ACTION OF A FROBENIUS-LIKE GROUP WITH KERNEL HAVING CENTRAL DERIVED SUBGROUP

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Abstract. A finite group FH is said to be Frobenius-like if it has a nontrivial nilpotent normal subgroup F with a nontrivial complement H such that $[F,h]=F$ for all nonidentity elements $h \in H$. Suppose that a finite group G admits a Frobenius-like group of automorphisms FH of coprime order with $[F',H]=1$. In case where $C_G(F)=1$ we prove that the groups G and $C_G(H)$ have the same nilpotent length under certain additional assumptions.

1. Introduction

A finite group A is defined in [5] to be Frobenius-like with kernel F and complement H, if it has a nontrivial nilpotent normal subgroup F, and a nontrivial subgroup H with $A = FH$, $F \cap H = 1$ and $[F,h]=F$ for all nonidentity elements $h \in H$. The title of this paper refers to Frobenius-like groups FH in which the derived subgroup F' of F is centralized by H. In this case one sees easily using the Three Subgroup Lemma that $F' \leqslant Z(FH)$. Extraspecial groups F admitting a group of automorphisms H which acts semiregularly on F/F' and centralizes F' provide examples FH of Frobenius-like groups which contain F' in the center. In [2] and [4] we investigated the structure of solvable groups G which admit a Frobenius-like group of the above type for which F' is of prime order. It is observed in the present paper that whenever the action of F' on the top Fitting factor of G is Frobenius, that is, $(G/F_{n-1}(G))F'$ is a Frobenius group; the condition that F' is of prime order can be replaced by the weaker condition that F' has a maximal subgroup of prime power order. It should be noted that the group F' is cyclic under the assumption that its action on the top Fitting factor of G is Frobenius. Namely we prove

Theorem *Let G be a finite group of odd order of nilpotent length n admitting a Frobenius-like group of automorphisms FH of odd coprime order with kernel F and complement H with $[F',H]=1$ such that $C_G(F)=1$. Suppose that F' has a maximal subgroup of prime power order and its action on the group $G/F_{n-1}(G)$ is Frobenius, then the nilpotent length of $C_G(H)$ is n.*

The key result which is crucial in proving this theorem is as follows.

Proposition *Let FH be a Frobenius-like group of odd order with $[F',H]=1$ acting on a q-group Q of class at most 2 for some odd prime q coprime to the order*

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of FH. Let V be a $kQFH$-module on which F acts fixed point freely where k is a field of characteristic not dividing $|QFH|$. Assume further that for any nonidentity $x \in F'$ and for any irreducible Q-submodule U of V, $[Q, x]$ acts nontrivially on U. Then we have

$$Ker(C_{[Q,F]}(H)\, on\, C_V(H)) = Ker(C_{[Q,F]}(H)\, on\, V).$$

Here we use alternative notation for the kernel of an action of a group A by automorphisms on a group B denoting $Ker(A\, on\, B) := C_A(B)$ in order to avoid cumbersome subscripts.

Remark It is well known that in the study of Hall-Higman type theorems some exceptional cases do really occur due to the existence of Fermat or/and Mersenne primes and nonabelian Sylow 2-subgroups. Although we have been able to prove the above theorems without assuming that the relevant groups are of odd order, but under conditions which help to avoid the exceptional situations, we have stated them only for groups of odd order, in order to make the arguments as clear as possible. Otherwise the hypotheses of the theorems would be so complicated that one would have difficulties in appreciating the main idea. More precisely, in views of [3] and [6], our result could actually be reformulated as Theorem* below using the following hypothesis *(*)*:

(*) Sylow 2-subgroups of H are cyclic and normal, F has no extraspecial sections of order p^{2m+1} where $p^m + 1 = |H_1|$ for some subgroup $H_1 \leqslant H$, and $|G|$ is not divisible by any prime q such that $q^f + 1$ divides $\exp(F')$ for some positive integer f.

Theorem* *Let G be a finite group of nilpotent length n admitting a Frobenius-like group of automorphisms FH of coprime order with kernel F and complement H such that $C_G(F) = 1$. Suppose that $[F', H] = 1$, F' has a maximal subgroup of prime power order and its action on the group $G/F_{n-1}(G)$ is Frobenius. If (*) holds then the nilpotent length of $C_G(H)$ is n.*

2. PROOF OF THE PROPOSITION

The following lemma will be used in the proof of the proposition.

