and $G$ be a finite group with $char(k) \nmid |G|$. Then every finite dimensional representation of $G$ is completely reducible.*

Now we are going to prove the above results. We need to recall notion of projection from 'Linear Algebra'. Let $V$ be a finite dimensional vector space over field $k$.

**Definition 3.5.** *An endomorphisms $\pi\colon V \to V$ is called a projection if $\pi^2 = \pi$.*

Let $W \subset V$ be a subspace. A subspace $W'$ is called a compliment of $W$ if $V = W \oplus W'$. It is a simple exercise in 'Linear Algebra' to show that such a compliment always exists (see exercise below) and there could be many of them.

**Lemma 3.6.** *Let $\pi$ be an endomorphism. Then $\pi$ is a projection if and only if there exists a decomposition $V = W \oplus W'$ such that $\pi(W) = 0$ and $\pi(W') = W'$ and $\pi$ restricted to $W'$ is identity.*

Proof. Let $\pi\colon V \to V$ such that $\pi(w, w') = w'$. Then clearly $\pi^2 = \pi$.

Now suppose $\pi$ is a projection. We claim that $V = ker(\pi) \oplus Im(\pi)$. Let $x \in ker(\pi) \cap Im(\pi)$. Then there exists $y \in V$ such that $\pi(y) = x$ and $x = \pi(y) = \pi^2(y) = \pi(\pi(x)) = \pi(0) = 0$. Hence $ker(\pi) \cap Im(\pi) = 0$. Now let $v \in V$. Then $v = (v - \pi(v)) + \pi(v)$ and we see that $\pi(v) \in Im(\pi)$ and $v - \pi(v) \in ker(\pi)$ since $\pi(v - \pi(v)) = \pi(v) - \pi^2(v) = 0$. Now let $x \in Im(\pi)$, say $x = \pi(y)$. Then $\pi(x) = \pi(\pi(y)) = \pi(y) = x$. This shows that $\pi$ restricted to $Im(\pi)$ is identity map. □

**Remark 3.7.** The reader familiar with 'Canonical Form Theory' will recognise the following. The minimal polynomial of $\pi$ is $X(X-1)$ which is a product of distinct linear factors. Hence $\pi$ is a diagonalizable linear transformation with eigen values 0 and 1. Hence with respect to some basis the matrix of $\pi$ is $\text{diag}\{0, \ldots, 0, 1, \ldots, 1\}$. This will give another proof of the Lemma.

**Proof of the Maschke's Theorem.** Let $\rho\colon G \to GL(V)$ be a representation. Let $W$ be a $G$-invariant subspace of $V$. Let $W_0$ be a compliment, i.e., $V = W_0 \oplus W$. We have to produce a compliment which is $G$-invariant. Let $\pi$ be a projection corresponding to this decomposition, i.e., $\pi(W_0) = 0$ and $\pi(w) = w$ for all $w \in W$. We define an endomorphism $\pi'\colon V \to V$ by 'averaging technique' as follows:

$$\pi' = \frac{1}{|G|} \sum_{t \in G} \rho(t)^{-1} \pi \rho(t).$$

We claim that $\pi'$ is a projection. We note that $\pi'(V) \subset W$ since $\pi\rho(t)(V) \subset W$ and $W$ is $G$-invariant. In fact, $\pi'(w) = w$ for all $w \in W$ since $\pi'(w) = \frac{1}{|G|} \sum_{t \in G} \rho(t)^{-1} \pi \rho(t)(w) =$

$\frac{1}{|G|}\sum_{t\in G}\rho(t)^{-1}\pi(\rho(t)(w)) = \frac{1}{|G|}\sum_{t\in G}\rho(t)^{-1}(\rho(t)(w)) = w$ (note that $\rho(t)(w) \in W$ and $\pi$ takes it to itself). Let $v \in V$. Then $\pi'(v) \in W$. Hence $\pi'^2(v) = \pi'(\pi'(v)) = \pi'(v)$ as we have $\pi'(v) \in W$ and $\pi'$ takes any element of $W$ to itself. Hence $\pi'^2 = \pi'$.

Now we write decomposition of $V$ with respect to $\pi'$, say $V = W' \oplus W$ where $W' = ker(\pi')$ and $Im(\pi') = W$. We claim that $W'$ is $G$-invariant which will prove the theorem. For this we observe that $\pi'$ is a $G$-invariant homomorphism, i.e., $\pi'(\rho(g)(v)) = \rho(g)(\pi'(v))$ for all $g \in G$ and $v \in V$.

$$\begin{aligned}
\pi'(\rho(g)(v)) &= \frac{1}{|G|}\sum_{t\in G}\rho(t)^{-1}\pi\rho(t)(\rho(g)(v)) \\
&= \frac{1}{|G|}\sum_{t\in G}\rho(g)\rho(g)^{-1}\rho(t)^{-1}\pi\rho(t)\rho(g)(v) \\
&= \rho(g)\frac{1}{|G|}\sum_{t\in G}\rho(tg)^{-1}\pi\rho(tg)(v) \\
&= \rho(g)(\pi'(v)).
\end{aligned}$$

This helps us to verify that $W'$ is $G$-invariant. Let $w' \in W'$. To show that $\rho(g)(w') \in W'$. For this we note that $\pi'(\rho(g)(w')) = \rho(g)(\pi'(w)) = \rho(g)(0) = 0$. This way we have produced $G$-invariant compliment of $W$.

For the converse let $char(k) \mid |G|$. We take the regular representation $V = k[G]$. Consider $W = \left\{\sum_{g\in G}\alpha_g g \mid \sum\alpha_g = 0\right\}$. We claim that $W$ is $G$-invariant but it has no $G$-invariant compliment. □

**Remark 3.8.** In the proof of Maschke's theorem one can start with a symmetric bilinear form and apply the trick of averaging to it. In that case the compliment will be the orthogonal subspace. Conceptually I like that proof better however it requires familiarity with bilinear form to be able to appreciate that proof. Later we will do that in some other context.

**Proof of the Proposition (Complete Reducibility).** Let $\rho\colon G \to GL(V)$ be a representation. We use induction on the dimension of $V$ to prove this result. Let $\dim(V) = 1$. It is easy to verify that one-dimensional representation is always irreducible. Let $V$ be of dimension $n \geq 2$. If $V$ is irreducible we have nothing to prove. So we may assume $V$ has a $G$-invariant proper subspace, say $W$ with $1 \leq \dim(W) \leq n-1$. By Maschke's Theorem we can write $V = W \oplus W'$ where $W'$ is also $G$-invariant. But now $\dim(W)$ and $\dim(W')$ bot are less than $n$. By induction hypothesis they can be written as direct sum of irreducible representations. This proves the proposition. □

**Exercise 3.9.** Let $V$ be a finite dimensional vector space. Let $W \subset V$ be a subspace. Show that there exists a subspace $W'$ such that $V = W \oplus W'$.

Hint: Start with a basis of $W$ and extend it to a basis of $V$.

**Exercise 3.10.** Show that when $V = W \oplus W'$ the map $\pi(w + w') = w$ is a projection map. Verify that $ker(\pi) = W'$ and $Im(\pi) = W$. The map $\pi$ is called a projection on $W$. Notice that this map depends on the chosen compliment $W'$.

**Exercise 3.11.**
(1) Compliment of a subspace is not unique. Let us consider $V = \mathbb{R}^2$. Take a line $L$ passing through the origin. It is a one dimensional subspace. Prove that any other line is a compliment.
(2) Let $W$ be the one dimensional subspace x-axis. Choose the compliment space as y-axis and write down the projection map. What if we chose the compliment as the line $x = y$?

The exercises below show that Maschke's theorem may not be true if we don't have finite group.

**Exercise 3.12.** Let $G = \mathbb{Z}$ and $V = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{R}\}$ be the sequence space (a vector space of infinite dimension). Define $\rho(1)(a_1, a_2 \ldots .) = (0, a_1, a_2, \ldots)$ and $\rho(n)$ by composing $\rho(1)$ $n$-times. Show that this is a representation of $\mathbb{Z}$. Prove that it has no invariant subspace.

**Exercise 3.13.** Consider a two dimensional representation of $\mathbb{R}$ as follows:

$$a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

It leaves one dimensional subspace fixed generated by $(1, 0)$ but it has no complementary subspace. Hence this representation is not completely reducible.

**Exercise 3.14.** Let $k = \mathbb{Z}/p\mathbb{Z}$. Consider two dimensional representation of the cyclic group $G = \mathbb{Z}/p\mathbb{Z}$ of order $p$ over $k$ of characteristic $p$ defined as in the previous example. Find a subspace to show that Maschke's theorem does not hold.

**Exercise 3.15.** Let $V$ be an irreducible representation of $G$. Let $W$ be a $G$-invariant subspace of $V$. Show that either $W = 0$ or $W = V$.

CHAPTER 4

# Schur's Lemma

**Definition 4.1** ($G$-map)**.** *Let $(\rho, V)$ and $(\rho', V')$ be two representations of $G$ over field $k$. A linear map $T\colon V \to V'$ is called a* **$G$-map (between two representations)** *if it satisfies the following:*

$$\rho'(t)T = T\rho(t)\forall t \in G.$$

The $G$-maps are also called **intertwiners**.

**Exercise 4.2.** Prove that two representations of $G$ are equivalent if and only if there exists an invertible $G$-map.

In the case representations are irreducible the $G$-maps are easy to decide. In the wake of Maschke's Theorem considering irreducible representations are enough.

**Proposition 4.3** (Schur's Lemma)**.** *Let $(\rho, V)$ and $(\rho', V')$ be two irreducible representations of $G$ (of dimension $\geq 1$). Let $T\colon V \to V'$ be a $G$-map. Then either $T = 0$ or $T$ is an isomorphism. Moreover if $T$ is nonzero then $T$ is an isomorphism if and only if the two representations are equivalent.*

Proof. Let us consider the subspace $ker(T)$. We claim that it is a $G$-invariant subspace of $V$. For this let us take $v \in ker(T)$. Then $T\rho(t)(v) = \rho'(t)T(v) = 0$ implies $\rho(t)(v) \in ker(T)$ for all $t \in G$. Now applying Maschke's theorem on the irreducible representation $V$ we get either $ker(T) = 0$ or $ker(T) = V$. In the case $ker(T) = V$ the map $T = 0$.

Hence we may assume $ker(T) = 0$, i.e., $T$ is injective. Now we consider the subspace $Im(T) \subset V'$. We claim that it is also $G$-invariant. For this let $y = T(x) \in Im(T)$. Then $\rho'(t)(y) = \rho'(t)T(x) = T\rho(t)(x) \in Im(T)$ for any $t \in G$. Hence $Im(T)$ is $G$-invariant. Again by applying Maschke's theorem on the irreducible representation $V'$ we get either $Im(T) = 0$ or $Im(T) = V'$. Since $T$ is injective $Im(T) \neq 0$ and hence $Im(T) = V'$. Which proves that in this case $T$ is an isomorphism. $\square$

**Exercise 4.4.** Let $V$ be a vector space over $\mathbb{C}$ and $T \in \mathrm{End}(V)$ be a linear transformation. Show that there exists a one-dimensional subspace of $V$ left invariant by $T$. Show by example that this need not be true if the field is $\mathbb{R}$ instead of $\mathbb{C}$.

Hint: Show that $T$ has an eigen-value and the corresponding eigen-vector will do the job.

**Corollary 4.5.** *Let $(\rho, V)$ be an irreducible representation of $G$ over $\mathbb{C}$. Let $T\colon V \to V$ be a $G$-map. Then $T = \lambda.Id$ for some $\lambda \in \mathbb{C}$ and $Id$ is the identity map on $V$.*

PROOF. Let $\lambda$ be an eigen-value of $T$ corresponding to the eigen-vector $v \in V$, i.e., $T(v) = \lambda v$. Consider the subspace $W = ker(T - \lambda.Id)$. We claim that $W$ is a $G$-invariant subspace. Since $T$ and scalar multiplications are $G$-maps so is $T - \lambda$. Hence the kernal is $G$-invariant (as we verified in the proof of Schur's Lemma). One can do this directly also see the exercise below.

Since $W \neq 0$ and is $G$-invariant we can apply Maschke's Theorem and get $W = V$. This gives $T = \lambda.Id$. $\square$

**Exercise 4.6.** Let $T$ and $S$ be two $G$-maps. Show that $ker(T + S)$ is a $G$-invariant subspace.

CHAPTER 5

# Representation Theory of Finite Abelian Groups over $\mathbb{C}$

Throughout this chapter $G$ denotes a finite Abelian group.

**Proposition 5.1.** *Let $k = \mathbb{C}$ and $G$ be a finite Abelian group. Let $(\rho, V)$ be an irreducible representation of $G$. Then,* $\dim(V) = 1$.

Proof. Proof is a simple application of the Schur's Lemma. We will break it in step-by-step exercise below. ☐

**Exercise 5.2.** With notation as in the proposition,

(1) for $g \in G$ consider $\rho(g): V \to V$. Prove that $\rho(g)$ is a $G$-map. (Hint: $\rho(g)(\rho(h)(v)) = \rho(gh)(v) = \rho(hg)(v) = \rho(h)(\rho(g)(v))$.)
(2) Prove that there exists $\lambda$ (depending on $g$) in $\mathbb{C}$ such that $\rho(g) = \lambda.Id$. (Hint: Use the corollary of Schur's Lemma.)
(3) Prove that the map $\rho: G \to GL(V)$ maps every element $g$ to a scalar map, i.e., it is given by $\rho(g) = \lambda_g.Id$ where $\lambda_g \in \mathbb{C}$.
(4) Prove that the dimension of $V$ is 1. (Hint: Take any one dimensional subspace of $V$. It is $G$-invariant. Use Maschke's theorem on it as $V$ is irreducible.)

