ON A FITTING LENGTH CONJECTURE WITHOUT THE COPRIMENESS CONDITION

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Abstract. Let A be a finite nilpotent group acting fixed point freely by automorphisms on the finite solvable group G. It is conjectured that the Fitting length of G is bounded by the number of primes dividing the order of A, counted with multiplicities. The main result of this paper shows that the conjecture is true in the case where A is cyclic of order $p^n q$, for prime numbers p and q coprime to 6 and G has abelian Sylow 2-subgroups.

1. Introduction

Let A be a finite group that acts on the finite solvable group G by automorphisms in such a way that $C_G(A)$ is trivial. A long-standing conjecture in the coprime case $(|G|, |A|) = 1$ states that the Fitting length $f(G)$ of G is bounded above by the length $\ell(A)$ of the longest chain of subgroups of A. This problem has been studied for various cases of A (see [10], [1]) and finally Turull settled the conjecture for almost all A in a sequence of papers. A complete list of the results related to this conjecture is given in [12].

The case that $|G|$ and $|A|$ are not necessarily coprime has also been studied. By a result due to Bell and Hartley [2], if A is any nonnilpotent finite group, then there exists a finite group G of arbitrarily large Fitting length on which A acts fixed point freely and noncoprimely. This led to a new Fitting length conjecture without the coprimeness condition which asserts that if A is a finite nilpotent group acting fixed point freely on a finite solvable group G by automorphisms then $f(G) \leq \ell(A)$. It should be noted that if A is solvable, $\ell(A)$ coincides with the number of primes dividing the order of A, counted with multiplicities. A special case of this conjecture has been treated firstly by K.N. Cheng [3] where A is cyclic of order a product of two primes. Later we proved that the conjecture is true in the case where A is cyclic of order a product of three distinct primes [5]. According to Dade's fundamental paper [4], a more appropriate way of studying the Fitting length of G is to consider the action of A, not on G, but on a sequence of sections of G which is called an *A-Fitting chain* of G. By pursuing the ideas of both [4] and [11], under the additional assumption that the Sylow 2-subgroups of G are abelian we settled two special cases of this conjecture in the first of which A is abelian of squarefree exponent coprime to 6 ([6], [7]) and in the second A is abelian

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of order a product of three odd primes [7]. The proof of these results is based on Theorem 3 of [6] assuring the existence of a sufficiently long A-Fitting subchain with sections centralized by a subgroup B of A of prime order. In the present article, we extend Theorem 3 of [6] to the case where B is cyclic of prime power order (see Theorem 3.3) and as an application we obtain our main result Theorem 4.2 :

Let A be a cyclic group of order $p^n q$, for prime numbers p and q coprime to 6*, acting by automorphisms on a finite solvable group G whose Sylow* 2*-subgroups are abelian. If A acts fixed point freely on G, then $f(G) \leq \ell(A)$.*

Throughout the article, all groups are finite. The notation and terminology are mostly standard with the following exceptions taken from [6] and [7]:

Let G be a group.

(i) We write $\tilde{G}$ for the Frattini factor group of G.

(ii) If H and K are subgroups of G, then for all $n \in \mathbb{N}$, we set $[H,K]^n = [H,K,...,K]$ where the last expression contains n-copies of K.

(iii) We write (S on G) whenever a group S acts on G. Suppose that G is solvable and S acts also on another solvable group H. We say (S on G) and (S on H) are *weakly equivalent* and write (S on G) $\equiv_w$ (S on H), if each nontrivial irreducible component of (S on G) is S-isomorphic to an irreducible component of (S on H) and vice versa.

2. Lemmas

In this section we establish the results that we need to prove our major theorems Theorem 3.1, Theorem 3.2, Theorem 3.3, and Theorem 3.4.

The following lemma contains the essential part of Theorem 3.1. It is obtained by combining Proposition 3.10 in [4] and Lemma 1.3 in [8].

Lemma 2.1. *Let $S \lhd S\langle\alpha\rangle$ where S is an s-group for some prime s, $\Phi(S) \leq Z(S)$, and $\langle\alpha\rangle$ is cyclic of order p^n for an odd prime p. Suppose that V is a $kS\langle\alpha\rangle$-module for a field k of characteristic different from s. Then $C_V(\alpha) \neq 0$ if one of the following is satisfied for $z = \alpha^{p^{n-1}}$:*

(i) $[Z(S), z]$ is nontrivial on V.

(ii) $[S, z]^{p-1}$ is nontrivial on V and $p = s$.

Furthermore, if $S\langle\alpha\rangle$ acts irreducibly on V or the characteristic of k is different from p, then we also have $(C$ on $C_V(\alpha)) \equiv_w (C$ on $V)$ where

$$C = C_D(\alpha) \quad for\ D = \begin{cases} S & , \quad \text{when (i) holds} \\ [S,z]^{p-1} & , \quad \text{when (ii) holds} \end{cases}.$$

Proof. We may assume that V is an irreducible $S\langle\alpha\rangle$-module. We suppose first that (i) holds. Since $[Z(S), z]$ is nontrivial on V and V is an irreducible $S\langle\alpha\rangle$-module, the stabilizer of any homogeneous component of $V|_{Z(S)}$ is S, whence

V is induced from an irreducible S-module W. It follows from Mackey's theorem that $V|_{C\times\langle\alpha\rangle}$ and $W|_C \otimes k\langle\alpha\rangle$ are $C \times \langle\alpha\rangle$-isomorphic. Hence $C_V(\alpha)$ and $W|_C \otimes C_{k\langle\alpha\rangle}(\alpha)$ are also $C \times \langle\alpha\rangle$-isomorphic. Thus as $C_{k\langle\alpha\rangle}(\alpha) \neq 0$, we have $C_V(\alpha) \neq 0$. Moreover $(C \text{ on } V) \equiv_w (C \text{ on } C_V(\alpha))$, since both V and $C_V(\alpha)$ are multiples of the same C-module $W|_C$. This proves the claim when (i) holds.