Lemma 2.1. *Let FH be a Frobenius-like group with $[F', H] = 1$. For any subgroup S of F which is normalized but not centralized by H, the group $[S, H]H$ is a Frobenius-like group with kernel $[S, H]$.*

Proof. Set $R = [S, H]$ and $\hat{R} = R/\Phi(R)$. Pick a nonidentity element $h \in H$. Clearly, we have $[\hat{R}, h] = \widehat{[R, h]}$. If $[\hat{R}, h] \neq \hat{R}$ then $C_R(h) = R \cap C_F(h) = R \cap F'$ is not contained in $\Phi(R)$ and is centralized by H. This contradicts the fact that $[R, H] = R$. Therefore $[R, h] = R$ for each nonidentity $h \in H$, establishing the claim. □

Now we are ready to present a proof for the proposition.

Suppose that the proposition is false and choose a counterexample with minimum $\dim_k V + |QFH|$. To ease the notation we set $K = Ker(C_{[Q,F]}(H)\, on\, C_V(H))$. We proceed over several steps.

(1) We may assume that k is a splitting field for all subgroups of QFH.

Proof. We consider the QFH-module $\bar{V} = V \otimes_k \bar{k}$ where $\bar{k}$ is the algebraic closure of k. Notice that $dim_k V = dim_{\bar{k}} \bar{V}$ and $C_{\bar{V}}(H) = C_V(H) \otimes_k \bar{k}$. Therefore once the proposition has been proven for the group QFH on $\bar{V}$, it becomes true for QFH on V also. □

(2) We have $Q = [Q, F]$ and hence $C_Q(F) \leqslant Q' \leqslant Z(Q)$.

Proof. If there is a nonidentity $x \in F'$ such that $[[Q, F], x]$ acts trivially on an irreducible $[Q, F]$-submodule X of V, then $[Q, x, x] = [Q, x]$ acts trivially on X. Since $[Q, x] \trianglelefteq Q$, the module $X^Q = \Sigma_{y\in Q} X^y$ is centralized by $[Q, x]$ and hence there is an irreducible Q-submodule of V which is centralized by $[Q, x]$. Therefore the action of $[Q, F]FH$ on V satisfies the hypothesis of the proposition. So if $[Q, F] \neq Q$, the proposition holds for the group $[Q, F]FH$ on V by induction. That is, the conclusion of the proposition is true as we have $[Q, F, F] = [Q, F]$. This contradiction shows that $[Q, F] = Q$ and hence $C_Q(F) \leqslant Q' \leqslant Z(Q)$ as claimed. □

(3) V is an irreducible QFH-module on which Q acts faithfully.

Proof. As $char$(k) is coprime to the order of Q and $K \neq 1$, there is a QFH-composition factor W of V on which K acts nontrivially. If $W \neq V$, then the proposition is true for the group QFH on W by induction. That is,

$$Ker(C_Q(H)\, on\, C_W(H)) = Ker(C_Q(H)\, on\, W)$$

and hence

$$K = Ker(K\, on\, C_W(H)) = Ker(K\, on\, W)$$

which is a contradiction with the assumption that K acts nontrivially on W. Hence $V = W$.

We next set $\overline{Q} = Q/Ker(Q\, on\, V)$ and consider the action of the group $\overline{Q}FH$ on V assuming $Ker(Q\, on\, V) \neq 1$. An induction argument gives $Ker(C_{\overline{Q}}(H)\, on\, C_V(H)) = Ker(C_{\overline{Q}}(H)\, on\, V)$. This leads to a contradiction as $C_{\overline{Q}}(H) = \overline{C_Q(H)}$ due to the coprime action of H on Q. Thus we may assume that Q acts faithfully on V. □

By Clifford's theorem the restriction of the QFH-module V to the normal subgroup Q is a direct sum of Q-homogeneous components.

(4) Let Ω denote the set of Q-homogeneous components of V. Then F acts transitively on Ω and H fixes an element of Ω.