**Proposition 5.3.** *Let $k = \mathbb{C}$ and $G$ be a finite Abelian group. Let $\rho: G \to GL(V)$ be a representation of dimension $n$. Prove that we can choose a basis of $V$ such that $\rho(G)$ is contained in diagonal matrices.*

Proof. Since $V$ is a representation of finite group we can use Maschke's theorem to write it as direct sum of $G$-invariant irreducible ones, say $V = W_1 \oplus \ldots \oplus W_r$. Now using Schur's lemma we conclude that $\dim(W_i) = 1$ for all $i$ and hence in turn we get $r = n$. By choosing a vector in each $W_i$ we get the required result. ☐

**Corollary 5.4.** *Let $G$ be a finite group (possibly non-commutative). Let $\rho: G \to GL(V)$ be a representation. Let $g \in G$. Then there exists a basis of $V$ such that the matrix of $\rho(g)$ is diagonal.*

Proof. Consider $H = < g > \subset G$ and $\rho: H \to GL(V)$ the restriction map. Since $H$ is Abelian, using above proposition, we can simultaneously diagonalise elements of $H$. This proves the required result. ☐

**Remark 5.5.** In 'Linear Algebra' we prove the following result: A commuting set of diagonalizable matrices over $\mathbb{C}$ can be simultaneously diaognalised. The proposition above is a version of the same result. We also give a warning about corollary above that if we have a finite subgroup $G$ of $GL_n(\mathbb{C})$ then we can take a conjugate of $G$ in such a way that a particular element becomes diagonal.

Now this leaves us the question to determine all irreducible representations of an Abelian group $G$. For this we need to determine all group homomorphisms $\rho\colon G \to \mathbb{C}^*$.

**Exercise 5.6.** Let $G$ be a finite group (not necessarily Abelian). Let $\chi\colon G \to \mathbb{C}^*$ be a group homomorphism. Prove that $|\chi(g)| = 1$ and hence $\chi(g)$ is a root of unity.

Let $\widehat{G}$ be the set of all group homomorphisms from $G$ to the multiplicative group $\mathbb{C}^*$. Let us also denote $\widehat{\widehat{G}}$ for the group homomorphisms from $\widehat{G}$ to $\mathbb{C}^*$.

**Exercise 5.7.** With the notation as above,

(1) Prove that for $G = G_1 \times G_2$ we have $\widehat{G} \cong \widehat{G}_1 \times \widehat{G}_2$.
(2) Let $G = \mathbb{Z}/n\mathbb{Z}$. Prove that $\widehat{G} = \{\chi_k \mid 0 \leq k \leq n-1\}$ is a group generated by $\chi_1$ of order $n$ where $\chi_1(r) = e^{\frac{2\pi i r}{n}}$ and $\chi_k = \chi_1^k$. Hence $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$.
(3) Use the structure theorem of finite Abelian groups to prove that $G \cong \widehat{G}$.
(4) Prove that $G$ is naturally isomorphic to $\widehat{\widehat{G}}$ given by $g \mapsto e_g$ where $e_g(\chi) = \chi(g)$ for all $\chi \in G$.

**Exercise 5.8** (Fourier Transform)**.** For $f \in \mathbb{C}[\mathbb{Z}/n\mathbb{Z}] = \{f \mid f\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}\}$ we define $\hat{f} \in \mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ by,

$$\hat{f}(q) = \frac{1}{n}\sum_{k=0}^{n-1} f(k)e(-kq) = \frac{1}{n}\sum_{k=0}^{n-1} f(k)\chi_q(-k).$$

Show that $f(k) = \sum_{q=0}^{n-1} \hat{f}(q)e(kq) = \sum_{q=0}^{n-1} \hat{f}(q)\chi_q(k)$ and $\frac{1}{n}\ \sum_{k=0}^{n-1} |f(k)|^2 = \sum_{q=0}^{n-1} |\hat{f}(q)|^2$.

**Exercise 5.9.** On $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ let us define an inner product by $\langle f, f'\rangle = \frac{1}{n}\sum_{j=0}^{n-1} f(j)\bar{f}'(j)$ where bar denotes complex conjugation. Prove that $\{\chi_k \mid 0 \leq k \leq n-1\}$ form an orthonormal basis of $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$. Let

$$f = \sum_{\chi \in \widehat{\mathbb{Z}/n\mathbb{Z}}} c_\chi \chi.$$

Calculate the coefficients using the inner product and compare this with previous exercise.

Now we show that the converse of the Proposition 5.1 is also true.

**Theorem 5.10.** *Let $G$ be a finite group. Every irreducible representation of $G$ over $\mathbb{C}$ is $1$ dimensional if and only if $G$ is an Abelian group.*

PROOF. Let all irreducible representations of $G$ over $\mathbb{C}$ be of dimension 1. Consider the regular representation $\rho\colon G \to GL(V)$ where $V = \mathbb{C}[G]$). We know that if $|G| \geq 2$ this representation is reducible and is an injective map (also called faithful representation). Using Maschke's theorem we can write $V$ as a direct sum of irreducible ones and they are given to be of dimension 1. Hence there exists a basis (check why?) $\{v_1, \ldots, v_n\}$ of $V$ such that subspace generated by each basis vectors are invariant. Hence $\rho(G)$ consists of diagonal matrices with respect to this basis which is an Abelian group. Hence $G \cong \rho(G)$ is an Abelian group. $\square$

## 5.1. Example of representation over $\mathbb{Q}$

**5.1.1. An Irreducible Representation of $\mathbb{Z}/p\mathbb{Z}$.** Consider $G = \mathbb{Z}/p\mathbb{Z}$ where $p$ is an odd prime. Let $K = \mathbb{Q}(\zeta)$ where $\zeta$ is a primitive $p$th root of unity. Let us consider the left multiplication map $l_\zeta\colon K \to K$ given by $x \mapsto \zeta x$. Consider the basis $\{1, \zeta, \zeta^2, \ldots, \zeta^{p-2}\}$ of $K$. Then $l_\zeta(1) = \zeta, l_\zeta(\zeta^i) = \zeta^{i+1}$ for $1 \leq i \leq p-1$ and $l_\zeta(\zeta^{p-2}) = \zeta^{p-1} = -(1 + \zeta + \zeta^2 + \ldots + \zeta^{p-2})$ and the matrix of $l_\zeta$ is:

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

The map $1 \mapsto l_\zeta$ defines a representation $\rho\colon G \to GL_{p-1}(\mathbb{Q})$. It is an irreducible representation of $G$. For this we note that $K = \mathbb{Q}(\zeta)$ is a simple

**5.1.2. An Irreducible Representation of the Dihedral Group $D_{2p}$.** Notice that the Galois group $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/(p-1)\mathbb{Z}$ comes with a natural representation on $K$. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. Then $\sigma$ is a $\mathbb{Q}$-linear map which gives representation $\mathrm{Gal}(K/Q) \cong \mathbb{Z}/(p-1)\mathbb{Z} \to GL_{p-1}(\mathbb{Q})$. If we consider slightly different basis of $K$, namely, $\{\zeta, \zeta^2, \ldots, \zeta^{p-1}\}$ then the matrix of each $\sigma$ is a permutation matrix. In fact this way $\mathrm{Gal}(K/\mathbb{Q}) \hookrightarrow S_{p-1}$, the symmetric group. However this representation is not irreducible (the element $\zeta + \zeta^2 + \cdots + \zeta^{p-1}$ is invariant and gives decomposition).

Notice that the Galois automomrphism $\sigma\colon K \to K$ given by $\zeta \mapsto \zeta^{-1}$ is an order 2 element. We claim that $\sigma$ normalizes $l_\zeta$, i.e., $\sigma l_\zeta \sigma = l_{\zeta^{-1}}$. Since $\sigma l_\zeta \sigma(\zeta^i) = \sigma l_\zeta(\zeta^{-i}) = \sigma(\zeta^{1-i}) = \zeta^{i-1} = l_{\zeta^{-1}}(\zeta^i)$. Let us denote the subgroup generated by $\sigma$ as $H$ and the subgroup generated by $l_\zeta$ by $K$. Then $HK$ is a group of order $2p$ where $K$ is a normal subgroup of order $p$. Hence $HK \cong D_{2p}$. This gives representation of order $p-1$ of $D_{2p} = \langle r, s \mid r_p = 1 = s^2, srs = r^{-1}\rangle$ given by $D_{2p} \to GL(K)$ such that $r \mapsto l_\zeta$ and $s \mapsto \sigma$.

**Exercise 5.11.** Prove that the representation constructed above are irreducible.

**Exercise 5.12.** Write down the above representation concretely for $D_6$ and $D_{10}$.

CHAPTER 6

# The Group Algebra $k[G]$

Let $R$ be a ring (possibly non-commutative) with 1.

**Definition 6.1.** *A (left)* **module** $M$ *over a ring* $R$ *is an Abelian group* $(M, +)$ *with a map (called scalar multiplication)* $R \times M \to M$ *satisfying the following:*

(1) $(r_1 + r_2)m = r_1 m + r_2 m$ *for all* $r_1, r_2 \in R$ *and* $m \in M$*.*
(2) $r(m_1 + m_2) = rm_1 + rm_2$ *for all* $r \in R$ *and* $m_1, m_2 \in M$*.*
(3) $r_1(r_2 m) = (r_1 r_2)m$ *for all* $r_1, r_2 \in R$ *and* $m \in M$*.*
(4) $1.m = m$ *for all* $m \in M$*.*

Notice that this definition is same as definition of a vector space over a field. Analogous to definitions there we can define submodules and module homomorphisms.

**Example 6.2.** If $R = k$ (a field) or $D$ (a division ring) then the modules are nothing but vector spaces over $R$.

**Example 6.3.** Let $R$ be a PID (a commutative ring such as $\mathbb{Z}$ or polynomial ring $k[X]$ etc). Then $R \times R \ldots \times R$ and $R/I$ for an ideal $I$ are modules over $R$. The structure theory of modules over PID states that any module is a direct sum of these kinds. However over a non-PID things could be more complicated.

**Example 6.4.** In the non-commutative situation the simple/semisimple rings are studied.

A module $M$ over a ring $R$ is called **simple** if it has no proper submodules. And a module $M$ is called **semisimple** if every submodule of $M$ has a direct compliment. It is also equivalent to saying that $M$ is a direct sum of simple modules.

A ring $R$ is called **semisimple** if every module over it is semisimple. And a ring $R$ is called **simple**1 if it has no proper two-sided ideal.

**Exercise 6.5.** (1) Is $\mathbb{Z}$ a semisimple ring or simple ring?
(2) When $\mathbb{Z}/n\mathbb{Z}$ a semisimple or simple ring?
(3) Prove that the ring $M_n(D)$ where $D$ is a division ring is a simple ring and the module $D^n$ thought as one of the columns of this ring is a module over this ring.

All of the representation theory definitions can be very neatly interpreted in module theory language. Given a field $k$ and a group $G$, we form the ring

$$k[G] = \left\{ \sum_{g \in G} \alpha_g g \mid \alpha_g \in k \right\}$$

called the **group ring of** $G$. We can also define $k[G] = \{f \mid f\colon G \to k\}$. We define following operations on $k[G]$: $\left(\sum_{g\in G} \alpha_g g\right) + \left(\sum_{g\in G} \beta_g g\right) = \sum_{g\in G} (\alpha_g + \beta_g)\, g,\ \lambda \left(\sum_{g\in G} \alpha_g g\right) = \sum_{g\in G} (\lambda \alpha_g) g$ and the multiplication by

$$\left(\sum_{g\in G} \alpha_g g\right) \cdot \left(\sum_{g\in G} \beta_g g\right) = \sum_{g\in G} \left(\sum_{t\in G} \alpha_t \beta_{t^{-1}g}\right) g = \sum_{g\in G} \left(\sum_{ts=g\in G} \alpha_t \beta_s\right) g.$$

With above operations $k[G]$ is an algebra called the **group algebra** of $G$ (clearly it's a ring). A representation $(\rho, V)$ for $G$ is equivalent to taking a $k[G]$-module $V$ (see the exercises below).

**Exercise 6.6.** Prove that $k[G]$ is a ring as well as a vector space of dimension $|G|$. In fact it is a $k$-algebra.

**Exercise 6.7.** Let $k$ be a field. Let $G$ be a group. Then,

(1) $(\rho, V)$ is a representation of $G$ if and only if $V$ is a $k[G]$-module.
(2) $W$ is a $G$-invariant subspace of $V$ if and only if $W$ is a $k[G]$-submodule of $V$.
(3) The representations $V$ and $V'$ are equivalent if and only if $V$ is isomorphic to $V'$ as $k[G]$-module.
(4) $V$ is irreducible if and only if $V$ is a simple $k[G]$-module.
(5) $V$ is completely reducible if and only if $V$ is a semisimple module.

We can rewrite Maschke's Theorem and Schur's Lemma in modules language:

**Theorem 6.8** (Maschke's Theorem)**.** *Let $G$ be a finite group and $k$ a field. The ring $k[G]$ is semisimple if and only if $char(k) \nmid |G|$.*

**Proposition 6.9** (Schur's Lemma)**.** *Let $M, M'$ be two non-isomorphic simple $R$ module. Then $\mathrm{Hom}_R(M, M') = \{0\}$. Moreover, $\mathrm{Hom}_R(M, M)$ is a division ring.*

PROOF. Proof is left as an exercise. $\square$

**Exercise 6.10.** Let $D$ be a finite dimensional division algebra over $\mathbb{C}$ then $D = \mathbb{C}$.

**Exercise 6.11.** Let $R$ be a finite dimensional algebra over $\mathbb{C}$ and $M$ a simple module over $R$. Suppose $M$ is a finite dimensional over $\mathbb{C}$. Then $\mathrm{Hom}_R(M, M) \cong \mathbb{C}$.

CHAPTER 7

# Constructing New Representations

Here we will see how we can get new representations out of the known ones. All of the representations are considered over a field $k$.

## 7.1. Subrepresentation and Sum of Representations

Let $(\rho, V)$ and $(\rho', V')$ be two representations of the group $G$. We define $(\rho \oplus \rho', V \oplus V')$ by $(\rho \oplus \rho')(t)(v, v') = (t(v), t(v'))$. This is called sum of the two representations. If we have a representation $(\rho, V)$ and $W$ is a $G$-invariant subspace then we can define a subrepresentation $(\tilde{\rho}, W)$ by $\tilde{\rho}(t)(w) = \rho(t)(w)$.