We suppose next that (ii) holds, and that $[Z(S), z]$ is trivial on V. Now $D = [S, z]^{p-1}$. For $S_1 = [S, z]^{p-2}$, Lemma 5.37 of [4] implies that

$$[D, S_1] \leq [\phi(S), z]^{p-2} \leq [\phi(S), z].$$

Then D is contained in $Z(S_1)$ modulo $[Z(S), z]$. Next, we notice that there is a collection $\{V_1, ..., V_l\}$ of irreducible $S_1\langle\alpha\rangle$-submodules such that $V = \oplus_{i=1}^{l} V_i$ since $V|_{S_1\langle\alpha\rangle}$ is completely reducible. Pick $i \in \{1, ..., l\}$ and let $\overline{S_1} = S_1/\text{Ker}(S_1$ on $V_i)$. We have $\overline{D} \leq Z(\overline{S_1})$ and hence $\overline{C} \leq Z(\overline{S_1}\langle\alpha\rangle)$. It follows that $V_i|_{\overline{C}}$ is homogeneous. If D is trivial on V_i, then so is C and hence the equivalence $(C \text{ on } C_{V_i}(\alpha)) \equiv_w (C \text{ on } V_i)$ holds. Suppose now that D is not trivial on V_i. We then have $C_{V_i}(\alpha) \neq 0$ by Lemma 1.3 of [8] applied to the action of $\overline{S}_1\langle\alpha\rangle$ on V_i. It follows that $(C \text{ on } C_{V_i}(\alpha)) \equiv_w (C \text{ on } V_i)$ since $V_i|_{\overline{C}}$ is homogeneous. Thus as i is arbitrary in $\{1, ..., l\}$ and $V = \oplus_{i=1}^{l} V_i$, we have the equivalence $(C$ on $C_V(\alpha)) \equiv_w (C$ on $V)$. This completes the proof. □

Next we describe a slightly modified version of Lemma 5.30 in [4].

Lemma 2.2. *Let $S \lhd SA$ and let V be an irreducible kSA-module for a field k. If E is an A-invariant subgroup of $Z(S)$ and U is a nonzero EA-submodule of V, then $Ker(E \text{ on } V) = Ker(E \text{ on } U)$.*

Proof. The module V is a direct sum of homogeneous S-modules permuted by A as $V|_S$ is completely reducible. Since $E \leq Z(S)$ it follows that $W|_E$ is homogeneous for each S-homogeneous component W in this decomposition. It should be noted that $U|_E$ is a submodule of $V|_E$ and that $V|_E$ is completely reducible as $E \lhd SA$.

Let now $\{X_1, ..., X_r\}$ be a complete set of nonisomorphic irreducible E-submodules of V, and let the $W_1, ..., W_r$ be E-homogeneous components of $V|_E$ containing modules isomorphic to $X_1, ..., X_r$, respectively. Then $U|_E = \oplus_{i=1}^{r} U \cap W_i$. We may assume that $U \cap W_1 \neq 0$. Since the module X_1^a is an irreducible submodule of $U|_E$ for each $a \in A$ and the E-modules, $U \cap W_1, ..., U \cap W_r$ are permuted by A, we have $U \cap W_i \neq 0$ for each $i = 1, ..., r$. Consequently, the only irreducible E-constituents of U are precisely the elements of $\{X_1, ..., X_r\}$ and $\text{Ker}(E \text{ on } U) = \text{Ker}(E \text{ on } V)$ as desired. □

The following lemma is crucial in proving Theorem 3.1.

Lemma 2.3. *Let $S \lhd S\langle\alpha\rangle$ where $\langle\alpha\rangle$ is cyclic of prime power order p^n. Let V be a $kS\langle\alpha\rangle$-module for a field k of characteristic different from p, and let Ω be an $S\langle\alpha\rangle$-stable subset of V^*. Set $V_0 = \bigcap \{ \, Ker\, f \mid f \in \Omega - C_\Omega(z) \, \}$*

for $z = \alpha^{p^{n-1}}$. *If there exists a nonzero* f *in* Ω *and* $x \in S$ *such that* $f(V_0) \neq 0$ *and* $[x, a, z] \notin C_S(f)$ *for each* $1 \neq a \in \langle \alpha \rangle$, *then* $C_V(\alpha) \nsubseteq V_0$.

Proof. The assumption $f(V_0) \neq 0$ implies that $f \in C_\Omega(z)$ and hence $C_S(f)$ is normalized by $\langle z \rangle$. As $[x, a, z] \notin C_S(f)$ for each $1 \neq a \in \langle \alpha \rangle$, we have $[x, a] \notin C_S(f)$ for each $1 \neq a \in \langle z \rangle$. In particular, $bxf \notin C_\Omega(z)$ for each $b \in \langle z \rangle$. Pick an element $a \in \langle \alpha \rangle$. If $axf \notin C_\Omega(z)$, then $xf \in C_\Omega(z)$, which is a contradiction. So it must be true that $axf \notin C_\Omega(z)$, for any $a \in \langle \alpha \rangle$. It is now straightforward to verify that $g = \sum_{a \in \langle \alpha \rangle} axf \in C_\Omega(\alpha)$ and therefore $[V, \alpha] \subseteq \mathrm{Ker}\, g$. On the other hand $V = [V, \alpha] \oplus C_V(\alpha)$ and hence either $g = 0$ or $C_V(\alpha) \nsubseteq \mathrm{Ker}\, g$. If the latter holds, that is, $V_0 \subseteq \mathrm{Ker}(axf)$ for each $a \in \langle \alpha \rangle$, we then have $C_V(\alpha) \nsubseteq V_0$ as claimed. If the former holds, then $x^{-1}g = 0$ and hence $f = -\sum_{1 \neq a \in \langle \alpha \rangle} x^{-1}axf$. Since $[x, a, z] \in C_S(f)$, $x^{-1}axf \notin C_\Omega(z)$ for each $1 \neq a \in \langle \alpha \rangle$. This forces that $f(V_0) = 0$, contrary to our assumption. So the lemma is established. □

We now restate Theorem 4.1 of [10] in the most appropriate form for our purposes.

Lemma 2.4. *Let* $S \lhd SA$ *where* A *is an abelian group and* S *is an* s*-group for some prime* s *which is coprime to* $|A|$. *Assume that* S *is abelian when* $s = 2$. *Let* V *be an irreducible* kSA*-module where* k *is a splitting field for all subgroups of* SA *and is of characteristic not dividing* $|SA|$. *Suppose that* S *acts nontrivially and* A *acts fixed point freely on* V. *Then there is a nontrivial subgroup* D *of* A *such that* $[S, D]$ *acts trivially on* V.

3. Proofs of Major Theorems

We are now ready to prove the results that are indispensable for the proof of the main result Theorem 4.2 of this paper.

The next theorem generalizes Theorem 2.1.A in [11], Theorem 2.1 in [7] (see also Theorem 2 in [6]).