Proof. Let Ω_1 be an F-orbit on Ω and set $H_1 = Stab_H(\Omega_1)$. Suppose first that $H_1 = 1$. Pick an element W from Ω_1. Clearly, we have $Stab_H(W) \leqslant H_1 = 1$ and hence the sum $X = \sum_{h\in H} W^h$ is direct. It is straightforward to verify that $C_X(H) = \left\{\sum_{h\in H} v^h : v \in W\right\}$. By definition, K acts trivially on $C_X(H)$. Note also that K normalizes each W^h as $K \leqslant Q$. It follows now that K is trivial on X. Notice that the action of H on the set of F-orbits on Ω is transitive, and $K \leqslant C_Q(H)$. Hence K is trivial on the whole of V, which is a contradiction. Thus $H_1 \neq 1$.

The group H acts transitively on $\{\Omega_i : i = 1, 2, \ldots, s\}$, the collection of F-orbits on Ω. Let now $V_i = \bigoplus_{W \in \Omega_i} W$ for $i = 1, 2, \ldots, s$. Suppose now that H_1 is a proper subgroup of H, equivalently, $s > 1$. By induction the proposition holds for the group QFH_1 on V_1, that is,

$$Ker(C_Q(H_1)\, on\, C_{V_1}(H_1)) = Ker(C_Q(H_1)\, on\, V_1).$$

In particular, we have

$$Ker(C_Q(H)\, on\, C_{V_1}(H_1)) = Ker(C_Q(H)\, on\, V_1).$$

On the other hand we observe that

$$C_V(H) = \{u^{x_1} + u^{x_2} + \cdots + u^{x_s} : u \in C_{V_1}(H_1)\}$$

where $x_1, \ldots, x_s$ is a complete set of right coset representatives of H_1 in H. By definition, K acts trivially on $C_V(H)$ and normalizes each V_i. Then K is trivial on $C_{V_1}(H_1)$ and hence on V_1. As K is normalized by H we see that K is trivial on each V_i and hence on V, a contradiction. Therefore $H_1 = H$ so that $\Omega = \Omega_1$ and F acts transitively on Ω as desired.

Let now $S = Stab_{FH}(W)$ and $F_1 = F \cap S$. Then $|F : F_1| = |\Omega| = |FH : S|$ and so $|S : F_1| = |H|$. Notice next that as $(|F_1|, |H|) = 1$ there exists a complement, say S_1, of F_1 in S with $|H| = |S_1|$ by Schur-Zassenhaus theorem. Therefore by passing, if necessary, to a conjugate of W in Ω, we may assume that $S = F_1H$, that is, W is H-invariant. This establishes the claim. $\square$

From now on W will denote an H-invariant element in Ω the existence of which is established by (4). It should be noted that the group $Z(Q/Ker(Q\, on\, W))$ acts by scalars on the homogeneous Q-module W, and so $[Z(Q), H] \leqslant Ker(Q\, on\, W)$ as W is stabilized by H. Set $L = K \cap Z(C_Q(H))$. Since $1 \neq K \trianglelefteq C_Q(H)$, the group L is nontrivial.

(5) Set $U = \sum_{x \in F'} W^x$ and $F_2 = Stab_F(U)$. Then $[L, Q] \leqslant C_Q(U)$.

Proof. Note that $Z_2(Q) = Q$ by the hypothesis and $Q = [Q, H]C_Q(H)$ as $(|Q|, |H|) = 1$. We have $[Q, L, H] \leqslant [Z(Q), H] \leqslant C_Q(W)$. We also have $[L, H, Q] = 1$ as $[L, H] = 1$. It follows now by the Three Subgroup Lemma that $[H, Q, L] \leqslant C_Q(W)$. On the other hand $[C_Q(H), L] = 1$ by the definition of L. Thus $[L, Q] \leqslant C_Q(W)$. Since the group $[L, Q]$ is F' -invariant as $[F', H] = 1$, we conclude that $[L, Q] \leqslant C_Q(U)$. $\square$

(6) $F_2 = F_1F'$ is a proper subgroup of F, and K^x acts trivially on U for every $x \in F - F_2$. Moreover, $C_V(H) \neq 0$.