## 7.2. Adjoint Representation

Let $V$ be a vector space over $k$ with a basis $\{e_1, \ldots, e_n\}$. A linear map $f\colon V \to k$ is called a **linear functional**. We denote $V^* = \mathrm{Hom}_k(V, k)$, the set of all linear functionals. We define operations on $V^*$: $(f_1 + f_2)(v) = f_1(v) + f_2(v)$ and $(\lambda f)(v) = \lambda f(v)$ and it becomes a vector space. The vector space $V^*$ is called the dual space of $V$.

**Exercise 7.1.** With the notation as above,

(1) Show that $V^*$ is a vector space with basis $e_i^*$ where $e_i^*(e_j) = \delta_{ij}$. Hence it has same dimension as $V$. After fixing a basis of $V$ we can obtain a basis this way of $V^*$ which is called **dual basis** with respect t the given one.
(2) Show that $V$ is naturally isomorphic to $V^{**}$.

Let $T\colon V \to V$ be a linear map. We define a map $T^*\colon V^* \to V^*$ by $T^*(f)(v) = f(T(v))$.

**Exercise 7.2.** Fix a basis of $V$ and consider the dual basis of $V^*$ with respect to that. Let $A = (a_{ij})$ be the matrix of $T$. Show that the matrix of $T^*$ with respect to the dual basis is ${}^tT$, the transpose matrix.

Let $\rho\colon G \to GL(V)$ be a representation. We define the adjoint representation $(\rho^*, V^*)$ as follows: $\rho^*\colon G \to GL(V^*)$ where $\rho^*(g) = \rho(g^{-1})^*$. In the matrix form if we have a repesentation $\tau\colon G \to GL_n(k)$ then $\tau^*\colon G \to GL_n(k)$ is given by $\tau^*(g) = {}^t\tau(g)^{-1}$.

**Exercise 7.3.** With the notation as above,

(1) Show that $\rho^*$ is a representation of $G$ of same dimension as $\rho$.

(2) Fix a basis and suppose $\tau$ is the matrix form of $\rho$ then show that the matrix form of $\rho^*$ is $\tau^*$.
(3) Prove that if $\rho$ is irreducible then show is $\rho^*$.

The last exercise will become easier in the case of complex representations once we define characters as we will have a simpler criterion to test when a representation is irreducible.

## 7.3. Tensor Product of two Representations

Let $V$ and $V'$ be two vector spaces over $k$. First we define tensor product of two vector spaces. Tensor product of $V$ and $V'$ is a vector space $V \otimes V' = \{\sum_{i=1}^r v_i \otimes v_i' \mid v_i \in V, v_i' \in V'\}$ with following properties:

(1) $(\sum_{i=1}^r v_i \otimes v_i') + (\sum_{i=1}^s w_i \otimes w_i') = v_1 \otimes v_1' + \cdots + v_r \otimes v_r' + w_1 \otimes w_1' + \cdots + w_s \otimes w_s'$.
(2) $(v_1 + v_2) \otimes v' = v_1 \otimes v' + v_2 \otimes v'$ and $v \otimes (v_1' + v_2') = v \otimes v_1' + v \otimes v_2'$.
(3) $\lambda (\sum_{i=1}^r v_i \otimes v_i') = \sum_{i=1}^r \lambda v_i \otimes v_i' = \sum_{i=1}^r v_i \otimes \lambda v_i'$.

Let $\{e_1, \ldots, e_n\}$ be a basis of $V$ and $\{e_1', \ldots, e_m'\}$ be that of $V'$. Then $\{e_i \otimes e_j' \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis of the vector space $V \otimes V'$ hence of dimension $nm$.
**Warning:** Elements of $V \otimes V'$ are not exactly of kind $v \otimes v'$ but they are finite sum of these ones!

Let $(\rho, V)$ and $(\rho', V')$ be two representations of the group $G$. We define $(\rho \otimes \rho', V \otimes V')$ by $(\rho \otimes \rho')(t)(\sum v \otimes v') = \sum(\rho(t)(v) \otimes \rho'(t)(v'))$.

**Exercise 7.4.** Choose a basis of $V$ and $V'$. Let $A = (a_{ij})$ be the matrix of $\rho(t)$ and $B = (b_{lm})$ be that of $\rho'(t)$. What is the matrix of $(\rho \otimes \rho')(t)$?

**Exercise 7.5.** Let $(\rho, V)$ be a representation of $G$. Let $\{e_1, \ldots, e_n\}$ be a basis of $V$.

(1) Consider an automorphism $\theta$ of $V \otimes V$ defined by $\theta(e_i \otimes e_j) = e_j \otimes e_i$. Show that $\theta^2 = \theta$.
(2) Consider the subspace $Sym^2(V) = \{z \in V \otimes V \mid \theta(z) = z\}$ and $\wedge^2(V) = \{z \in V \otimes V \mid \theta(z) = -z\}$. Prove that $V \otimes V = Sym^2(V) \oplus \wedge^2(V)$. Write down a basis of $Sym^2(V)$ and $\wedge^2(V)$ and find its dimension.
(3) Prove that $V \otimes V = Sym^2(V) \oplus \wedge^2(V)$ is a $G$-invariant decomposition.

If $(\rho, V)$ is a representation of $G$ then $V^{\otimes n} = V \otimes \cdots \otimes V, Sym^n V, \wedge^n(V)$ are also representations of $G$. This way starting from one representation we can get many representations. Though even if you start from an irreducible representation the above constructed representations need not be irreducible (for example $V \otimes V$ given above) but often they contain other irreducible representations. Writing down direct sum decompositions of tensor representaion is a important topic of study. Often it happens that we need much smaller number of representations (called **fundamental representations**) of which tensor products contain all irreducible representations.

## 7.4. Restriction of a Representation

Let $(\rho, V)$ be a representation of the group $G$. Let $H$ be a subgroup. Then $(\rho, V)$ is a representation of $H$ also denoted as $(\rho_H, V)$. Let $N$ be a normal subgroup of $G$. Then any representation of $G/N$ gives rise to a representation of $G$. Moreover if the representation of $G/N$ is irreducible then the representation of $G$ remains irreducible.

CHAPTER 8

# Matrix Elements

Now onwards we assume the field $k = \mathbb{C}$. We also denote $\mathbb{S}_1 = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$. Let $G$ be a finite group. Then $\mathbb{C}[G] = \{f \mid f\colon G \to \mathbb{C}\}$ is a vector space of dimension $|G|$. Let $f_1, f_2$ be two functions from $G$ to $\mathbb{C}$, i.e., $f_1, f_2 \in \mathbb{C}[G]$. We define a map $(,)\colon \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}$ as follows,

$$(f_1, f_2) = \frac{1}{|G|} \sum_{t \in G} f_1(t) f_2(t^{-1}).$$

Note that $(f_1, f_2) = (f_2, f_1)$.

Let $\rho\colon G \to GL(V)$ be a representation. We can choose a basis and get a map in matrix form $\rho\colon G \to GL_n(\mathbb{C})$ where $n$ is the dimesnion of the representation. This means we have,

$$g \mapsto \begin{bmatrix} a_{11}(g) & a_{12}(g) & \cdots & a_{1n}(g) \\ a_{21}(g) & a_{22}(g) & \cdots & a_{2n}(g) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(g) & a_{n2}(g) & \cdots & a_{nn}(g) \end{bmatrix}$$

where $a_{ij}\colon G \to \mathbb{C}$, i.e, $a_{ij} \in \mathbb{C}[G]$. The maps $a_{ij}$'s are called **matrix elements** of $\rho$. Thus to a representation $\rho$ we can associate a subspace $\mathcal{W}$ of $\mathbb{C}[G]$ spanned by $a_{ij}$. In what follows now we will explore relations between these subspaces $\mathcal{W}$ associated to irreducible representations of a finite group $G$. Let $\rho_1, \ldots, \rho_r, \cdots$ be irreducible representations of $G$ of dimension $n_1, \cdots, n_r, \cdots$ respectively. We don't know yet whether there are finitely many irreducible representations which we will prove later. Let $\mathcal{W}_1, \cdots, \mathcal{W}_r, \cdots$ be associated subspaces of $\mathbb{C}[G]$ to the irreducible representations.

**Theorem 8.1.** *Let $(\rho, V)$ and $(\rho', V')$ be two irreducible representations of $G$ of dimension $n$ and $n'$ respectively. Let $a_{ij}$ and $b_{ij}$ be the corresponding matrix elements with respect to fixed basis of $V$ and $V'$. Then,*

(1) $(a_{il}, b_{mj}) = 0$ *for all* $i, j, l, m$.

(2) $(a_{il}, a_{mj}) = \frac{1}{n}\delta_{ij}\delta_{lm} = \begin{cases} \frac{1}{n} & \textit{if } i = j \textit{ and } l = m \\ 0 & \textit{otherwise} \end{cases}$.

Proof. Let $T\colon V \to V'$ be a linear map. Define

$$T^0 = \frac{1}{|G|} \sum_{t \in G} \rho(t) T (\rho'(t))^{-1}.$$

Then $T^0$ is a $G$-linear map. Using Lemma 4.3 we get $T^0 = 0$. Let us denote the matrix of $T$ by $x_{lm}$. Then $ij$th entry of $T^0$ is zero for all $i$ and $j$, i.e.,

$$\frac{1}{|G|}\sum_{t\in G}\sum_{l,m} a_{il}(t)x_{lm}b_{mj}(t^{-1}) = 0.$$

Since $T$ is arbitrary linear transformation the entries $x_{lm}$ are arbitrary complex number hence can be treated as indeterminate. Hence coefficients of $x_{lm}$ are 0. This gives $\frac{1}{|G|}\sum_{t\in G} a_{il}(t)b_{mj}(t^{-1}) = 0$ hence $(a_{il}, b_{mj}) = 0$ for all $i, j, l, m$.

Now let us consider $T\colon V \to V$, a linear map. Again define $T^0 = \frac{1}{|G|}\sum_{t\in G}\rho(t)T(\rho(t))^{-1}$ which is a $G$-map. From Corollary 4.5 we get that $T^0 = \lambda.Id$ where $\lambda = \frac{1}{n}tr(T) = \frac{1}{n}\sum_l x_{ll} = \frac{1}{n}\sum_{l,m} x_{lm}\delta_{lm}$ since $n.\lambda = tr(T^0) = \frac{1}{|G|}\sum_{t\in G} tr(T) = tr(T)$. Now using matrix elements we can write $ij$th term of $T^0$:

$$\frac{1}{|G|}\sum_{t\in G}\sum_{lm} a_{il}(t)x_{lm}a_{mj}(t^{-1}) = \lambda\delta_{ij} = \frac{1}{n}\sum_{l,m} x_{lm}\delta_{lm}\delta_{ij}.$$

Again $T$ is an arbitrary linear map so its matrix elements $x_{lm}$ can be treated as indeterminate. Comparing coefficients of $x_{lm}$ we get:

$$\frac{1}{|G|}\sum_{t\in G} a_{il}(t)a_{mj}(t^{-1}) = \frac{1}{n}\delta_{lm}\delta_{ij}.$$

Which gives $(a_{il}, a_{mj}) = \frac{1}{n}\delta_{lm}\delta_{ij}$. □

**Exercise 8.2.** Prove that, in both cases, $T^0$ is a $G$-map.

**Corollary 8.3.** *If $f \in \mathcal{W}_i$ and $f' \in \mathcal{W}_j$ with $i \neq j$ then $(f, f') = 0$, i.e., $(\mathcal{W}_i, \mathcal{W}_j) = 0$.*

CHAPTER 9

# Character Theory

We have $\mathbb{C}[G]$, space of all complex valued functions on $G$ which is a vector space of dimension $|G|$. We define an **inner product** $\langle , \rangle \colon \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}$ by

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{t \in G} f_1(t)\overline{f_2(t)}.$$

**Exercise 9.1.** With the notation as above,

(1) Prove that $\langle , \rangle$ is an inner product on $\mathbb{C}[G]$.
(2) If $f_1$ and $f_2$ take value in $\mathbb{S}_1 \subset \mathbb{C}$ then $\langle f_1, f_2 \rangle = (f_1, f_2)$.

**Definition 9.2** (Character)**.** *Let $(\rho, V)$ be a representation of $G$. The* **character** *of (corresponding to) representation $\rho$ is a map $\chi \colon G \to \mathbb{C}$ defined by $\chi(t) = tr(\rho(t))$ where $tr$ is the trace of corresponding matrix.*

Strictly speaking it is $\chi_\rho$ but for the simplicity of notation we write $\chi$ only when it is clear which representation it corresponds to.

**Exercise 9.3.** Let $A, B \in GL_n(k)$. Prove the following:

(1) $tr(AB) = tr(BA)$.
(2) $tr(A) = tr(BAB^{-1})$.

**Exercise 9.4.** Usually we define trace of a matrix. Show that in above definition character is a well defined function. That is, prove that $\chi(t)$ doesn't change if we choose different basis and calculate trace of $\rho(t)$. This is another way to say that trace is invariant of conjugacy classes of $GL_n(k)$.

**Exercise 9.5.** If $\rho$ and $\rho'$ are two isomorphic representations, i.e., $G$-equivalent, then the corresponding characters are same. The converse of this statement is also true which we will prove later.

**Proposition 9.6.** *If $\rho$ is a representation of dimension $n$ and $\chi$ is the corresponding character then,*

(1) *$\chi(1) = n$ is the dimension of the representation.*
(2) *$\chi(t^{-1}) = \overline{\chi(t)}$ for all $t \in G$ where bar denotes complex conjugation.*
(3) *$\chi(tst^{-1}) = \chi(s)$ for all $t, s \in G$, i.e., character is constant on the conjugacy classes of $G$.*

(4) *For* $f \in \mathbb{C}[G]$ *we have* $(f, \chi) = \langle f, \chi \rangle$.