Theorem 3.1. *Let* $S \lhd S\langle \alpha \rangle$ *where* S *is an* s*-group with* $\phi(\phi(S)) = 1$, $\phi(S) \leq Z(S)$ *and* $\langle \alpha \rangle$ *is cyclic of order* p^n *for primes* s *and* p. *Assume that either* $s = p \geq 5$ *or* $s \neq p$ *and* p *is odd. Assume further that if* $s = 2$, *either* S *is abelian or* p *is not a Fermat prime.*

Let V *be a* $kS\langle \alpha \rangle$*-module such that* $[S, z]^{p-1}$ *acts nontrivially on each irreducible submodule of* $V|_S$ *where* $z = \alpha^{p^{n-1}}$, *for a field* k *of characteristic not dividing* ps, *and let* Ω *be an* $S\langle \alpha \rangle$*-stable subset of* V^* *spanning* V^*.
Set $V_0 = \bigcap\{\, Ker\, f \mid f \in \Omega - C_\Omega(z)\,\}$. *Then*

$$C_V(\alpha) \nsubseteq V_0 \quad \textit{and} \quad (C_D(\alpha) \text{ on } C_V(\alpha)/C_{V_0}(\alpha)) \equiv_w (C_D(\alpha) \text{ on } V)$$

$$\textit{where } D = \begin{cases} [S, z]^{p-1} & , \quad \textit{when } s = p \\ S & , \quad \textit{otherwise} \end{cases}.$$

Proof. Assume that the theorem is false and choose a counter-example with minimum $dimV + |S\langle\alpha\rangle|$. We shall proceed in several steps. To simplify the notation we set $X = C_V(\alpha)/C_{V_0}(\alpha)$ and $C = C_D(\alpha)$.

Claim 1. We may assume that S acts faithfully, $S\langle\alpha\rangle$ acts irreducibly on V and that k is a splitting field for all subgroups of $S\langle\alpha\rangle$.

By induction applied to the action of $\overline{S}\langle\alpha\rangle$ on V, we get $C_V(\alpha) \nsubseteq V_0$ and $(C_{\overline{D}}(\alpha)$ on $X) \equiv_w (C_{\overline{D}}(\alpha)$ on $V)$ for $\overline{S} = S/\mathrm{Ker}(S$ on $V)$. We then have $(C$ on $X) \equiv_w (C$ on $V)$ as $\overline{C} \leq C_{\overline{D}}(\alpha)$. This contradiction shows that S is faithful on V.

Since V is completely reducible as an $S\langle\alpha\rangle$-module, we have a collection $\{V_1, ..., V_l\}$ of irreducible $S\langle\alpha\rangle$-submodules of V such that $V = \oplus_{i=1}^{l} V_i$. By hypothesis $[S, z]^{p-1}$ acts nontrivially on each irreducible constituent of $V_i|_S$, and hence it acts nontrivially on each V_i, for $i = 1, ..., l$. It is easy to observe that $\Omega_i = \Omega|_{V_i}$ is an $S\langle\alpha\rangle$-stable subset of V_i^* and $\langle\Omega_i\rangle = V_i^*$. We define $(V_i)_0$ similar to V_0 as $(V_i)_0 = \bigcap\{\ \mathrm{Ker}\, h \mid h \in \Omega_i - C_{\Omega_i}(z)\ \}$. If V is not irreducible as an $S\langle\alpha\rangle$-module, it follows by induction that $C_{V_i}(\alpha) \nsubseteq (V_i)_0$ and that $(C$ on $C_V(\alpha)/C_{(V_i)_0}(\alpha)) \equiv_w (C$ on $V_i)$ for $i = 1, ..., l$. Notice that $(V_i)_0 \supseteq V_i \cap V_0$. Thus we have $(C$ on $X_i) \equiv_w (C$ on $V_i)$ for $X_i = C_{V_i}(\alpha)/C_{V_i\cap V_0}(\alpha)$. Since $V = \oplus_{i=1}^{l} V_i$ and $X \cong \oplus_{i=1}^{l} X_i$, we conclude that $(C$ on $X) \equiv_w (C$ on $V)$, a contradiction. Therefore we can regard V as an irreducible $S\langle\alpha\rangle$-module.

Claim 2. For each $x \in Z(S)$, there is a nontrivial element a in $\langle\alpha\rangle$ such that $[x, a, z] = 1$. In particular, $[Z(S), z, z] = 1$.

Assume the claim to be false. Note that $S_1 = Z(S)C$ is an $\langle\alpha\rangle$-invariant subgroup of S, $C \lhd S_1\langle\alpha\rangle$ and $V|_{S_1\langle\alpha\rangle}$ is completely reducible. Let V_i be an irreducible $S_1\langle\alpha\rangle$-submodule of V and let W be a C-homogeneous component of V_i. Since $Z(S)\langle\alpha\rangle \leq C_{S_1\langle\alpha\rangle}(C) \leq N_{S_1\langle\alpha\rangle}(W)$, $V_i|_C$ is homogeneous. Moreover, by Lemma 2.2 applied to the action of $S\langle\alpha\rangle$ on V, we get $\mathrm{Ker}(Z(S)$ on $V_i) = \mathrm{Ker}(Z(S)$ on $V) = 1$.

We observe next that $C_{Z(S)}(f) = 1$ for each $0 \neq f \in C_\Omega(\alpha)$: To see this, consider $\langle f\rangle = \{cf \mid c \in k\}$, as a $C_{Z(S)}(f)\langle\alpha\rangle$-submodule of V^*.
By applying Lemma 2.2 to the action of $S\langle\alpha\rangle$ on V^* we conclude that $C_{Z(S)}(f) = \mathrm{Ker}(C_{Z(S)}(f)$ on $V^*) = 1$, as desired.

Since $[Z(S), z] \neq 1$, $[Z(S_1), z]$ is nontrivial on V_i. Then Lemma 2.1 implies that $C_{V_i}(\alpha) \neq 0$. If $C_{V_i}(\alpha) \nsubseteq V_0$ holds, as $V_i|_C$ is homogeneous we have $(C$ on $C_{V_i}(\alpha)/C_{V_i\cap V_0}(\alpha)) \equiv_w (C$ on $V_i)$. Hence there is at least one irreducible $S_1\langle\alpha\rangle$-submodule V_i of the completely reducible module $V|_{S_1\langle\alpha\rangle}$ such that $C_{V_i}(\alpha) \subseteq V_0$. Notice that $C_{V_i}(\alpha) \neq 0$ implies $V_i \cap V_0 \neq 0$. We define $(V_i)_0$ similar to V_0 as $(V_i)_0 = \bigcap\{\ \mathrm{Ker}\, h \mid h \in \Omega_i - C_{\Omega_i}(z)\ \}$ for $\Omega_i = \Omega|_{V_i}$. Since $V_i \cap V_0 \subseteq (V_i)_0$ we have $(V_i)_0 \neq 0$. We can choose $f \in \Omega$ such that $f((V_i)_0) \neq 0$ with $f_i = f|_{V_i} \in C_{\Omega_i}(z)$. Since V_i is completely reducible as an $S_1\langle z\rangle$-module, we have a collection $\{U_{i1}, ..., U_{it}\}$ of irreducible $S_1\langle z\rangle$-submodules such that $V_i = \oplus_{j=1}^{t} U_{ij}$, and thus $V_i^\star = \oplus_{j=1}^t U_{ij}^*$. Set $f_{ij} = f_i|_{U_{ij}}$ and consider $\langle f_{ij}\rangle = \{cf_{ij} \mid c \in k\}$ as a $C_{Z(S)}(f_{ij})\langle z\rangle$-submodule of U_{ij}^*. An application of Lemma 2.2 gives that $C_{Z(S)}(f_{ij}) = \mathrm{Ker}(C_{Z(S)}(f_{ij})$ on $U_{ij}^*)$ for each $j = 1, ..., t$. In fact, we have