Proof. For $F_2 = Stab_F(U)$, clearly we have $F' \leqslant F_2$ and $F_1 = Stab_F(W) \leqslant F_2$. This gives $F_2 = F_1F'$. Assume that $F = F_2$. This forces the equality $V = U$ as F is transitive on Ω by (4). In fact we have $F = F_1 = F_2$ and so $V = W = U$ as $F' \leqslant \Phi(F)$. Then $[L^{F_2}, Q] \leqslant C_Q(V) = 1$ by (5) and hence $L^{F_2} \leqslant Z(Q)$. Now $Z(Q)$ and hence L acts by scalars on the homogeneous Q-module V. Notice that $C_V(H) \neq 0$ by Corollary 3.4 in [4] applied to the action of FH on V. Since $L/Ker(L\, on\, V)$ acts faithfully and by scalars on V, we get $L \leqslant Ker(L\, on\, V)$, which is not the case. Consequently, in any case $F \neq F_2$.

Pick $x \in F - F_2$ and suppose that there exists $1 \neq h \in H$ such that $(U^x)^h = U^x$ holds. Then $[h, x^{-1}] \in F_2$ and so $F_2x = F_2x^h = (F_2x)^h$. The Frobenius action of H on F/F_2 gives that $x \in F_2$, a contradiction. That is, for each $x \in F - F_2$, $Stab_H(U^x) = 1$. In particular, H-orbit of U^x is regular and hence we conclude that $C_V(H) \neq 0$.

Set now $U_1 = U^x$ for some $x \in F - F_2$. The sum $Y = \sum_{h\in H} {U_1}^h$ is direct by the preceding paragraph. It is straightforward to verify that $C_Y(H) = \left\{\sum_{h\in H} v^h : v \in U_1\right\}$. By definition, K acts trivially on $C_Y(H)$. Note also that K normalizes each ${U_1}^h$ for every $h \in H$ as $K \leqslant Q$. It follows now that K is trivial on Y and hence trivial on U^x for every $x \in F - F_2$ which is equivalent to that K^x acts trivially on U for all $x \in F - F_2$ as desired. □

(7) $L \leqslant Z(Q)$ and hence the group $LC_Q(W)/Ker(Q\,on\,W)$ acts by scalars on W.

Proof. Recall that $[L, Q] \leqslant Ker(Q\,on\,U)$ by (5). This gives $[L^{F_2}, Q] \leqslant Ker(Q\,on\,U)$. On the other hand $[L^x, Q] \leqslant [Ker(Q\,on\,U), Q] \leqslant Ker(Q\,on\,U)$ for any $x \in F - F_2$ by (6). Then we have $[L^F, Q] \leqslant Ker(Q\,on\,U)$. It follows that $[L^F, Q] \leqslant Ker(Q\,on\,V) = 1$, that is $L^F \leqslant Z(Q)$. □

(8) $C_U(H) = 0, [U, [F_2, H]] = 0,$ and hence $[Q, [F_2, H]] \leqslant Ker(Q\,on\,U)$.

Proof. It should be noted that the group $[F_2, H]H$ is Frobenius-like by Lemma. If $[U, [F_2, H]] \neq 0$ then Corollary 3.4 in [4] applied to the action of $[F_2, H]H$ on U gives that $C_U(H) \neq 0$. This forces that $C_W(H) \neq 0$ and hence L acts trivially on W, which is a contradiction. Therefore we have $C_U(H) = 0$ and $[U, [F_2, H]] = 0$. As a consequence, $[U, [F_2, H], Q] = 0 = [Q, U, [F_2, H]]$. It follows by the three subgroup lemma that $[Q, [F_2, H]] \leqslant Ker(Q\,on\,U)$. □

(9) $[F_2, H] = [F_1, H]$ and $[F_1, H] \cap F' = 1$.

Proof. By (8), $[F_1, H] \cap F' \leqslant Ker(Z(F)\,on\,W)$ and hence trivial. □

(10) The theorem follows.

Proof. Notice that $C_W(F_1) = 0 = C_U(F_2)$ as $C_V(F) = 0$. Suppose first that $F_1 = F_2 = [F_1, H]F'$. In case $C_W(F') \neq 0$ we apply Corollary 3.4 in [4] to the action of the Frobenius group $(F_1/F')H$ on $C_W(F')$ and see that $C_W(F')_H$ is free. Since $C_W(H) = 0$ by (8) we get $C_W(F') = 0$. Suppose next that $F_1 \neq F_2 = F_1F'$. In fact $F_2 = [F_1, H] \times F'$. Notice that $C_U(F_2) = 0 = [U, [F_1, H]]$ whence $C_U(F') = 0$. As $QF' \lhd QF_2H$ and U is an irreducible QF_2H-module, we can consider the decomposition of U into its QF'-homogeneous components. Let Y be one of these components. Recall that F' is cyclic. By Theorem 4.1 in [6] applied to the action of QF' on Y we get $x \in F'$ such that $[Q, x] \leqslant Ker(Q\,on\,Y)$. As $[Q, x]$ is FH-invariant we have $[Q, x] \leqslant Ker(Q\,on\,U)$ and so $[Q, x] \leqslant Ker(Q\,on\,V) = 1$. This contradiction completes the proof. □