PROOF. For the proof of part two we use Corollary 5.4 to calculate $\chi(t)$. From that corollary every $\rho(t)$ can be diagonalised (at a time not simultaneously which is enough for our purposes), say diag$\{\omega_1, \cdots, \omega_n\}$. Since $t$ is of finite order, say $d$, we have $\rho(t)^d = 1$. That is each $\omega_j^d = 1$ means the diagonal elements are $d$th root of unity. We know that roots of unity satisfy $\omega_j^{-1} = \overline{\omega_j}$. Hence $\chi(t^{-1}) = tr(\rho(t)^{-1}) = \omega_1^{-1} + \cdots + \omega_n^{-1} = \overline{\omega_1} + \cdots + \overline{\omega_n} = \overline{\chi(t)}$.

We can also prove this result by upper triangulation theorem, i.e., every matrix $\rho(t)$ can be conjugated to an upper triangular matrix. And the fact that trace is same for the conjugate matrices. □

**Definition 9.7** (Class Function). *A function* $f : G \to \mathbb{C}$ *is called a* **class function** *if* $f$ *is constant on the conjugacy classes of* $G$. *We denote the set of class functions on* $G$ *by* $\mathcal{H}$.

**Exercise 9.8.** With the notation as above,

(1) Prove that $\mathcal{H}$ is a subspace of $\mathbb{C}[G]$.
(2) The dimension of $\mathcal{H}$ is the number of conjugacy classes of $G$.
(3) Let $c_g = \sum_{x \in G} xgx^{-1} \in \mathbb{C}[G]$. The center of the group algebra $\mathbb{C}[G]$ is spanned by $c_g$.
(4) Let $\chi$ be a character corresponding to some representation of $G$. Then, $\chi \in \mathcal{H}$.

**Proposition 9.9.** *Let* $(\rho, V)$ *and* $(\rho', V')$ *be two representations of the group* $G$ *and* $\chi, \chi'$ *be the corresponding characters. Then,*

(1) *The character of the sum of two representations is equal to the sum of characters, i.e.,* $\chi_{\rho \oplus \rho'} = \chi + \chi'$.
(2) *The character of the tensor product of two representations is the product of two characters, i.e.,* $\chi_{\rho \otimes \rho'} = \chi \chi'$.

PROOF. Proof is a simple exercise involving matrices. □

This way we can define sum and product of characters which is again a character.

**Exercise 9.10.** Let $(\rho, V)$ be a representation of the group $G$. Then $V \otimes V$ is also a representation of $G$. We look at the map $\theta : V \otimes V \longrightarrow V \otimes V$ defined by $\theta(e_i \otimes e_j) = e_j \otimes e_i$. This gives rise to the decomposition $V \otimes V = Sym^2(V) \oplus \wedge^2(V)$ which is a $G$-decomposition. Calculate the characters of $Sym^2(V)$ and $\wedge^2(V)$.

**Exercise 9.11.** Let $\chi$ be the character of a representation $\rho$. Show that the character of the adjoint representation $\rho^*$ is given by $\overline{\chi}$.

CHAPTER 10

# Orthogonality Relations

Let $G$ be a finite group. Let $W_1, W_2, \ldots, W_h, \ldots$ be irreducible representations of $G$ of dimension $n_1, n_2, \ldots, n_h, \ldots$ over $\mathbb{C}$. Let $\chi_1, \chi_2, \ldots, \chi_h, \ldots$ are corresponding characters, called **irreducible characters of** $G$. We will fix this notation now onwards. We will prove that the number of irreducible characters and hence the number of irreducible representations if finite and equal to the number of conjugacy classes.

In the last chapter we introduced an inner product $\langle , \rangle$ on $\mathbb{C}[G]$. We also observed that character of any representation belongs to $\mathcal{H}$, the space of class functions.

**Theorem 10.1.** *The set of irreducible characters* $\{\chi_1, \chi_2, \ldots\}$ *form an orthonormal set of* $(\mathbb{C}[G], \langle , \rangle)$. *That is,*

(1) *If* $\chi$ *is a character of an irreducible representation then* $\langle \chi, \chi \rangle = 1$.
(2) *If* $\chi$ *and* $\chi'$ *are two irreducible characters of non-isomorphic representations then* $\langle \chi, \chi' \rangle = 0$.

Proof. Let $\rho$ and $\rho'$ be non-isomorphic irreducible representations and $(a_{ij})$ and $(b_{ij})$ be the corresponding matrix elements. Then $\chi(g) = \sum_i a_{ii}(g)$ and $\chi'(g) = \sum_j b_{jj}(g)$. Then $\langle \chi, \chi' \rangle = (\chi, \chi') = (\sum_i a_{ii}, \sum_j b_{jj}) = \sum_{i,j}(a_{ii}, b_{jj}) = 0$ from Theorem 8.1 and Proposition 9.6 part 4. Using the similar argument we get $\langle \chi, \chi \rangle = (\chi, \chi) = (\sum_i a_{ii}, \sum_j b_{jj}) = \sum_i (a_{ii}, b_{ii}) = \sum_{i=1}^{n} \frac{1}{n} = 1$ where $n$ is the dimension of the representation $\rho$. □

**Corollary 10.2.** *The number of irreducible characters are finite.*

Proof. Since the irreducible characters form an orthonormal set they are linearly independent. Hence their number has to be less than the dimension of $\mathbb{C}[G]$ which is $|G|$, hence finite. □

Once we prove that two representations are isomorphic if and only if their characters are same this corollary will also give that there are finitely many non-isomorphic irreducible representations.

We are going to use above results to analyse general representation of $G$ and identify its irreducible components.

**Theorem 10.3.** *Let* $(\rho, V)$ *be a representation of* $G$ *with character* $\chi$. *Let* $V$ *decomposes into a direct sum of irreducible representations:*

$$V = V_1 \oplus \cdots \oplus V_m.$$

*Then the number of $V_i$ isomorphic to $W_j$ (a fixed irreducible representation) is equal to the scalar product $\langle \chi, \chi_j \rangle$.*

PROOF. Let $\phi_1, \ldots, \phi_m$ be the characters of $V_1, \ldots, V_m$. Then $\chi = \phi_1 + \cdots + \phi_m$. Also $\langle \chi, \chi_j \rangle = \sum_i \langle \phi_i, \chi_j \rangle = \sum_{\phi_i = \chi_j} \langle \phi_i, \chi_j \rangle =$ the number of $V_i$ isomorphic to $W_j$. □

**Corollary 10.4.** *With the notation as above,*

(1) *The number of $V_i$ isomorphic to a fixed $W_j$ does not depend on the chosen decomposition.*
(2) *Let $(\rho, V)$ and $(\rho', V')$ be two representations with characters $\chi$ and $\chi'$ respectively. Then $V \cong V'$ if and only if $\chi = \chi'$.*

PROOF. The proof of part 1 is clear from the theorem above. For the proof of part 2 it is clear that if $V \cong V'$ we get $\chi = \chi'$. Now suppose $\chi = \chi'$. Let $V \cong W_1^{n_1} \oplus \cdots \oplus W_h^{n_h}$ and $V' \cong W_1^{m_1} \oplus \cdots \oplus W_h^{m_h}$ be the decomposition as direct sum of irreducible representations (we can do this using Maschke's Theorem) where $n_i, m_j \geq 0$. Suppose $\chi_1, \chi_2, \ldots, \chi_h$ be the irreducible characters of $W_1, \ldots, W_h$. Then $\chi = n_1\chi_1 + \cdots + n_h\chi_h$ and $\chi' = m_1\chi_1 + \cdots + m_h\chi_h$. However as $\chi_i$'s form an orthonormal set they are linearly independent. Hence $\chi = \chi'$ implies $n_i = m_i$ for all $i$. Hence $V \cong V'$. □

From this corollary it follows that the number of irreducible representations are same as the number of irreducible characters which is less then or equal to $|G|$. In fact later we will prove that this number is equal to the number of conjugacy classes. The above analysis also helps to identify whether a representation is irreducible by use of the following:

**Theorem 10.5** (Irreducibility Criteria)**.** *Let $\chi$ be the character of a representation $(\rho, V)$. Then $\langle \chi, \chi \rangle$ is a positive integer and $\langle \chi, \chi \rangle = 1$ if and only if $V$ is irreducible.*

PROOF. Let $V \cong W_1^{n_1} \oplus \cdots \oplus W_h^{n_h}$. Then $\chi = n_1\chi_1 + \cdots + n_h\chi_h$ and

$$\langle \chi, \chi \rangle = \langle n_1\chi_1 + \cdots + n_h\chi_h, n_1\chi_1 + \cdots + n_h\chi_h \rangle = \sum_i n_i^2.$$

Hence $\langle \chi, \chi \rangle = 1$ if and only if one of the $n_i = 1$, i.e, $\chi = \chi_i$ for some $i$. Hence the result. □

**Exercise 10.6.** Let $\chi$ be an irreducible character. Show that $\overline{\chi}$ is so. Hence a representation $\rho$ is irreducible if and only if $\rho^*$ is so.

**Exercise 10.7.** Use the above criteria to show that the "Permutations representation", "Regular Representation" (defined in the second chater) are not irreducible if $|G| > 1$.

**Exercise 10.8.** Let $\rho$ be an irreducible representation and $\tau$ be an 1-dimensional representation. Show that $\rho \otimes \tau$ is irreducible.

CHAPTER 11

# Main Theorem of Character Theory

## 11.1. Regular Representation

Let $G$ be a finite group and $\chi_1, \dots, \chi_h$ be the irreducible characters of dimension $n_1, \dots, n_h$ respectively. Let $L$ be the left regular representation of $G$ with corresponding character $l$.

**Exercise 11.1.** The character $l$ of the regular representation is given by $l(1) = |G|$ and $l(t) = 0$ for all $1 \neq t \in G$.

**Theorem 11.2.** *Every irreducible representation $W_i$ of $G$ is contained in the regular representation with multiplicity equal to the dimension of $W_i$ which is $n_i$. Hence $l = n_1\chi_1 + \dots + n_h\chi_h$.*

Proof. In the view of Theorem 10.3 the number of times $\chi_i$ is contained in the regular representation is given by $\langle l, \chi_i\rangle = \frac{1}{|G|}l(1)\chi_i(1) = \frac{1}{|G|}|G|n_i = n_i$. □

If we are asked to construct irreducible representations of a finite group we don't know where to look for them. This theorem ensures a natural place, namely the regular representation where we could find them. For this reason it is also called "God Given Representation".

**Exercise 11.3.** With the notation as above,

(1) The degree $n_i$ satisfy $\sum_{i=1}^{h} n_i^2 = |G|$.
(2) If $1 \neq s \in G$ we have $\sum_{i=1}^{h} n_i\chi_i(s) = 0$.

Hints: This follows from the formula for $l$ as in the theorem by evaluating at $s = 1$ and $s \neq 1$.

These relations among characters will be useful to determine character table of the group $G$.

## 11.2. The Number of Irreducible Representations

Now we will prove the main theorem of the character theory.

**Theorem 11.4** (Main Theorem). *The number of irreducible representations of $G$ (upto isomorphism) is equal to the number of conjugacy classes of $G$.*

The proof of this theorem will follow from the following proposition. We know that the irreducible characters $\chi_1, \ldots, \chi_h \in \mathcal{H}$ and they form an orthonormal set (see Theorem 10.1). We prove that, in fact, they form an orthonormal basis of $\mathcal{H}$ and generate as an algebra whole of $\mathbb{C}[G]$.

**Proposition 11.5.** *The irreducible characters of $G$ form an orthonormal basis of $\mathcal{H}$, the space of class functions.*

To prove this we will make use of the following:

**Lemma 11.6.** *Let $f \in \mathcal{H}$ be a class function on $G$. Let $(\rho, V)$ be an irreducible representation of $G$ of degree $n$ with character $\chi$. Let us define $\rho_f = \sum_{t \in G} f(t)\rho(t) \in \mathrm{End}(V)$. Then, $\rho_f = \lambda.Id$ where $\lambda = \frac{|G|}{n}\langle f, \overline{\chi}\rangle$.*

Proof. We claim that $\rho_f$ is a $G$-map and use Schur's Lemma to prove the result. For any $g \in G$ we have,

$$\rho(g)\rho_f\rho(g^{-1}) = \sum_{t \in G} f(t)\rho(g)\rho(t)\rho(g^{-1}) = \sum_{t \in G} f(t)\rho(gtg^{-1}) = \sum_{s \in G} f(g^{-1}sg)\rho(s) = \rho_f.$$

Hence $\rho_f$ is a $G$-map. From Schur's Lemma (see 4.5) we get that $\rho_f = \lambda.Id$ for some $\lambda \in \mathbb{C}$. Now we calculate trace of both side:

$$\lambda.n = tr(\rho_f) = \sum_{t \in G} f(t)tr(\rho(t)) = \sum_{t \in G} f(t)\chi(t) = |G|\frac{1}{|G|}\sum_{t \in G} f(t)\overline{\chi(t^{-1})} = |G|\langle f, \overline{\chi}\rangle.$$

Hence we get $\lambda = \frac{|G|}{n}\langle f, \overline{\chi}\rangle$. □

**Proof of the Proposition 11.5.** We need to prove that irreducible characters span $\mathcal{H}$ as being orthonormal they are already linearly independent. Let $f \in \mathcal{H}$. Suppose $f$ is orthogonal to each irreducible $\chi_i$, i.e., $\langle f, \chi_i\rangle = 0$ for all $i$. Then we will prove $f = 0$.

Since $\langle f, \chi_i\rangle = 0$ it implies $\langle f, \overline{\chi_i}\rangle = 0$ for all $i$. Let $\rho_i$ be the corresponding irreducible representation. Then from previous lemma ${\rho_i}_f = \left(\frac{|G|}{n}\langle f, \overline{\chi_i}\rangle\right).Id = 0$ for all $i$. Now let $\rho$ be any representation of $G$. From Maschke's Theorem it is direct sum of $\rho_i$'s, the irreducible ones. Hence $\rho_f = 0$ for any $\rho$.