$$C_{Z(S)}(f_i) = \bigcap_{j=1}^{t} \mathrm{Ker}(C_{Z(S)}(f_{ij}) \text{ on } U_{ij}^*) \leq \mathrm{Ker}(C_{Z(S)}(f_i) \text{ on } V_i^*) = 1.$$

Note also that, by our assumption, for each $1 \neq a \in \langle\alpha\rangle$ there exists $x \in Z(S)$ with $[x, a, z] \neq 1$. By Lemma 2.3 applied to the action of $S_1\langle\alpha\rangle$ on V_i with f_i and Ω_i, it follows that $C_{V_i}(\alpha) \nsubseteq (V_i)_0$. On the other hand $C_{V_i}(\alpha) \subseteq V_0$ as $V_i \cap V_0 \subseteq (V_i)_0$. This contradiction completes the proof.

Claim 3. $s \neq p$.

Suppose that this claim is false. Then obviously $D = [S, z]^{p-1} \neq 1$, and hence $C = C_D(\alpha) \neq 1$. To simplify the notation we set $S_1 = [S, z]^{p-3}[\phi(S), z]$. We observe by Lemma 5.37 in [4] that $[[S, z]^{p-3}, D] \leq [\phi(S), z]^{p-3} = 1$ as $[Z(S), z, z] = 1$. Therefore $D \leq Z(S_1)$.

We have a collection $\{V_1, ..., V_l\}$ of irreducible $S_1\langle\alpha\rangle$-modules such that $V = \oplus_{i=1}^l V_i$. It should be noted that $C \lhd S_1\langle\alpha\rangle$ and hence $V|_C$ is completely reducible. In particular, $V_i|_C$ is homogeneous for each $i \in \{1, ..., l\}$ by the fact that $C \leq Z(S_1\langle\alpha\rangle)$.

We choose $i \in \{1, ..., l\}$ such that C acts nontrivially on V_i and set $X_i = C_{V_i}(\alpha)/C_{V_i\cap V_0}(\alpha)$. Now $(C \text{ on } X_i) \equiv_w (C \text{ on } C_{V_i}(\alpha))$, as $V_i|_C$ is homogeneous. Then $V_i \cap V_0 \neq 0$, and hence there exists $f \in C_\Omega(z)$ such that $f(V_i \cap V_0) \neq 0$. By the fact that $V_i|_{S_1\langle z\rangle}$ is completely reducible, there is a collection $\{U_{i1}, ..., U_{it}\}$ of irreducible $S_1\langle z\rangle$-submodules such that $V_i = \oplus_{j=1}^t U_{ij}$. Note that $V_i^* = \oplus_{j=1}^t U_{ij}^*$. Let $f|_{V_i} = f_i$ and $f_{ij} = f_i|_{U_{ij}}$ for some j such that D is nontrivial on U_{ij}. We consider $\langle f_{ij}\rangle = \{cf_{ij} \mid c \in k\}$ as a $(C_{DN}(f_{ij}))\langle z\rangle$-submodule of U_{ij}^* for $N = [\phi(S), z]$. By Lemma 2.2, it follows that $C_{DN}(f_{ij}) = \mathrm{Ker}(C_{DN}(f_{ij})$ on $U_{ij}^*)$. Then $C_{DN}(f_{ij})$ is properly contained in DN. We set $\overline{S} = S/N$, and take $\overline{y} \in \overline{D} - \overline{C_{DN}(f_{ij})}$. This implies the existence of $\overline{x} \in \overline{[S, z]}^{p-3} = [\overline{S}, z]^{p-3} = \overline{S}_1$ with $\overline{y} = [\overline{x}, z, z]$. By Lemma 5.37 in [4],

$$[[\overline{S}, z]^{p-2}, \overline{S}_1] = [[\overline{S}, z]^{p-2}, [\overline{S}, z]^{p-3}] \leq [\overline{\phi(S), z}]^{p-4} \leq \overline{N} = \overline{1}$$

and hence $\overline{C_{DN}(f_{ij})} \lhd \overline{S_1}$.

Let $\overline{x} = xN$ for $x \in [S, z]^{p-3}$. We claim that for any $1 \neq b \in \langle\alpha\rangle$, $[x, b, z] \notin C_S(f_{ij})$: Towards a contradiction, we assume that there exists $1 \neq b \in \langle\alpha\rangle$ such that $[x, b, z] \in C_S(f_{ij})$. Now $\overline{y} \in \overline{C_T(f_{ij})}$, for $T = [S, z]^{p-2}$. Since

$$\overline{C_T(f_{ij})N \cap DN} = \overline{C_{TN}(f_{ij}) \cap DN} = \overline{C_{DN}(f_{ij})}$$

we have $\overline{y} \in \overline{C_{DN}(f_{ij})}$, which is not possible. Thus $[x, b, z] \notin C_S(f_{ij})$ for any $1 \neq b \in \langle\alpha\rangle$.

After this preparation we apply Lemma 2.3 to the action of $S_1\langle\alpha\rangle$ on V_i with $\Omega_i = \Omega|_{V_i}$ and f_i, and obtain $C_{V_i}(\alpha) \not\subseteq V_0$. This gives

$$C_V(\alpha) \not\subseteq V_0 \quad \text{and} \quad (C \text{ on } C_V(\alpha)/C_{V_0}(\alpha)) \equiv_w (C \text{ on } C_V(\alpha))$$

by using the fact that $(C \text{ on } X_i) \equiv_w (C \text{ on } C_{V_i}(\alpha))$ holds. Applying Lemma 2.1 to the action of $S\langle\alpha\rangle$ on V, we see that $(C \text{ on } C_V(\alpha)) \equiv_w (C \text{ on } V)$, whence the equivalence $(C \text{ on } V) \equiv_w (C \text{ on } C_V(\alpha)/C_{V_0}(\alpha))$ holds. So the claim is established.

Claim 4. The theorem follows.