The following example shows that the faithfulness of F' on $Q/Ker(Q\,on\,U)$ for any irreducible Q-submodule U of V is indispensable.

Example. Let p be an odd prime so that $p-1$ is divisible by three distinct primes q, r and s with s dividing $r + 1$; for example $p = 211, q = 7, r = 5$ and $s = 3$. Let V_1 be the additive group of the field with p elements. The multiplicative group of this field contains a cyclic group of order qrs which acts by multiplication on V_1 by automorphisms. Let us denote the cyclic subgroups of this group of orders q, r and s respectively by Q_1, F_1 and H. So $Q_1 \times F_1 \times H \leqslant \mathbb{Z}_p^*$ and the semidirect product $V_1(Q_1 \times F_1 \times H)$ is a Frobenius group of order $pqrs$.

Let $F = E_1 \times E_2$ where E_i, $i = 1, 2$, are extraspecial groups of order r^3 and exponent r. Let $Z(E_i) = \langle z_i \rangle, i = 1, 2$, and $\tau = z_1 z_2$. Identify $\langle \tau \rangle$ ($\cong \mathbb{Z}_5$) with F_1.

Each E_i and hence F admits an automorphism of order s acting trivially on F' and regularly on F/F'. We identify the group generated by this automorphism with H.

Since H is of odd order and acts semiregularly on F/F' we can choose a transversal T for F_1 in F such that

(1) $\langle z_1 \rangle \subset T$,
(2) T is closed with respect to taking inverses,
(3) T is H-invariant.

Then T is a disjoint union of H-orbits $T_j, j = 1, 2, \ldots, m$, where $T_i, i = 1, 2, \ldots, r$, are orbits of length 1, with $T_1 \cup T_2 \cup \ldots \cup T_r = \langle z_1 \rangle$ and $T_i, i = r+1, r+2, \ldots, m$, are the orbits of length s.

The group F_1 acts by automorphisms on $G = V_1 Q_1$. This allows to define an action of F by automorphisms on the group $B = \{(g_t)_{t \in T} : g_t \in G, t \in T\}$ with componentwise operation by

$$[(g_t)_{t \in T}]^\sigma := (g_{t \cdot u^{-1}}^{a(t,u^{-1})^{-1} z})_{t \in T}$$

where $z \in F_1$, $u \in T$ and $\sigma = zu$ where for any x and y in T we have a uniquely determined element $x \cdot y \in T$ and a uniquely determined element $a(x, y)$ of F_1 such that

$$xy = a(x, y)\ x \cdot y.$$

Consider the semidirect product BF. The action of H on G and on F can be extended to an action of H on BF as follows:

$$[(g_t)_{t \in T} \sigma]^h := [(g_{hth^{-1}})^h]_{t \in T} \sigma^h$$

Let $V_0 = \{(x_t)_{t \in T} : x_t \in V_1, t \in T\}$, $Q_0 = \{\ (x_t)_{t \in T} : x_t \in Q_1, t \in T\}$. From now on we denote $\{(x_t)_{t \in T} : x_u \in V_1, x_t = 1$ for any $u \neq t \in T\}$ by $V_u, u \in T$ (this needs of course the identification of the group V_1 with the subgroup of V_0 defined as $\{(x_t)_{t \in T} : x_1 \in V_1, x_t = 1$ for $1 \neq t \in T\} \cong V_1$) and $\{(x_t)_{t \in T} : x_u \in Q_1, x_t = 1$ for any $u \neq t \in T\}$ by $Q_u, u \in T$ (this needs of course the identification of the group Q_1 with the subgroup of Q_0 defined as $\{(x_t)_{t \in T} : x_1 \in Q_1, x_t = 1$ for $1 \neq t \in T\} \cong Q_1$).