In particular we can take the regular representation $L\colon G \to GL(\mathbb{C}[G])$ for $\rho$ and we get $L_f = 0$. Hence $L_f(e_1) = 0$ implies $\sum_{t \in G} f(t)L(t)(e_1) = 0$, i.e., $\sum_{t \in G} f(t)e_t = 0$ hence $f(t) = 0$ for all $t \in G$. Hence $f = 0$. □

**Proposition 11.7.** *Let $s \in G$ and let $r_s$ be the number of elements in the conjugacy class of $s$. Let $\{\chi_1, \ldots, \chi_h\}$ be irreducible representations of $G$. Then,*

(1) *We have $\sum_{i=1}^h \chi_i(s)\overline{\chi_i(s)} = \frac{|G|}{r_s}$.*
(2) *For $t \in G$ not conjugate to $s$, we have $\sum_{i=1}^h \chi_i(s)\overline{\chi_i(t)} = 0$.*

PROOF. Let us define a class function $f_t$ by $f_t(t) = 1$ and $f_t(g) = 0$ if $g$ is not conjugate to $t$. Since irreducible characters span $\mathcal{H}$ (see 11.5) we can write $f_t = \sum_{i=1}^{h} \lambda_i \chi_i$ where $\lambda_i = \langle f_t, \chi_i \rangle = \frac{1}{|G|} r_t \overline{\chi_i(t)}$. Hence $f_t(s) = \frac{r_t}{|G|} \sum_{i=1}^{h} \overline{\chi_i(t)} \chi_i(s)$ for any $s \in G$. This gives the required result by taking $s$ conjugate to $t$ and not conjugate to $t$. □

**Exercise 11.8.** Let $G$ be a finite group. The elements $s, t \in G$ are conjugate if and only if $\chi(s) = \chi(t)$ for all (irreducible) characters $\chi$ of $G$.

**Exercise 11.9.** A normal subgroup of $G$ is disjoint union of conjugacy classes.

**Exercise 11.10.** Any normal subgroup can be obtained by looking at the intersection of kernel ($\{g \in G \mid \chi(g) = 1\}$) of some characters.

CHAPTER 12

# Examples

Let $G$ be a finite group. Let $\rho_1, \ldots, \rho_h$ be irreducible representations of $G$ over $\mathbb{C}$ with corresponding characters $\chi_1, \ldots, \chi_h$. We know that the number $h$ is equal to the number of conjugacy classes in $G$. We can make use of this information about $G$ and make a **character table** of $G$. The character table is a matrix of size $h \times h$ of which rows are labelled as characters and columns as conjugacy classes.

| | $r_1 = 1$<br>$g_1 = e$ | $r_2$<br>$g_2$ | $\cdots$<br>$\cdots$ | $r_h$<br>$g_h$ |
|---|---|---|---|---|
| $\chi_1$ | $n_1 = 1$ | 1 | $\cdots$ | 1 |
| $\chi_2$ | $n_2$ | | | |
| $\vdots$ | $\vdots$ | | | |
| $\chi_h$ | $n_h$ | | | |

where $g_1, g_2, \ldots$ denote representative of the conjugacy class and $r_i$ denotes the number of elements in the conjugacy class of $g_i$. The following proposition summarizes the results proved about characters. We also recall the inner product on $\mathbb{C}[G]$ defined by $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{t \in G} f_1(t)\overline{f_2(t)}$.

**Proposition 12.1.** *With the notation as above we have,*

1. *The number of conjugacy classes is same as the number of irreducible characters which is same as the number of non-isomorphic irreducible representations.*
2. *Two representations are isomorphic if and only if their characters are equal.*
3. *A representation $\rho$ with character $\chi$ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*
4. $|G| = n_1^2 + n_2^2 + \cdots + n_h^2$ *where $n_1 = 1$.*
5. *The characters form orthonormal basis of $\mathcal{H}$, i.e.,*

$$\sum_{t \in G} \chi_i(t)\overline{\chi_j(t)} = \sum_{g_l} r_l \chi_i(g_l)\overline{\chi_j(g_l)} = \delta_{ij}|G|.$$

*That is, the rows of the charcater table are orthonormal.*

(6) *The columns of the character table also form an orthogonal set, i.e.,*

$$\sum_{i=1}^{h} \chi_i(g_l)\overline{\chi_i(g_l)} = \frac{|G|}{r_l}$$

*and*

$$\sum_{i=1}^{h} \chi_i(g_l)\overline{\chi_i(g_m)} = 0$$

*where $g_l$ and $g_m$ are representative of different conjugacy class.*

(7) *The character table matrix is an invertible matrix.*

(8) *The degree of irreducible representations divide the order of th group, i.e., $n_i \mid |G|$.*

The proof of the last statement will be done later.

**Warning :** If character table of two groups are same that does not imply that the groups are isomorphic. Look at the character tables of $Q_8$ and $D_4$. In fact, all non-Abelian groups of order $p^3$ have same character table (find a reference for this).

## 12.1. Groups having Large Abelian Subgroups

First we will give another proof of the Theorem 5.10 using Character Theory.

**Theorem 12.2.** *Let $G$ be a finite group. Then $G$ is Abelian if and only if all irreducible representations are of dimension 1, i.e., $n_i = 1$ for all $i$.*

PROOF. With the notation as above let $G$ be Abelian. We have

$$|G| = n_1^2 + \cdots + n_h^2$$

where $h = |G|$. Hence the only solution to the equation is $n_i = 1$ for all $i$. Now suppose $n_i = 1$ for all $i$. Then the above equation implies $h = |G|$. Hence each conjugacy class is has size 1 and the group is Abelian. □

**Proposition 12.3.** *Let $G$ be a group and $A$ be an Abelian subgroup. Then $n_i \leq \frac{|G|}{|A|}$ for all $i$.*

PROOF. Let $\rho\colon G \to GL(V)$ be an irreducible representation. We can restrict $\rho$ to $A$ and denote it by $\rho_A\colon A \to GL(V)$ which may not be irreducible. Let $W$ an invariant subspace of $V$ for $A$. The above theorem implies $\dim(W) = 1$. Say $W =< v >$ where $v \neq 0$. Consider $V' =< \{\rho(g)v \mid g \in G\} >\subset V$. Clearly $V'$ is $G$-invariant and as $V$ is irreducible $V' = V$. Notice that $\rho(ga)v = \lambda\rho(g)v$, i.e., $\rho(ga)v$ and $\rho(g)v$ are linearly dependent. Hence $V' =< \{\rho(g_1)v, \ldots, \rho(g_m)v\} >$ where $g_i$ are representatives of the coset $g_iA$ in $G/A$. This implies $\dim(V) \leq m = \frac{|G|}{|A|}$. □

**Corollary 12.4.** *Let $G = D_n$ be the dihedral group with $2n$ elements. Any irreducible representation of $D_n$ has dimension 1 or 2.*

### 12.2. Character Table of Some groups

**Example 12.5** (Cyclic Group)**.** Let $G = \mathbb{Z}/n\mathbb{Z}$. All representations are one dimensional hence give character. The characters are $\chi_1, \dots, \chi_n$ given by $\chi_r(s) = e^{\frac{2\pi i}{n} rs}$ for $0 \leq s \leq n$.

**Example 12.6** ($S_3$)**.**

$$S_3 = \{1, (12), (23), (13), (123), (132)\}$$

We already know the one dimensional representations of $S_3$ which are trivial representation and the sign representation. The characters for these representations are itself. Now we use the formula $6 = |G| = n_1^2 + n_2^2 + n_3^2 = 1 + 1 + n_3^2$ gives $n_3 = 2$. Now we use $\langle \chi_3, \chi_1 \rangle = \frac{1}{6}\{1.2.1 + 3.a.1 + 2.b.1\} = 0$ and $\langle \chi_3, \chi_2 \rangle = \frac{1}{6}\{1.2.1 + 3.a.(-1) + 2.b.1 = 0\}$. Hence solving the above equations $3a + 2b = -2$ and $-3a + 2b = -2$ gives $a = 0$ and $b = -1$.

| | $r_1 = 1$<br>$g_1 = 1$ | $r_2 = 3$<br>$g_2 = (12)$ | $r_3 = 2$<br>$g_3 = (123)$ |
|---|---|---|---|
| $\chi_1$ | $n_1 = 1$ | 1 | 1 |
| $\chi_2$ | $n_2 = 1$ | $-1$ | 1 |
| $\chi_3$ | $n_3 = 2$ | $a = 0$ | $b = -1$ |

**Example 12.7** ($Q_8$)**.**

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

The commutator subgroup of $Q_8 = \{1, -1\}$ and $Q_8/\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence it has 4 one dimensional representations which are lifted from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Notice that the first $\mathbb{Z}/2\mathbb{Z}$ component is the image of $i$ and second one of $j$. Now we use $8 = |G| = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 1 + 1 + 1 + 1 + n_5^2$ gives $n_5 = 2$. Again using orthogonality of $\chi_5$ with other known $\chi_i$'s we get following equations:

$$\begin{aligned}
1.2.1 + 1.a.1 + 2.b.1 + 2.c.1 + 2.d.1 &= 0 \\
1.2.1 + 1.a.1 + 2.b.1 + 2.c.(-1) + 2.d.(-1) &= 0 \\
1.2.1 + 1.a.1 + 2.b.(-1) + 2.c.1 + 2.d.(-1) &= 0 \\
1.2.1 + 1.a.1 + 2.b.(-1) + 2.c.(-1) + 2.d.1 &= 0
\end{aligned}$$

This gives the solution $a = -2, b = 0, c = 0$ and $d = 0$.

| | $r_1 = 1$ <br> $g_1 = 1$ | $r_2 = 1$ <br> $g_2 = -1$ | $r_3 = 2$ <br> $g_3 = i$ | $r_4 = 2$ <br> $g_4 = j$ | $r_5 = 2$ <br> $g_5 = k$ |
|---|---|---|---|---|---|
| $\chi_1$ | $n_1 = 1$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | $n_2 = 1$ | 1 | 1 | $-1$ | $-1$ |
| $\chi_3$ | $n_3 = 1$ | 1 | $-1$ | 1 | $-1$ |
| $\chi_4$ | $n_4 = 1$ | 1 | $-1$ | $-1$ | 1 |
| $\chi_5$ | $n_5 = 2$ | $a = -2$ | $b = 0$ | $c = 0$ | $d = 0$ |

**Example 12.8** ($D_4$).

$$D_4 = \{r, s \mid r^4 = 1 = s^2, rs = sr^{-1}\}$$

The commutator subgroup of $D_4$ is $\{1, r^2\} = \mathcal{Z}(D_4)$. And $D_4/\mathcal{Z}(D_4) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ hence there are 4 one dimensional representations as in the case of $Q_8$. We can also compute rest of it as we did in $Q_8$. We also observe that the character table is same as $Q_8$.

| | $r_1 = 1$ <br> $g_1 = 1$ | $r_2 = 1$ <br> $g_2 = r^2$ | $r_3 = 2$ <br> $g_3 = r$ | $r_4 = 2$ <br> $g_4 = s$ | $r_5 = 2$ <br> $g_5 = sr$ |
|---|---|---|---|---|---|
| $\chi_1$ | $n_1 = 1$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | $n_2 = 1$ | 1 | 1 | $-1$ | $-1$ |
| $\chi_3$ | $n_3 = 1$ | 1 | $-1$ | 1 | $-1$ |
| $\chi_4$ | $n_4 = 1$ | 1 | $-1$ | $-1$ | 1 |
| $\chi_5$ | $n_5 = 2$ | $a = -2$ | $b = 0$ | $c = 0$ | $d = 0$ |

**Example 12.9** ($A_4$).

$$A_4 = \{1, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$$

We note that the 3 cycles are not conjugate to their inverses. Here we have

$$H = \{1, (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

a normal subgroup of $A_4$ and $A_4/H \cong \mathbb{Z}/3\mathbb{Z}$. This way the 3 one dimensional irreducible representations of $\mathbb{Z}/3\mathbb{Z}$ lift to $A_4$ as we have maps $A_4 \to A_4/H \cong \mathbb{Z}/3\mathbb{Z} \to GL_1(\mathbb{C})$ given by $\omega$ where $\omega^3 = 1$. We use $1^2 + 1^2 + 1^2 + n_4^2 = 12 = |A_4|$ to get $n_4 = 3$. Now we take inner product of $\chi_4$ with others and get the equations:

$$\begin{aligned} 1.3.1 + 3.a.1 + 4.b.1 + 4.c.1 &= 0 \\ 1.3.1 + 3.a.1 + 4.b.\overline{\omega} + 4.c.\overline{\omega}^2 &= 0 \\ 1.3.1 + 3.a.1 + 4.b.\overline{\omega}^2 + 4.c.\overline{\omega} &= 0 \end{aligned}$$

This gives us the character table.

| | $r_1 = 1$<br>$g_1 = 1$ | $r_2 = 3$<br>$g_2 = (12)(34)$ | $r_3 = 4$<br>$g_3 = (123)$ | $r_4 = 4$<br>$g_4 = (132)$ |
|---|---|---|---|---|
| $\chi_1$ | $n_1 = 1$ | 1 | 1 | 1 |
| $\chi_2$ | $n_2 = 1$ | 1 | $\omega$ | $\omega^2$ |
| $\chi_3$ | $n_3 = 1$ | 1 | $\omega^2$ | $\omega$ |
| $\chi_4$ | $n_4 = 3$ | $a = -1$ | $b = 0$ | $c = 0$ |

**Example 12.10** ($S_4$)**.** The group $S_4$ has 2 one dimensional representations given by trivial and sign. We recall that the permutation representation (Example 2.15) gives rise to the subspace $V = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n \mid x_1 + x_2 + \cdots + x_n = 0\}$ which is an $n-1$ dimensional irreducible representation of $S_n$. We will make use of this to get a 3 dimensional representation of $S_4$ and corresponding character $\chi_3$. A basis of the space $V$ is $\{e_1 - e_2, e_2 - e_3, e_3 - e_4\}$ and the action is given by:

$$\begin{aligned}(12): \qquad (e_1 - e_2) &\mapsto e_2 - e_1 = -(e_1 - e_2)\\ (e_2 - e_3) &\mapsto e_1 - e_3 = (e_1 - e_2) + (e_2 - e_3)\\ (e_3 - e_4) &\mapsto (e_3 - e_4)\end{aligned}$$

So the matrix is $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\chi_3((12)) = 1$. The action of $(12)(34)$ is $e_1 - e_2 \mapsto e_2 - e_1 = -(e_1 - e_2)$, $e_2 - e_3 \mapsto e_1 - e_4 = (e_1 - e_2) + (e_2 - e_3) + (e_3 - e_4)$ and $e_3 - e_4 \mapsto e_4 - e_3 = -(e_3 - e_4)$ and the matrix is $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$. So $\chi_3((12)(34)) = -1$. The action of $(123)$ is $e_1 - e_2 \mapsto e_2 - e_3$, $e_2 - e_3 \mapsto e_3 - e_1 = -(e_1 - e_2) - (e_2 - e_3)$ and $e_3 - e_4 \mapsto e_1 - e_4 = (e_1 - e_2) + (e_2 - e_3) + (e_3 - e_4)$. So the matrix is $\begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\chi_3((123)) = 0$. And the action of $(1234)$ is $e_1 - e_2 \mapsto e_2 - e_3$, $e_2 - e_3 \mapsto e_3 - e_4$ and $e_3 - e_4 \mapsto e_4 - e_1 = -(e_1 - e_2) - (e_2 - e_3) - (e_3 - e_4)$. So the matrix is $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ and $\chi_3((1234)) = -1$. This gives $\chi_3$. We can get another character $\chi_4 = \chi_3 . \chi_2$ corresponding to the representation $Perm \otimes sgn$. We can check that this is different from others so corresponds to a new representation and also irreducible as $\langle \chi_4, \chi_4 \rangle = 1$.