We observed that $s \neq p$ and $[Z(S), z] = [Z(S), z, z] = 1$. Thus S is a nonabelian group which is a central product of the subgroups $[S, z]$ and $C_S(z)$.

Our next goal is to show that $[\phi(S), \alpha] = 1$: Assume otherwise. Every $\phi(S)$-homogeneous component of V is stabilized by $S\langle z\rangle$. If $V|_{\phi(S)}$ is homogeneous, then $\phi(S)$ acts by scalars on V, and hence $[\phi(S), \alpha] = 1$. Thus there is a proper subgroup B of $\langle\alpha\rangle$ which is the stabilizer of every $\phi(S)$-homogeneous component of V. By induction applied to the action of SB on V, we obtain

$$(C_S(B) \text{ on } C_V(B)/C_{V_0}(B)) \equiv_w (C_S(B) \text{ on } V) \text{ and } C_V(B) \not\subseteq V_0.$$

This gives the equivalence $(C \text{ on } C_V(B)/C_{V_0}(B)) \equiv_w (C \text{ on } V)$, as we have $C = C_D(\alpha) = C_S(\alpha) \subseteq C_S(B)$. Since $V = W^{S\langle\alpha\rangle}$ for some irreducible SB-module W, the $C \times \langle\alpha\rangle$-modules $V|_{C\times\langle\alpha\rangle}$ and $W|_{CB} \otimes k(\langle\alpha\rangle/B)$ are isomorphic. It is straightforward to verify that

$$C_V(B) \cong C_W(B) \otimes k(\langle\alpha\rangle/B) \text{ and } C_V(\alpha) \cong C_W(B) \otimes C_{k(\langle\alpha\rangle/B)}(\alpha).$$

Then

$$(C \,\text{on}\, C_V(B)) \equiv_w (C \,\text{on}\, C_V(\alpha)) \,\text{and hence}\, (C \,\text{on}\, V) \equiv_w (C \,\text{on}\, C_V(\alpha)/C_{V_0}(\alpha)).$$

This forces that $C_V(\alpha) \subseteq V_0$. Pick an element $v_1 \in C_V(B) - V_0$. Then there exists $f \in \Omega - C_\Omega(z)$ such that $f(v_1) \neq 0$. Let $\langle\alpha\rangle = \bigcup_{i=1}^{m} Ba_i$ be a coset decomposition of $\langle\alpha\rangle$ with respect to B, and set $v = \sum_{i=1}^{m} v_1^{a_i}$. It is straightforward to verify that $v \in C_V(\alpha)$ and hence $v \in V_0$. Let $\{W_1, \ldots, W_m\}$ be the collection of $\Phi(S)$-homogeneous components of V and λ_i be the irreducible character of $\Phi(S)$ corresponding to W_i for $i = 1, \ldots, m$. Then, for each $x \in \Phi(S)$, we have

$$0 = (xf)(v) = f(x^{-1}v) = \sum_{i=1}^{m} f(x^{-1}v_i) = \sum_{i=1}^{m} \lambda_i(x^{-1})f(v_i)$$

where λ_i, $i = 1, \ldots, m$, are linear characters with $\lambda_i(x)v_i = xv_i$. Since λ_i, $i = 1, \ldots, m$, are linearly independent, we get $f(v_i) = 0$ for all i, a contradiction as $f(v_1) \neq 0$. This shows that $[\phi(S), \alpha] = 1$.

Set $S_1 = [S, z]$. Then as $Z(S) \leq C_S(z)$ and $[S_1, C_S(z)] = 1$, we have $[Z(S_1), z] = 1$. Note that S_1 is nonabelian and $1 \neq S_1' \leq \phi(S_1) \leq \phi(S)$. Since $V = \langle C_V(x) \mid 1 \neq x \in \phi(S)\rangle$, it is easy to verify that $\phi(S)$ is cyclic of order s and hence

$S_1' = \phi(S_1)$. We also have $Z(S_1) = \phi(S_1)$, because otherwise $1 \neq Z(S_1)/\phi(S_1) \leq C_{S_1/\phi(S_1)}(z) = 1$, a contradiction. Thus S_1 is extraspecial.

Applying Lemma 2.4 to the action of $S_1\langle\alpha\rangle$ on V, we get $C_V(\alpha) \neq 0$. We claim next that $C_V(\alpha) \nsubseteq V_0$: Since $V_0 \neq 0$, there exists $0 \neq f \in C_\Omega(z)$ with $f(V_0) \neq 0$. We define $C^*(f) = \{y \in S_1 \mid yf = cf \text{ for some } c \in k\}$. Let $y_1, y_2 \in C^*(f)$. Now $[y_1, y_2]$ has eigenvalue 1 on V^*. Since $[y_1, y_2] \in \phi(S_1)$ is cyclic of order s and since $\phi(S_1)$ acts faithfully and by scalars on V, we have $[y_1, y_2] = 1$. That is, $C^*(f)$ is abelian. So $C^*(f)$ is contained in a maximal abelian subgroup of S_1. If $|S_1| = s^{2t+1}$, a maximal abelian subgroup of S_1 has order s^{t+1}. Now, $|N| \leq s^t$ for $N = C^*(f)/\phi(S_1)$. We define the map $\alpha_b : \tilde{S}_1 \to \tilde{S}_1$ by $\alpha_b(\tilde{x}) = [\tilde{x}, b]$ for each $1 \neq b \in \langle\alpha\rangle$. Since $C_{\tilde{S}_1}(z) = 1$, the map α_b is injective. For any irreducible $\langle\alpha\rangle$-submodule U_1 of $\tilde{S}_1$, we have $\tilde{S}_1 = U_1 \perp U_2 \perp \ldots \perp U_l$; an orthogonal sum of irreducible $\langle\alpha\rangle$-modules each of which is a symplectic space of dimension $2m$ for some positive integer m on which α acts regularly and faithfully. We see that $|\alpha|$ divides $s^m + 1$ and hence $|\alpha| - 1 = p^n - 1 \leq s^t$. Since α_b is injective, we have the equality $\bigcup_{1 \neq b \in \langle\alpha\rangle} \alpha_b^{-1}(N - \{1\}) \cup \{1\} = \bigcup_{1 \neq b \in \langle\alpha\rangle} \alpha_b^{-1}(N)$. Obviously this set contains at most $(|\alpha| - 1)(|N| - 1) + 1 < s^{2t} = |\tilde{S}_1|$ elements and hence there exists $\tilde{x} = x\phi(S_1) \in \tilde{S}_1$ with $[\tilde{x}, b] \notin N$, that is, $\tilde{x} \notin \alpha_b^{-1}(N)$ for each $1 \neq b \in \langle\alpha\rangle$. Now $[x, b] \notin C^*(f)$ for each $1 \neq b \in \langle\alpha\rangle$. On the other hand $[S_1, z] = S_1$ implies $Z(S_1) = C_{S_1}(z) = \phi(S_1) \leq C^*(f)$, that is $C_{S_1/C^*(f)}(z) = 1$. It follows that $[x, b, z] \notin C_S(f)$ for each $1 \neq b \in \langle\alpha\rangle$, because otherwise $[x, b]C^*(f)$ is centralized by z. After this preparation
we apply Lemma 2.3 and conclude that $C_V(\alpha) \nsubseteq V_0$, as desired.