To find $\chi_5$ we can use orthogonality relations and get equations.

| | $r_1 = 1$ <br> $g_1 = 1$ | $r_2 = 6$ <br> $g_2 = (12)$ | $r_3 = 3$ <br> $g_3 = (12)(34)$ | $r_4 = 8$ <br> $g_4 = (123)$ | $r_5 = 6$ <br> $g_5 = (1234)$ |
|---|---|---|---|---|---|
| $\chi_1$ | $n_1 = 1$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | $n_2 = 1$ | $-1$ | 1 | 1 | $-1$ |
| $\chi_3$ | $n_3 = 3$ | 1 | $-1$ | 0 | $-1$ |
| $\chi_4$ | $n_4 = 3$ | $-1$ | $-1$ | 0 | 1 |
| $\chi_5$ | $n_5 = 2$ | $a = 0$ | $b = 2$ | $c = -1$ | $d = 0$ |

**Example 12.11.** In the above example of $S_4$ we see that the representations $\rho_3$ and $\rho_4$ are adjoint of each other. This we can see by looking at their characters. This gives an example of representation which is not isomorphic to its adjoint.

**Example 12.12** ($D_n$, $n$ even)**.**

$$D_n = \{a, b \mid a^n = 1 = b^2, ab = ba^{-1}\}$$

and the conjugacy classes are $\{1\}, \{a^{\frac{n}{2}}\}, \{a^j, a^{-j}\}(1 \leq j \leq \frac{n}{2} - 1), \{a^j b \mid j \ even\}$ and $\{r^j b \mid j \ odd\}$. The subgroup generatd by $a^2$ is normal and hence there are 4 one dimensional representations. Rest of them are two dimensional (refer to Corollary 12.4) representations defined in the Example 2.7. Using them we can make character table.

**Example 12.13** ($D_n$, $n$ odd)**.**

$$D_n = \{a, b \mid a^n = 1 = b^2, ab = ba^{-1}\}$$

and the conjugacy classes are $\{1\}, \{a^j, a^{-j}\}(1 \leq j \leq \frac{n-1}{2}), \{a^j b \mid 0 \leq j \leq n - 1\}$. The subgroup generatd by $a$ is normal and hence there are 2 one dimensional representations. Rest of them are two dimensional (refer to Corollary 12.4) representations defined in the Example 2.7. Using them we can make character table.

CHAPTER 13

# Characters of Index 2 Subgroups

In this chapter we will see how we can use characters of a group $G$ to get characters of its index 2 subgroups. We apply this for $S_5$ and its subgroup $A_5$.

## 13.1. The Representation $V \otimes V$

Let $G$ be a finite group. Suppose that $(\rho, V)$ and $(\rho', V')$ are representations of $G$. Then we can get a new representation of $G$ from these representations defined as follows (recall from section 7.3):

$$\begin{aligned} \rho \otimes \rho' \colon G &\rightarrow GL(V \otimes V') \\ (\rho \otimes \rho')(g)(v \otimes v') &= \rho(g)(v) \otimes \rho'(g)(v') \end{aligned}$$

Now suppose $\rho$ and $\rho'$ are representations over $\mathbb{C}$ and $\chi$ and $\chi'$ are the corresponding characters, then the character of $\rho \otimes \rho'$ is $\chi\chi'$ given by $(\chi\chi')(g) = \chi(g)\chi'(g)$. However even if $\rho$ and $\rho'$ are irreducible $\rho \otimes \rho'$ need not be irreducible.

Let $(\rho, V)$ be a representation of $G$. We consider $(\rho \otimes \rho, V \otimes V)$. As defined above it is a representation of $G$ with character $\chi^2$ where $\chi^2(g) = \chi(g)^2$. Recall from section 7.3 it can be decomposed as

$$V \otimes V = Sym^2(V) \oplus \Lambda^2(V).$$

We prove below that both $Sym^2(V)$ and $\Lambda^2(V)$ are $G$-spaces.

**Theorem 13.1.** *Let $(\rho, V)$ be a representation of $G$. Then $V \otimes V = Sym^2(V) \oplus \Lambda^2(V)$ where each of the subspaces $Sym^2(V)$ and $\Lambda^2(V)$ are $G$-invariant.*

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis of $V$. Then $\{v_i \otimes v_j \mid 1 \leq i, j \leq n\}$ is a basis of $V \otimes V$ with dimension $n^2$. Consider the linear map $\theta$ defined on the basis of $V \otimes V$ by $\theta(v_i \otimes v_j) = v_j \otimes v_i$ and extended linearly. We observe that $\theta$ can also be defined without the help of any basis by $\theta(v \otimes w) = w \otimes v$ since if $v = \sum a_i v_i$ and $w = \sum b_j v_j$, then $\theta(v \otimes w) = \theta\left(\sum a_i v_i \otimes \sum b_j v_j\right) = \sum a_i b_j \theta(v_i \otimes v_j) = \sum a_i b_j (v_j \otimes v_i) = \sum (b_j v_j \otimes a_i v_i) = (w \otimes v)$. Observe that $\theta^2 = 1$ and we take the subspaces of $V \otimes V$ corresponding to eigen values 1 and $-1$:

$$\begin{aligned} Sym^2(V) &= \{x \in V \otimes V \mid \theta(x) = x\} \\ \Lambda^2(V) &= \{x \in V \otimes V \mid \theta(x) = -x\} \end{aligned}$$

Note that $\{(v_i\otimes v_j+v_j\otimes v_i) \mid 1\le i\le j\le n\}$ is a basis for $Sym^2(V)$ and hence its dimension is $\frac{n(n+1)}{2}$ and $\{(v_i\otimes v_j-v_j\otimes v_i) \mid 1\le i<j\le n\}$ is a basis for $\Lambda^2(V)$ and hence dimension is $\frac{n(n-1)}{2}$.

We claim that $Sym^2(V)$ and $\Lambda^2(V)$ are $G$-invariant. Suppose that $v\otimes w\in Sym^2(V)$ and $g\in G$ then we have

$$\begin{aligned}
\theta(\rho(g)(v\otimes w)) &= \theta\left(\rho(g)\left(\sum \lambda_{ij}(v_i\otimes v_j+v_j\otimes v_i)\right)\right)\\
&= \theta\left(\sum \lambda_{ij}(\rho(g)v_i\otimes\rho(g)v_j+\rho(g)v_j\otimes\rho(g)v_i)\right)\\
&= \sum \lambda_{ij}\left[\theta(\rho(g)v_i\otimes\rho(g)v_j)+\theta(\rho(g)v_j\otimes\rho(g)v_i)\right]\\
&= \sum \lambda_{ij}[\rho(g)v_j\otimes\rho(g)v_i+\rho(g)v_i\otimes\rho(g)v_j)]\\
&= \rho(g)\left(\sum \lambda_{ij}(v_j\otimes v_i+v_i\otimes v_j)\right)\\
&= \rho(g)(v\otimes w).
\end{aligned}$$

We also see that $Sym^2(V)\cap\Lambda^2(V)=\{0\}$. Also their individual dimensions add up to $\frac{n(n+1)}{2}+\frac{n(n-1)}{2}=n^2=\dim(V\otimes V)$, hence we have $V\otimes V=Sym^2(V)\oplus\Lambda^2(V)$. □

**Theorem 13.2.** *The characters of $Sym^2(V)$ and $\Lambda^2(V)$ are $\chi_S$ and $\chi_A$ respectively given by*

$$\begin{aligned}
\chi_S(g) &= \frac{1}{2}\left(\chi^2(g)+\chi(g^2)\right)\\
\chi_A(g) &= \frac{1}{2}\left(\chi^2(g)-\chi(g^2)\right)
\end{aligned}$$

Proof. Suppose that $|G|=d$. Then for any $g\in G$, $(\rho(g))^d=I$. Thus $m(X)$, the minimal polynomial of $\rho(g)$, divides the polynomial $p(X)=X^d-1$. Since $p(X)$ has distinct roots so will $m(X)$ and hence $\rho(g)$ is diagonalisable.

Let $\{e_1,\cdots e_n\}$ be an eigen basis for $V$ and let $\{\lambda_1,\cdots,\lambda_n\}$ be the corresponding eigen values. Then from the proof of previous theorem it follows that $\{(e_i\otimes e_j-e_j\otimes e_i) \mid i<j\}$ is an eigen basis for $\Lambda^2(V)$ with corresponding eigen values $\{\lambda_i\lambda_j \mid i<j\}$. We now have

$$\chi_A(g)=Tr(\rho\otimes\rho)(g)=\sum_{i<j}\lambda_i\lambda_j=\frac{1}{2}\left(\left(\sum\lambda_i\right)^2-\sum\left(\lambda_i^2\right)\right)=\frac{1}{2}\left(\chi^2(g)-\chi(g^2)\right).$$

In similar way one can calculate $\chi_S$. □

## 13.2. Character Table of $S_5$

With the theory developed above, we are now ready to construct the character table for the symmetric group $S_5$. We record below some facts about $S_n$ and in particular about $S_5$ which will be the starting point to make the character table:

(1) $|S_5|=5!=120$.

(2) Two elements $\sigma$ and $\sigma'$ of $S_n$ are conjugate if and only if their cycle structure is same when they are written as product of disjoint cycles.
(3) The conjugacy classes in $S_5$ along with their cardinality are as mentioned below:

| $\|(g_i)\|$ | 1 | 10 | 20 | 15 | 30 | 20 | 24 |
|---|---|---|---|---|---|---|---|
| $g_i$ | 1 | (12) | (123) | (12)(34) | (1234) | (12)(345) | (12345) |

(4) For any $S_n$, we already know 2 one dimensional (and hence irreducible) representations: One being the trivial representation and the other the sign representation which sends each transposition to -1.
(5) For any $S_n$, we know an irreducible representation of dimension $n-1$ (as in the Example 2.15 obtained by the action of $S_n$ on the subspace $V$ of $\mathbb{C}^n$ given by $V = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n \mid x_1 + x_2 + \cdots + x_n = 0\}$).

We fill this information into the character table:

| $\|(g_i)\|$ | 1 | 10 | 20 | 15 | 30 | 20 | 24 |
|---|---|---|---|---|---|---|---|
| $g_i$ | 1 | (12) | (123) | (12)(34) | (1234) | (12)(345) | (12345) |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 |
| $\chi_3$ | 4 | 2 | 1 | 0 | 0 | $-1$ | $-1$ |

We need to find 4 more irreducible characters. We now deploy the ways described at the beginning to find new irreducible representations.

Taking tensor product of the trivial representation with any other representation $(\rho, V)$ gives a representation isomorphic to $(\rho, V)$ (since $\chi_1\chi_V = \chi_V$). Hence we only need to consider tensor product of the second and the third representation whose character is $\chi_2\chi_3$.

$$\begin{aligned}\langle \chi_2\chi_3, \chi_2\chi_3\rangle &= \frac{1}{120}((4)^2 + 10(-2)^2) + 20(1)^2 + 20(1)^2 + 24(-1)^2) \\ &= 1 \\ &= \langle \chi_3, \chi_3\rangle\end{aligned}$$

Thus this representation turns out to be irreducible. Let $\chi_4 = \chi_2\chi_3 \neq \chi_3$. We include this character $\chi_4$ into the character table:

| $\|(g_i)\|$ | 1 | 10 | 20 | 15 | 30 | 20 | 24 |
|---|---|---|---|---|---|---|---|
| $g_i$ | 1 | (12) | (123) | (12)(34) | (1234) | (12)(345) | (12345) |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 |
| $\chi_3$ | 4 | 2 | 1 | 0 | 0 | $-1$ | $-1$ |
| $\chi_4$ | 4 | $-2$ | 1 | 0 | 0 | 1 | $-1$ |

We now consider $\chi = \chi_3$ and for this we examine the representations $\chi_S$ and $\chi_A$:

| $\lvert(g_i)\rvert$ | 1 | 10 | 20 | 15 | 30 | 20 | 24 |
|---|---|---|---|---|---|---|---|
| $g_i$ | 1 | (12) | (123) | (12)(34) | (1234) | (12)(345) | (12345) |
| $\chi_A$ | 6 | 0 | 0 | $-2$ | 0 | 0 | 1 |
| $\chi_S$ | 10 | 4 | 1 | 2 | 0 | 1 | 0 |

We check that $\chi_A$ is irreducible as $\langle \chi_A, \chi_A \rangle = \frac{1}{120}((6)^2 + 15(-2)^2 + 24(1)^2) = 1$.

Thus $\chi_5 = \chi_A$ is the fifth irreducible character of $S_5$. Suppose that $\chi_6$ and $\chi_7$ are the other 2 irreducible characters. Since every representation of a finite group can be written as a direct sum of irreducible ones we have,

$$\chi_S = m_1\chi_1 + m_2\chi_2 + \cdots + m_7\chi_7$$

where $m_i = \langle \chi_S, \chi_i \rangle$. Calculations show that $\langle \chi_S, \chi_S \rangle = 3$, $\langle \chi_S, \chi_1 \rangle = 1$ and $\langle \chi_S, \chi_3 \rangle = 1$. We also have $\sum m_i^2 = \langle \chi_S, \chi_S \rangle = 3$. Thus $\chi_S = \chi_1 + \chi_3 + \psi$, where $\psi$ is an irreducible character. We rewrite $\psi = \chi_S - \chi_1 - \chi_3$ explicitly

| $\lvert(g_i)\rvert$ | 1 | 10 | 20 | 15 | 30 | 20 | 24 |
|---|---|---|---|---|---|---|---|
| $g_i$ | 1 | (12) | (123) | (12)(34) | (1234) | (12)(345) | (12345) |
| $\psi$ | 5 | 1 | $-1$ | 1 | $-1$ | 1 | 0 |

Since $\psi$ is irreducible, $\langle \psi, \psi \rangle = \frac{1}{120}(5^2 + 10 + 20 + 15 + 30 + 20) = 1$ as expected. Let $\chi_6 = \psi$ then $\chi_7 = \chi_2\chi_6$ will be another new irreducible character. We have thus found the character table of $S_5$:

*Character Table of $S_5$*

| $\lvert(g_i)\rvert$ | 1 | 10 | 20 | 15 | 30 | 20 | 24 |
|---|---|---|---|---|---|---|---|
| $g_i$ | 1 | (12) | (123) | (12)(34) | (1234) | (12)(345) | (12345) |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 |
| $\chi_3$ | 4 | 2 | 1 | 0 | 0 | $-1$ | $-1$ |
| $\chi_4$ | 4 | $-2$ | 1 | 0 | 0 | 1 | $-1$ |
| $\chi_5$ | 6 | 0 | 0 | $-2$ | 0 | 0 | 1 |
| $\chi_6$ | 5 | 1 | $-1$ | 1 | $-1$ | 1 | 0 |
| $\chi_7$ | 5 | $-1$ | $-1$ | 1 | 1 | $-1$ | 0 |

## 13.3. Index two subgroups

Let $(\rho, V)$ be a representation of $G$. Then $\rho\colon G \to GL(V)$ is a group homomorphism. Suppose $H$ is a subgroup of $G$. Then we can restrict the map $\rho$ to $H$ denoted as $\rho|_H$ to obtain a representation for $H$. Note that even if $(\rho, V)$ is an irreducible representation, $(\rho|_H, V)$ need not be irreducible. The following theorem shows the intimate connection between the characters of a group $G$ and any of its subgroup $H$. Recall from Chapter 9 we define an inner product on $\mathbb{C}[G]$ denoted as $\langle , \rangle$. We denote this inner product on $\mathbb{C}[H]$ by $\langle , \rangle_H$ thinking of $H$ as a group in itself.

**Proposition 13.3.** *Let $G$ be a group and $H$ its subgroup. Let $\psi$ be a non zero character of $H$. Then there exists an irreducible character $\chi$ of $G$ such that $\langle \chi|_H, \psi\rangle_H \neq 0$ where $\langle , \rangle_H$ represents inner product on $\mathbb{C}[H]$ as explained above.*

PROOF. Let $\chi_1, \chi_2, \ldots, \chi_h$ be the irreducible characters of $G$. We have the regular representation of $G$: $\chi_{reg} = \sum \chi_i(1)\chi_i$ and $\chi_{reg}(1) = |G|$, $\chi_{reg}(g) = 0, g \neq 1$. Thus we have the following:

$$\begin{aligned}
\langle \chi_{reg}|_H, \psi\rangle_H &= \frac{1}{|H|}\left(\chi_{reg}(1)\psi(1)\right) \\
&= \frac{|G|\psi(1)}{|H|} \neq 0 \\
\therefore \langle \chi_{reg}|_H, \psi\rangle_H &= \sum_{i=1}^{h} \langle \chi_i|_H, \psi\rangle_H \neq 0
\end{aligned}$$

Hence atleast one of the $\langle \chi_i, \psi\rangle_H \neq 0$. $\square$

**Theorem 13.4.** *Let $G$ be a group and $H$ a subgroup. Let $\chi$ be an irreducible character of $G$. Suppose $\psi_1, \psi_2, \ldots, \psi_k$ are all irreducible characters of $H$ and $\chi|_H = d_1\psi_1 + d_2\psi_2 + \cdots + d_k\psi_k$ for some $d_1, d_2, \ldots, d_k$ integers. Then*

$$\sum_{i=1}^{k} d_i^2 \leq [G:H]$$

*and equality occurs if and only if $\chi(g) = 0$ for all $g \notin H$.*

PROOF. Thinking of $H$ as a group we have $\langle \chi|_H, \chi|_H\rangle_H = \sum d_i^2$. Since $\chi$ is an irreducible character of $G$ we have,

$$\begin{aligned}
\langle \chi, \chi\rangle &= \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\chi(g)} \\
1 &= \frac{1}{|G|}\left(\sum_{h\in H}\chi(h)\overline{\chi(h)} + \sum_{g\notin H}\chi(g)\overline{\chi(g)}\right) \\
1 &= \frac{|H|}{|G|}\langle \chi|_H, \chi|_H\rangle_H + \frac{1}{|G|}\sum_{g\notin H}\chi(g)\overline{\chi(g)}
\end{aligned}$$

Rewriting the above we get,

$$\begin{aligned} \frac{|H|}{|G|}\langle \chi|_H, \chi|_H\rangle_H &= 1 - \frac{1}{|G|}\sum_{g\notin H}\chi(g)\overline{\chi(g)} \\ \sum d_i^2 = \langle \chi|_H, \chi|_H\rangle_H &= \frac{|G|}{|H|} - \frac{1}{|H|}\sum_{g\notin H}\chi(g)\overline{\chi(g)} \leq \frac{|G|}{|H|}. \end{aligned}$$

Moreover equality occurs if and only if $\frac{1}{|H|}\sum_{g\notin H}\chi(g)\overline{\chi(g)} = 0$, i.e., $\sum_{g\notin H}|\chi(g)|^2 = 0$ which happens if and only if $|\chi(g)| = 0$ and hence $\chi(g) = 0$ for all $g \notin H$. □

We will apply the above theorm for index two subgroups and get the following,

**Corollary 13.5.** *Let $G$ be a group and $H$ a subgroup of index 2. Let $\chi$ be an irreducible character of $G$. Then one of the following happens :*

(1) *$\chi|_H = \psi$ is an irreducible character of $H$. This happens if and only if there exists $g \in G$ and $g \notin H$ such that $\chi(g) \neq 0$.*
(2) *$\chi|_H = \psi_1 + \psi_2$ where $\psi_1$ and $\psi_2$ are irreducible characters of $H$. This happens if and only if $\chi(g) = 0$ for all $g \notin H$.*

### 13.4. Character Table of $A_5$

We will write down the character table of $A_5$. For this we will use the Corollary 13.5 and the character table of $S_5$ derived earlier in this chapter. The conjugacy classes of $A_5$ and their corresponding cardinality are as follows:

| $\|(g_i)\|$ | 1 | 20 | 15 | 12 | 12 |
|---|---|---|---|---|---|
| $g_i$ | 1 | (123) | (12)(34) | (12345) | (13452) |

From the character table of $S_5$ it follows that $\chi_1|_H = \chi_2|_H$, $\chi_3|_H = \chi_4|_H$ and $\chi_6|_H = \chi_7|_H$ are irreducible characters of $A_5$. Hence we get the character table of $A_5$ partially.

| $\|(g_i)\|$ | 1 | 20 | 15 | 12 | 12 |
|---|---|---|---|---|---|
| $g_i$ | 1 | (123) | (12)(34) | (12345) | (13452) |
| $\psi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\psi_2$ | 4 | 1 | 0 | $-1$ | $-1$ |
| $\psi_3$ | 5 | $-1$ | 1 | 0 | 0 |
| $\psi_4$ | $n_4 = 3$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
| $\psi_5$ | $n_5 = 3$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ |

We know that $1^2 + 4^2 + 5^2 + {n_4}^2 + {n_5}^2 = 60$ and hence ${n_4}^2 + {n_5}^2 = 18$. The only possible integral solutions of this equation are $n_4 = n_5 = 3$.

Now we see that the only irreducible character of $S_5$ whose restriction to $A_5$ is not irreducible is $\chi_5$. Since $\psi_1, \psi_2, \psi_3$ are all obtained by restriction of the characters other than $\chi_5$ it follows from Theorem 13.4 that only possibly $\langle \psi_4, \chi_5|_{A_5}\rangle_{A_5} \neq 0$ and $\langle \psi_5, \chi_5|_{A_5}\rangle_{A_5} \neq 0$.

Now from Corollary 13.5 it follows that $\chi_5|_{A_5}$ is sum of two characters of $A_5$ and hence $\chi_5|_{A_5} = \psi_4 + \psi_5$. Thus we have $a_1 = -b_1$, $a_2 = -2 - b_2$, $a_3 = 1 - b_3$ and $a_4 = 1 - b_4$.

Now we use orthogonality relations for characters of $A_5$ and get $\langle \psi_4, \psi_1 \rangle = 3 + 20a_1 + 15a_2 + 12a_3 + 12a_4 = 0$, $\langle \psi_4, \psi_2 \rangle = 12 + 20a_1 - 12a_3 - 12a_4 = 0$, and $\langle \psi_4, \psi_3 \rangle = 15 - 20a_1 + 15a_2 = 0$. Solving these equations we get $a_1 = 0$, $a_2 = -1$ and $a_3 + a_4 = 1$. Hence we have the following:

| $\lvert(g_i)\rvert$ | 1 | 20 | 15 | 12 | 12 |
|---|---|---|---|---|---|
| $g_i$ | 1 | (123) | (12)(34) | (12345) | (13452) |
| $\psi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\psi_2$ | 4 | 1 | 0 | $-1$ | $-1$ |
| $\psi_3$ | 5 | $-1$ | 1 | 0 | 0 |
| $\psi_4$ | 3 | 0 | $-1$ | $a_3$ | $a_4 = 1 - a_3$ |
| $\psi_5$ | 3 | 0 | $-1$ | $b_3 = 1 - a_3 = a_4$ | $b_4 = 1 - a_4 = a_3$ |

**Proposition 13.6.** *Every element of $A_5$ is conjugate to their own inverse. Hence the entries of the character table are real numbers.*

Proof. Clearly it is enough to prove that the representatives of the conjugacy classes are conjugate to their own inverse. It is clear for the element $1, (123)$ and $(12)(34)$. For others we check that

$$\begin{aligned} (12345)^{-1} &= (54321) = (15)(24)(12345)(15)(24) \\ (13452)^{-1} &= (25431) = (12)(35)(13452)(12)(35) \end{aligned}$$

Now we know that $\chi(g^{-1}) = \overline{\chi(g)}$ and $g$ being conjugate to $g^{-1}$ this is also equal to $\chi()g$. Hence $\chi(g) = \overline{\chi(g)}$ gives $\chi(g) \in \mathbb{R}$ for all $g \in A_5$. □

The above proposition implies that $a_3, a_4, b_3$ and $b_4$ are real numbers. From $\langle \chi_4, \chi_4 \rangle = 1$ we get $a_3^2 + a_4^2 = 3$. Substituting $a_4 = 1 - a_3$ we get $a_3^2 - a_3 - 1 = 0$. And the solutions are

$$a_3 = \frac{1+\sqrt{5}}{2} = b_4, a_4 = \frac{1-\sqrt{5}}{2} = b_3$$

or

$$a_3 = \frac{1-\sqrt{5}}{2} = b_4, a_4 = \frac{1+\sqrt{5}}{2} = b_3.$$

Since the values of $\psi_4$ and $\psi_5$ on other conjugacy classes are the same, both the above solutions would give the same set of irreducible characters. Hence without loss of generality we may take the first set of solutions. This gives the complete character table of $A_5$as follows:

*Character Table of $A_5$*

| $\lvert(g_i)\rvert$ | 1 | 20 | 15 | 12 | 12 |
|---|---|---|---|---|---|
| $g_i$ | 1 | (123) | (12)(34) | (12345) | (13452) |
| $\psi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\psi_2$ | 4 | 1 | 0 | $-1$ | $-1$ |
| $\psi_3$ | 5 | $-1$ | 1 | 0 | 0 |
| $\psi_4$ | 3 | 0 | $-1$ | $\frac{1+\sqrt{5}}{2}$ | $\frac{1-\sqrt{5}}{2}$ |
| $\psi_5$ | 3 | 0 | $-1$ | $\frac{1-\sqrt{5}}{2}$ | $\frac{1+\sqrt{5}}{2}$ |

**Exercise 13.7.** Prove that every element of $S_n$ is conjugate to its own inverse and hence character table consists of real numbers.

**Remark :** In fact more is true that every element of $S_n$ is conjugate to all those powers of itself which generates same subgroup (called rational conjugacy). Hence it is true that characters of $S_n$ always take value in integers. However this is not true for $A_n$, for example check $A_4$ and $A_5$. They may not be even real valued.