Since $[\phi(S), \alpha] = 1$, the group S is a central product of C and $[S, \alpha]$. Notice next that $C \neq 1$, because otherwise $(C \text{ on } V) \equiv_w (C \text{ on } C_V(\alpha)/C_{V_0}(\alpha))$, a contradiction. On the other hand, as $[C, [S, \alpha]\langle\alpha\rangle] = 1$, the C-module $V|_C$ is homogeneous and hence the equivalence $(C \text{ on } C_V(\alpha)/C_{V_0}(\alpha)) \equiv_w (C \text{ on } V)$ holds as $C_V(\alpha) \nsubseteq V_0$. This final contradiction completes the proof of claim 4. □

The next theorem is a generalization of Theorem 2.1 in [7] (see also Theorem 2 in [6]).

Theorem 3.2. *Let $S \lhd S\langle\alpha\rangle$, where S is an s-group, $\langle\alpha\rangle$ is cyclic of order p^n for distinct primes s and p, $\Phi(\Phi(S)) = 1$ and $\Phi(S) \leq Z(S)$. Assume that if $s = 2$, either S is abelian or p is not a Fermat prime. Let V be an irreducible $kS\langle\alpha\rangle$-module on which $[S, z]$ acts nontrivially, where k is a field of characteristic different from s for $z = \alpha^{p^{n-1}}$. Then*

$$[V, z]^{p-1} \neq 0 \quad \textit{and} \quad (C_S(\alpha) \textit{ on } V) \equiv_w (C_S(\alpha) \textit{ on } C_{[V,z]^{p-1}}(\alpha)).$$

Proof. We use induction on $\dim V + |S\langle\alpha\rangle|$. Then we may assume that k is a splitting field for all subgroups of $S\langle\alpha\rangle$ and that S acts faithfully on V.
To simplify the notation we set $C = C_S(\alpha)$ and $U = [V, z]^{p-1}$.

Claim 1. $[Z(S), z] = 1$.

Assume the contrary. The stabilizer of any $Z(S)$-homogeneous component of $V|_{Z(S)}$ is S, whence V is induced from an irreducible S-module W. By Mackey's theorem, the $C\times\langle\alpha\rangle$-modules $V|_{C\times\langle\alpha\rangle}$ and $W|_C\otimes k\langle\alpha\rangle$ are isomorphic.It follows that the modules U and $W|_C\otimes[k\langle\alpha\rangle,z]^{p-1}$ are also $C\times\langle\alpha\rangle$-isomorphic. Then as $[k\langle\alpha\rangle,z]^{p-1}\neq 0$, we have $U\neq 0$. It is also clear that $C_U(\alpha)$ is $C\times\langle\alpha\rangle$-isomorphic to $W|_C\otimes C_{[k\langle\alpha\rangle,z]^{p-1}}(\alpha)$. Thus as both V and $C_U(\alpha)$ are multiples of the same C-module $W|_C$, we conclude that the equivalence $(C$ on $V)\equiv_w(C$ on $C_U(\alpha))$ holds. This contradiction proves the claim.

Claim 2. $[Z(S),\alpha]=1$.

Assume this claim is false. By hypothesis $[S,z]\neq 1$ and hence S is nonabelian. Let Ω be the set of all homogeneous components of $V|_{Z(S)}$. Note that for any $W\in\Omega$, $N_{S\langle\alpha\rangle}(W)=SN_{\langle\alpha\rangle}(W)$. Now the group $B=N_{\langle\alpha\rangle}(W)$ stabilizes each member of Ω. Furthermore $1\neq B\neq\langle\alpha\rangle$ as $[Z(S),z]=1$, but $[Z(S),\alpha]\neq 1$. We observe that $[S,z]$ is nontrivial on each $W\in\Omega$, because otherwise it is trivial on each member of Ω. Applying induction to the action of SB on W, we obtain

$$[W,z]^{p-1}\neq 0 \text{ and } (C_S(B) \text{ on } W)\equiv_w (C_S(B) \text{ on } C_{[W,z]^{p-1}}(B)).$$

It follows that $(C$ on $W)\equiv_w(C$ on $C_{[W,z]^{p-1}}(B))$. Since W is arbitrary in Ω, the equivalence $(C$ on $V)\equiv_w(C$ on $C_U(B))$ holds.

By Mackey's theorem, the $C\times\langle\alpha\rangle$-modules $V|_{C\times\langle\alpha\rangle}$ and $W|_{CB}\otimes k(\langle\alpha\rangle/B)$ are isomorphic. We then have $U\cong[W,z]^{p-1}\otimes k(\langle\alpha\rangle/B)$ which yields that $C_U(B)\cong C_{[W,z]^{p-1}}(B)\otimes k(\langle\alpha\rangle/B)$ and $C_U(\alpha)\cong C_{[W,z]^{p-1}}(B)\otimes C_{k(\langle\alpha\rangle/B)}(\alpha)$. It is now easy to verify that $(C$ on $C_U(B))\equiv_w(C$ on $C_U(\alpha))$, which leads to the equivalence $(C$ on $V)\equiv_w(C$ on $C_U(\alpha))$, a contradiction.

Claim 3. $U=[V,z]^{p-1}=0$ *and* $\alpha=z$.

Since $[Z(S),\alpha]=1$, we have $C\neq 1$ and the group S is a central product of the subgroups $[S,\alpha]$ and C. In particular $C\lhd S\langle\alpha\rangle$ and $V|_C$ is homogeneous. Then, if $U\neq 0$, we have $C_U(\alpha)\neq 0$ and $(C$ on $V)\equiv_w(C$ on $C_U(\alpha))$, which is a contradiction. Therefore, $U=0$.

Assume now that $\alpha\neq z$. Let M be an irreducible $S\langle z\rangle$-component of V on which $[S,z]$ acts nontrivially. By induction applied to the action of $S\langle z\rangle$ on M, we obtain $[M,z]^{p-1}\neq 0$, which is contrary to the fact that $U=0$. Thus $\alpha=z$.

Claim 4. The theorem follows.

Recall that the group S is nonabelian and $1\neq\phi(S)\leq Z(S\langle\alpha\rangle)$, as $[Z(S),\alpha]=1$. It is also clear that $\phi(S)$ is cyclic of order s, since V is an irreducible $kS\langle\alpha\rangle$-module on which S acts faithfully. As the group S is a central product of S_1 and C for $S_1=[S_1,\alpha]$, we observe that $S_1'=Z(S_1)=\phi(S_1)=C_{S_1}(\alpha)$ is cyclic of order s, that is, S_1 is extraspecial.

Now $V=U_1\otimes_k U_2$ where U_1 is an irreducible $S_1\langle\alpha\rangle$-module on which S_1 acts faithfully and U_2 is an irreducible C-module. Since $[U_1,\alpha]^{p-1}\leq U=0$, by

Theorem IX.3.2 in [9] applied to the action of $S_1\langle\alpha\rangle$ on U_1, we get $s = 2$, contrary to the fact that S_1 is nonabelian. □

0,2cm

We now can strengthen Theorem 2.3 in [7](see also Theorem 3 in [6]).

Theorem 3.3. *Let $G \lhd GA$, let $\langle\alpha\rangle \trianglelefteq A$ be of prime power order p^n, and let $P_1, ..., P_t$ be an A-Fitting chain of G such that $[P_1, z] \neq 1$ where $z = \alpha^{p^{n-1}}$, P_i is a p_i-group for a prime p_i, and $t \geq 3$. Assume that $p \geq 5$ whenever $p_i = p$ for some $i \in \{1, ..., t\}$. Assume further that if $p_i = 2$, either P_i is abelian or p is not a Fermat prime. Then there are sections $D_{i_0}, ..., D_t$ of $P_{i_0}, ..., P_t$, respectively, forming an A-Fitting chain of G such that α centralizes each D_j*

for $j = i_0, ..., t$ where $i_0 = \begin{cases} 2 & , \text{ if } \quad p_1 \neq p \\ 3 & , \text{ if } \quad p_1 = p \end{cases}$.

Proof. The procedure of Theorem 3 in [6] is adopted again with the modifications that Lemma 2.1, Theorem 3.1 and Theorem 3.2 in [6] are replaced with Lemma 2.1, Theorem 3.1 and Theorem 3.2 of this paper, respectively, and the subgroups F_i of E_i for $i = 1, ..., t + 1$, are now defined as follows:

$$\begin{aligned} F_1 &= \{1\} \\ F_i &= C_{E_i}(\alpha) && , \text{ if } p_i \neq p \text{ and } i \geq 2 \\ F_2 &= C_{[E_2,z]^{p-1}}(\alpha) && , \text{ if } p_2 = p \\ F_i &= [C_{[E_i,z]^{p-1}}(\alpha), F_{i-1}] && , \text{ if } p_i = p \text{ and } i \geq 3 \end{aligned}$$

□

The next theorem is a slightly modified version of Lemma 4.5 in [7].

Theorem 3.4. *Let $S \lhd SA$ where S is a q-group for an odd prime q, $\Phi(S) \leq Z(S)$, and A is cyclic of order $p^n q$ for some prime p. Suppose that $[S, A_q]^{q-1} \nleq \Phi(S)$ and $[S, A_p] = S$ where A_p and A_q denote the Sylow p-and q-subgroups of A respectively. Let V be a kSA-module for a field k which is a splitting field for all subgroups of SA with characteristic not dividing $|SA|$. If $[S, A_q]^{q-1}$ acts nontrivially on V then $C_V(A) \neq 0$.*

Proof. Assume this claim to be false. To simplify the notation we set $\overline{S} = S/\text{Ker}(S \text{ on } V)$ and $D = [S, A_q]^{q-1}$. By Lemma 2.1 applied to the action of $\overline{S}A_q$ on V, we see that

$$C_V(A_q) \neq 0 \quad \text{and} \quad \text{Ker}(C_{\overline{D}}(A_q) \text{ on } C_V(A_q)) = \text{Ker}(C_{\overline{D}}(A_q) \text{ on } V).$$

Moreover we have

$$\text{Ker}([C_{\overline{D}}(A_q), A_p] \text{ on } C_V(A_q)) = \text{Ker}([C_{\overline{D}}(A_q), A_p] \text{ on } V).$$

Lemma 2.4 applied to the action of $[C_{\overline{D}}(A_q), A_p]A_p$ on $C_V(A_q)$ yields that $[C_{\overline{D}}(A_q), A_p] = 1$. Then $[\overline{D}, A_p] = 1$ by Thompson $A \times B$ Lemma and hence $\overline{D} \leq \Phi(\overline{S}) = \overline{\Phi(S)}$, which is contrary to our assumption. □

4. An Application

In this section we prove the main result of this paper as an application of the major theorems proved in Section 3.

Throughout the section, a sequence $P_1, ..., P_t$ of groups is called an *$\mathcal{F}$-chain of length t* if the following are satisfied:

(a) Each P_i, $i = 1, ..., t$, is a nontrivial p_i-group for some prime p_i.
(b) $[P_{i+1}, P_i] = P_{i+1}$, for $i = 1, ..., t-1$.
(c) $p_i \neq p_{i+1}$, for $i = 1, ..., t-1$.

It is easy to verify that the Fitting length of a finite solvable group G is the maximum of the lengths of all such $\mathcal{F}$-chains whose terms are sections of G.

Let A act on G by automorphisms. We call an $\mathcal{F}$-chain whose terms are all A-invariant sections of G, an *$\mathcal{F}(A)$-chain*.

Remark 4.1. When A normalizes a Sylow system of G, by a slight modification of Lemma 8.2 in [4], one can show the existence of an $\mathcal{F}(A)$-chain $P_1, ..., P_f$ of length $f = f(G)$, where $P_i = S_i/T_i$ is an A-invariant section of G, for each $i = 1, ..., f$, satisfying the following conditions:
(a) P_i is a nontrivial p_i-group for some prime p_i where $\Phi(P_i) \leq Z(P_i)$, $\Phi(\Phi(P_i)) = 1$ and P_i has exponent p_i when p_i is odd, for $i = 1, ..., f$.
(b) $p_i \neq p_{i+1}$, for $i = 1, ..., f-1$.
(c) $T_i = \mathrm{Ker}(S_i \text{ on } P_{i+1})$ for $i = 1, ..., f-1$, and $T_f = 1$ and $S_f \leq F(G)$.
(d) $[\Phi(P_{i+1}), S_i] = 1$ for $i = 1, ..., f-1$.
(e) $(\prod_{1 \leq j < i} S_j)A$ acts irreducibly on $\tilde{P}_i$.

Finally, we are ready to prove our main result.