CHAPTER 14

# Characters and Algebraic Integers

Let $G$ be a finite group and $\rho_1, \ldots, \rho_h$ be all irreducible representations of $G$ of dimension $n_1, \cdots, n_h$ with corresponding characters $\chi_1, \cdots, \chi_h$.

For any give representation $\rho \colon G \to GL(V)$ we can define an algebra homomorphism $\tilde{\rho} \colon \mathbb{C}[G] \to \mathrm{End}(V)$ by $\sum_g \alpha_g g \mapsto \sum_g \alpha_g \rho(g)$. We know that $\mathcal{Z}(\mathbb{C}[G])$, the center of $\mathbb{C}[G]$, is spanned by the elements $c_{g_1}, \ldots, c_{g_h}$ where

$$c_{g_i} = \sum_{\{t \in G | t = s g_i s^{-1}\}} t$$

i.e., sum of all conjugates of $g_i$ and $g_i$ are representatives of different conjugacy class.

**Exercise 14.1.** Let $\sum_g \alpha_g g \in \mathcal{Z}(\mathbb{C}[G])$ then $\alpha_g = \alpha_{sgs^{-1}}$ for any $s \in G$.

**Proposition 14.2.** *Let $\rho$ be an irreducible representation and $z \in \mathcal{Z}(\mathbb{C}[G])$. Then $\tilde{\rho}(z) = \lambda.Id$ for some $\lambda \in \mathbb{C}$.*

Proof. We claim that $\tilde{\rho}(z) \in \mathrm{End}(V)$ is a $G$-map. Let $z = \sum_g \alpha_g g$ then $\alpha_g = \alpha_{sgs^{-1}}$ for any $s \in G$. Then

$$\tilde{\rho}(z)(\rho(t)v) = \sum_g \alpha_g \rho(g)(\rho(t)v) = \rho(t) \sum_g \alpha_g \rho(t^{-1}gt)v = \rho(t) \sum_u \alpha_{tut^{-1}} \rho(u)v = \rho(t)\tilde{\rho}(z)v.$$

Since $\rho$ is irreducible Corollary 4.5 (Schur's Lemma) implies that $\tilde{\rho}(z) = \lambda.Id$ for some $\lambda \in \mathbb{C}$. $\square$

With notation as above let us denote $\tilde{\rho}_i(c_{g_j}) = \lambda_{ij} Id_{n_i}$ where $\lambda_{ij} \in \mathbb{C}$ and $Id_{n_i}$ denotes the identity matrix. We can take trace of both side and get,

$$n_i \lambda_{ij} = tr(\tilde{\rho}_i(c_{g_j})) = \sum_{\{t \in G | t = s g_j s^{-1}\}} tr(\rho_i(t)) = r_j \chi_i(g_j)$$

where $r_j$ is the number of conjugates of $g_j$. This gives,

$$\lambda_{ij} = \frac{r_j \chi_i(g_j)}{n_i} = r_j \frac{\chi_i(g_j)}{\chi_i(1)}.$$

**Proposition 14.3.** *Each $\lambda_{ij}$ is an algebraic integer.*

PROOF. Let us consider $M = c_{g_1}\mathbb{Z} \oplus \cdots \oplus c_{g_h}\mathbb{Z} \subset \mathcal{Z}(\mathbb{C}[G]) = c_{g_1}\mathbb{C} \oplus \cdots \oplus c_{g_h}\mathbb{C}$. Clearly $M$ is a $\mathbb{Z}$-submodule. It is a subring also. Take $c_{g_j}, c_{g_k} \in \mathcal{Z}(\mathbb{C}[G])$. Then $c_{g_j}c_{g_k} \in \mathcal{Z}(\mathbb{C}[G])$, in fact $c_{g_j}c_{g_k} \in M$. Hence we can write $c_{g_j}c_{g_k} = \sum_{l=1}^{h} a_{jkl}c_{g_l}$ where $a_{jkl}$ are integers. By applying $\tilde{\rho}_i$ we get,

$$(\lambda_{ij}Id_{n_i})(\lambda_{ik}Id_{n_i}) = \tilde{\rho}_i(c_{g_j})\tilde{\rho}_i(c_{g_k}) = \tilde{\rho}_i(c_{g_j}c_{g_k}) = \sum_{l=1}^{h} a_{jkl}\lambda_{il}Id_{n_i}.$$

This gives $\lambda_{ij}\lambda_{ik} = \sum_{l=1}^{h} a_{jkl}\lambda_{il}$ where each $a_{jkl} \in \mathbb{Z}$.

Now we take $N = \lambda_{i1}\mathbb{Z} \oplus \cdots \oplus \lambda_{ih}\mathbb{Z} \subset \mathbb{C}$ which is a finitely generated $\mathbb{Z}$-module and $\lambda_{ij}N \subset N$ for all $j$. This implies $\lambda_{ij}$ is an algebraic integer. □

**Lemma 14.4.** *For any $s \in G$ and $\chi$ character of a representation, $\chi(s)$ is an algebraic integer.*

PROOF. Let $s \in G$ be of order $d$ in $G$. Then $\rho(s)$ is of order less than or equal to $d$. Since the field is $\mathbb{C}$ we can choose a basis such that the matrix of $\rho(s)$, say $A$, becomes diagonal (see 5.4 and also proof in 9.6). Clearly $A^d = 1$ implies diagonal elements are root of the polynomial $X^d - 1$ hence are algebraic integer. As sum of algebraic integers is again an algebraic integer we get sum of diagonals of $A$ which is $\chi(s)$ is an algebraic integer. □

**Theorem 14.5.** *The order of an irreducible representation divides the order of the group, i.e., $n_i$ divides $|G|$ for all $i$.*

PROOF. Let $\rho_i$ be an irreducible representation of degree $n_i$ with character $\chi_i$. From the orthogonality relations we have,

$$\begin{aligned}
\frac{1}{|G|}\sum_{t\in G}\chi_i(t)\chi_i(t^{-1}) &= 1 \\
\sum_{j=1}^{h} r_j\chi_i(g_j)\chi_i(g_j^{-1}) &= |G| \\
\sum_{j=1}^{h} n_i\lambda_{ij}\chi_i(g_j^{-1}) &= |G| \\
\sum_{j=1}^{h} \lambda_{ij}\chi_i(g_j^{-1}) &= \frac{|G|}{n_i}
\end{aligned}$$

Left side of this equation is an algebraic integer (using Proposition 14.3 and Lemma 14.4) and right side is a rational number. Hence $\frac{|G|}{n_i}$ is an algebraic integer as well as algebraic number hence an integer. This implies $n_i$ divides $|G|$. □

CHAPTER 15

# Burnside's $pq$ Theorem

We continue with the notation in the last chapter and recall,

(1) $\lambda_{ij} = r_j \frac{\chi_i(g_j)}{n_i}$ is an algebraic integer.
(2) $\chi_i(t)$ is an algebraic integer.

**Lemma 15.1.** *With notation as above suppose $r_j$ and $n_i$ are relatively prime. Then either $\rho_i(g_j)$ is in the center of $\rho_i(G)$ or $tr(\rho_i(g_j)) = \chi_i(g_j) = 0$.*

Proof. As in the proof of Proposition 9.6 and Lemma 14.4 we can choose a basis such that the matrix of $\rho_i(g_j) = \text{diag}\{\omega_1, \cdots, \omega_{n_i}\}$ and $\chi_i(g_j) = \omega_1 + \cdots + \omega_{n_i}$ where $\omega_k$'s are $d$th root of unity ($d$ order of $g_j$). Now $|\omega_1 + \cdots + \omega_{n_i}| \leq 1 + \cdots + 1 = n_i$ hence $|\frac{\chi_i(g_j)}{n_i}| \leq 1$. In the case $|\frac{\chi_i(g_j)}{n_i}| = 1$ we must have $\omega_1 = \omega_2 = \ldots = \omega_{n_i}$. This implies the matrix of $\rho_i(g_j)$ is central and hence in this case $\rho_i(g_j)$ belongs in the center of $\rho_i(G)$.

Now suppose that $|\frac{\chi_i(g_j)}{n_i}| < 1$ and let $\alpha = \frac{\chi_i(g_j)}{n_i}$. Since $r_j$ and $n_i$ are relatively prime we can find integers $l, m \in \mathbb{Z}$ such that $r_j l + n_i m = 1$. Then $\lambda_{ij} = r_j \alpha$ gives $l\lambda_{ij} = (1 - n_i m)\alpha = \alpha - m\chi_i(g_j)$. Since $\lambda_{ij}$ and $\chi_i(g_j)$ are algebraic integers we get $\alpha$ is an algebraic integer. Let $\zeta$ be a primitive $d$th root of unity and let us consider the Galois extension $K = \mathbb{Q}(\zeta)$ of $\mathbb{Q}$. Let $\omega_k = \zeta^{a_k}$ then $\alpha = \frac{1}{n_i}(\zeta^{a_1} + \cdots + \zeta^{a_{n_i}})$. For $\sigma \in \text{Gal}(K/\mathbb{Q})$ the element $\sigma(\alpha)$ is also of the same kind hence $|\sigma(\alpha)| \leq 1$. This implies the norm of $\alpha$ defined by $N(\alpha) = \prod_{\sigma \in \text{Gal}(K/\mathbb{Q})} \sigma(\alpha)$ has $|N(\alpha)| < 1$. Since $\alpha$ is an algebraic integer so are $\sigma(\alpha)$ and hence the product $N(\alpha)$ is an algebraic integer. Since $N(\alpha)$ is also left invariant by all $\sigma \in \text{Gal}(K/\mathbb{Q})$ it is a rational number hence it must be an integer. However as $|N(\alpha)| < 1$ this gives $N(\alpha) = 0$ and which implies $\chi_i(g_j) = \alpha = 0$. $\square$

**Proposition 15.2.** *Let $G$ be a finite group and $C$ be conjugacy class of $g \in G$. If $|C| = p^r$, where $p$ is a prime and $r \geq 1$, then there exist a nontrivial irreducible representation $\rho$ of $G$ such that $\rho(C)$ is contained in the center of $\rho(G)$. In particular, $G$ is not a simple group.*

Proof. On contrary let us assume that $\rho_i(C)$ is not contained in the center of $\rho_i(G)$ for all irreducible representations $\rho_i$ of $G$. Then from previous Lemma if $p$ does not divide $n_i$ we have $\chi_i(g) = 0$ for $g \in C$.

Consider the character of the regular representation $\chi = \sum_{i=1}^{h} n_i\chi_i$. Then for any $1 \neq s \in G$ we have

$$0 = \chi(s) = \sum_{i=1}^{h} n_i\chi_i(s) = 1 + \sum_{i=2}^{h} n_i\chi_i(s).$$

Let us take $g \in C$ (note that $g \neq 1$ since $|C| > 1$). Then $\sum_{i=2}^{h} n_i\chi_i(g) = -1$. On the left hand side we have either $p|n_i$ or if $p \not| n_i$ then $\chi_i(g) = 0$. Rewriting the above expression we get $\sum_{i=2}^{h} \frac{n_i}{p}\chi_i(g) = -\frac{1}{p}$. Left hand side of the expression is an algebraic integer (since $\frac{n_i}{p}$ is an integer) and right hand side is a rational number hence it should be an integer which is a contradiction. Hence there exist a nontrivial irreducible representation $\rho_i$ such that $\rho_i(C)$ is contained in the center of $\rho_i(G)$.

Now let us take the above $\rho\colon G \to GL(V)$ and we have $\rho(C) \subset \mathcal{Z}(\rho(G))$ where $C$ is a conjugacy class of non-identity element. If $ker(\rho) \neq 1$ it will be normal subgroup of $G$ and $G$ is not simple. In case $ker(\rho) = 1$ we have $\rho$ an injective map and $\mathcal{Z}(\rho(G)) \neq 1$. But center is always a normal subgroup which implies $G$ is not simple. □

**Theorem 15.3** (Burnside's Theorem)**.** *Every group of order $p^a q^b$, where $p, q$ are distinct primes, is solvable.*

PROOF. We use induction on $a + b$. If $a + b = 1$ then $G$ is a $p$-group and hence $G$ is solvable. Now assume $a + b \geq 2$ and any group of order $p^r q^s$ with $r + s < a + b$ is solvable. Let $Q$ be a Sylow $q$-subgroup of $G$. If $Q = \{e\}$ then $b = 0$ and $G$ is a $p$-group and hence solvable. So let us assume $Q$ is nontrivial. Since $Q$ is a $q$-group (prime power order) it has nontrivial center. Let $1 \neq t \in \mathcal{Z}(Q)$. Then

$$t \in Q \subset C_G(t) \subset G$$

and hence $|C_G(t)| = p^l q^b$ for some $0 \leq l \leq a$ which gives $[G : C_G(t)] = p^{a-l}$. We claim that $G$ is not simple. If $G = C_G(t)$ then $t \in \mathcal{Z}(C_G(t)) = \mathcal{Z}(G)$, i.e., $\mathcal{Z}(G)$ is nontrivial normal subgroup and $G$ is not simple. Hence we may assume $|C_G(t)| = p^l q^b$ with $l < a$. Then using the formula $|C(t)| = \frac{|G|}{|C_G(t)|}$ we get $|C(t)| = p^{a-l}$. Here $C(t)$ denotes the conjugacy class of $t$. Using previous proposition we get $G$ is not simple.

Now we know that $G$ is not simple so it has a proper normal subgroup, say $N$. The order of $N$ and $G/N$ both satisfy the hypothesis hence they are solvable. Thus $G$ is solvable. □

CHAPTER 16

# Further Reading

## 16.1. Representation Theory of Symmetric Group

See Fulton and Harris chapter 4.

## 16.2. Representation Theory of $GL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q)$

See Fulton and Harris chapter 5.
See self-contained notes by Amritanshu Prasad on the subject available on Arxiv.

## 16.3. Wedderburn Structure Theorem

See "Non Commutative Algebra" by Farb and Dennis and also "Groups and Representations" by Alperin and Bell.

## 16.4. Modular Representation Theory

See the notes by Amritanshu Prasad on "Representations in Positive Characteristic" available on Arxiv.

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