Theorem 4.2. *Let A be a cyclic group of order $p^n q$, for prime numbers p and q coprime to 6, acting by automorphisms on a finite solvable group G whose Sylow 2-subgroups are abelian. If A acts fixed point freely on G, then $f(G) \leq \ell(A)$.*

Proof. By Theorem C in [7] we may assume that $n \geq 2$. Let $\ell = \ell(A)$ and $f = f(G)$. As indicated in Remark 4.1, there is an $\mathcal{F}(A)$-chain of length f in G. Since A is nilpotent, it is a Carter subgroup of the semidirect product of G by A. Therefore it is a Carter subgroup of any semidirect product of any A-invariant section of G by itself. This tells us that A acts fixed point freely on each section of this chain and hence it is sufficient to prove the following assertion which refers only to $\mathcal{F}(A)$-chains:

Let A be a cyclic group of order $p^n q$ where p and q are prime numbers coprime to 6, and let $P_1, ..., P_f$ be an $\mathcal{F}(A)$-chain of a finite solvable group G such that A acts fixed point freely on P_i for each $i = 1, ..., f$. Assume that P_i is abelian when $p_i = 2$. Then $f \leq \ell$.

Set $A = A_p \times A_q$ where $A_p = \langle \alpha \rangle$ and A_q are Sylow p- and q-subgroups of A. We proceed by induction on ℓ and deduce a contradiction over a series of steps. We may assume that $P_1, ..., P_f$ with $P_i = S_i/T_i$ satisfy the conditions (a)-(e) stated in Remark 4.1.

Notice that $[P_{f-1}, z_p] \neq 1$, because otherwise $P_1, ..., P_{f-2}, P_{f-1}$ is an $\mathcal{F}(A/\langle z_p\rangle)$-chain such that $A/\langle z_p\rangle$ acts fixed point freely on each of its terms, and hence $f-1 \leq \ell - 1$ by induction.

Step 1. $[P_i, z_p] = 1$, *for* $i = 1, ..., f-2$, where $z_p = \alpha^{p^{n-1}}$.

If $[P_{f-3}, z_p] \neq 1$, then Theorem 3.3 gives an A-subchain D_{f-1}, D_f of P_{f-1}, P_f where D_{f-1} and D_f are sections centralized by A_p. The fixed point free action of A_q on the semidirect product $D_f D_{f-1}$ leads to a contradiction. Then $[P_{f-3}, z_p] = 1$ and hence $[P_i, z_p] = 1$, for $i = 1, ..., f-2$. It remains to prove that $[P_{f-2}, z_p] = 1$: Assume otherwise. A similar argument using Theorem 3.3 shows that $p_{f-2} = p$. Then $C_{P_{f-2}}(A_q) = 1$.

We shall observe next that $[P_{f-3}, A_q] = 1$. If $p_{f-3} \neq q$, an application of Theorem 3.1 to the action of $P_{f-3}A_q$ on $\tilde{P}_{f-2}$ yields that $[P_{f-3}, A_q] = 1$. Hence we may assume that $p_{f-3} = q$. Then we have $f > 4$, because otherwise $[P_{f-3}, A_q] = 1$, as desired, by the irreducibility of $P_{f-3} = P_1$ as an A-module.

If $[P_{f-3}, A_q]^{q-1} \neq 1$, then we consider the action of $P_{f-3}A_q$ on $\tilde{P}_{f-2}$ again and get a contradiction by Lemma 2.1. We apply now Theorem 3.2 to the action of $P_{f-4}A_q$ on $\tilde{P}_{f-3}$ and get $[P_{f-4}, A_q] = 1$. Thus we have $[P_i, A_q] = 1$ for $i = 1, ..., f-4$. Since P_i, $i = 1, ..., f-3$, is a p'-group, we may assume that $\prod_{i=1}^{f-3} S_i$ is centralized by z_p. It follows by the irreducibility of $\tilde{P}_{f-2}$ as a $(\prod_{i=1}^{f-3} S_i)$ A-module that $[\tilde{P}_{f-2}, z_p] = 1$. We then have $[P_{f-2}, z_p] = 1$ by the three-subgroups lemma as $[\phi(P_{f-2}), S_{f-3}] = 1$. Consequently, $[P_i, z_p] = 1$, for $i = 1, ..., f-2$.

Step 2. $p_{f-1} \notin \{p, q\}$.

We suppose that this claim is false and that $p_{f-1} = p$. Then Thompson's $A \times B$ lemma gives $[C_{P_{f-1}}(z_p), P_{f-2}] \neq 1$. It is now easy to verify that the sequence $P_1, ..., P_{f-2}, [C_{P_{f-1}}(z_p), P_{f-2}]$ forms an $\mathcal{F}(A/\langle z_p\rangle)$-chain such that $A/\langle z_p\rangle$ acts fixed point freely on each of its terms. By induction we get $f-1 \leq \ell - 1$, which is a contradiction.

We now have to handle the case $p_{f-1} = q$, which is a little more troublesome. Obviously, $C_{P_{f-1}}(A_p) = 1$. Notice that $[P_{f-2}, A_q] \neq 1$, because otherwise $[P_i, \langle z_p\rangle A_q] = 1$, for $i = 1, ..., f-2$, and hence $f-2 \leq \ell - 2$ by induction. Theorem 3.2 applied to the action of $P_{f-2}A_q$ on $[\tilde{P}_{f-1}, z_p]$ yields that $[P_{f-1}, z_p, \alpha_q]^{q-1} \neq 1$. Applying now Theorem 3.4 to the action of $[P_{f-1}, z_p]A$ on P_f, we get $C_{P_f}(A) \neq 1$, a contradiction. Consequently, we observe that P_{f-1} is a $\{p, q\}'$-group.

Step 3. Final contradiction.

Let now $X = [P_{f-1}, z_p]$. It is known that $X \neq 1$. Applying Lemma 2.4 to the action of XA on P_f, we obtain $[X, A_q] = 1$. Then by the three-subgroups lemma we have $[A_q, P_{f-2}, X] = 1$. However, $P_{f-1} = [P_{f-1}, z_p]C_{P_{f-1}}(z_p)$ and hence

$[C_{P_{f-1}}(z_p), [P_{f-2}, A_q]] \neq 1$. In fact $[C_{P_{f-1}}(z_p), P_{f-2}] \neq 1$. It is now straightforward to verify that $P_1, ...P_{f-2}, [C_{P_{f-1}}(z_p), P_{f-2}]$ is an $\mathcal{F}(A/\langle z_p \rangle)$-chain such that $(A/\langle z_p \rangle)$ acts fixed point freely on each of its terms. Finally, we obtain $f-1 \leq \ell-1$ by induction. This completes the proof. □

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