Then V_0 is elementary abelian of order $p^{|T|}$, Q_0 is elementary abelian of order $q^{|T|}$, $V_0 \trianglelefteq B = V_0 Q_0$. Both V_0 and Q_0 are FH-invariant.

Each V_t is F_1-invariant and F_1 acts fixed point freely on V_t, $t \in T$. Thus $C_{V_0}(F) = C_{V_0}(F_1) = 1$. We also have $[Q_0, F_1] = 1$.

Let $H = \langle h \rangle$ and let $U = \{u, huh^{-1}, h^2uh^{-2}, \ldots, h^{s-1}uh^{-s+1}\}$ be an H-orbit on T of length s for some $u \in T$. Then H normalizes the subspace $\bigoplus_{t \in U} V_t = V_U$ and if $(a_1, a_2, \ldots, a_s) \in V_u \oplus V_{huh^{-1}} \oplus \cdots \oplus V_{h^{s-1}uh^{-s+1}}$ then $(a_1, a_2, \ldots, a_s)^h = (a_s^h, a_1^h, a_2^h, \ldots, a_{s-1}^h)$ and hence the set of fixed points of H on V_U consists of $(a, a^h, \ldots, a^{h^{s-1}}), a \in V_u$. On the other hand H acts fixed point freely on V_t for any $t \in \langle z_1 \rangle$. This shows that $C_{V_0}(H)$ is a subspace of dimension $m - r$ and gives how we can write down explicitly that subspace, making use of $V_0 = \bigoplus_{i=1}^m \bigoplus_{t \in T_i} V_t$.

Let $K_0 = \bigoplus_{i=1}^{r} \bigoplus_{t \in T_i} Q_t = \bigoplus_{i=1}^{r} Q_{z_1^i}$. Then $K_0 \leqslant C_{Q_0}(H)$ and $[K_0, C_{V_0}(H)] = 1$. Set $V = [V_0, F] = V_0$ and $Q = [Q_0, F]$ and consider the group $VQFH$. Here FH is a Frobenius-like group acting on the group VQ by automorphisms such that $C_{VQ}(F) = 1$. Let $K_1 = K_0 \cap Q$. Since $\langle z_1 \rangle$ acts on K_0 and $[K_0, \langle z_1 \rangle]$ is a maximal subspace of K_0 (of dimension $r-1$) we see that K_1 is nontrivial and does not centralize V. One can observe that $K_1 \leqslant Ker(C_Q(H) \text{ on } C_V(H))$.

3. proof of Theorem

As $(|G|, |FH|) = 1$ and the action of F' on the group $G/F_{n-1}(G)$ is Frobenius, there exists an irreducible FH-tower $\hat{P}_1, \dots, \hat{P}_n$ in the sense of [7] where

(a) $\hat{P}_i$ is an FH-invariant p_i-subgroup, p_i is a prime, $p_i \neq p_{i+1}$, for $i = 1, \dots, n-1$;
(b) $\hat{P}_i \leqslant N_G(\hat{P}_j)$ whenever $i \leqslant j$;
(c) $P_n = \hat{P}_n$ and $P_i = \hat{P}_i / C_{\hat{P}_i}(P_{i+1})$ for $i = 1, \dots, n-1$ and $P_i \neq 1$ for $i = 1, \dots, n$;
(d) $\Phi(\Phi(P_i)) = 1$, $\Phi(P_i) \leqslant Z(P_i)$, and $\exp(P_i) = p_i$ when p_i is odd for $i = 1, \dots, n$;
(e) $[\Phi(P_{i+1}), P_i] = 1$ and $[P_{i+1}, P_i] = P_{i+1}$ for $i = 1, \dots, n-1$;
(f) $(\Pi_{j<i} \hat{P}_j)FH$ acts irreducibly on $P_i/\Phi(P_i)$ for $i = 1, \dots, n$;
(g) the action of F' on P_1 is Frobenius.

We observe that $C_{P_1}(H) \neq 1$ by Corollary 3.4 in [4] applied to the action of FH on $P_1/\Phi(P_1)$. So it is clear that the sequence $C_{\hat{P}_1}(H), [\hat{P}_2, C_{\hat{P}_1}(H)], \hat{P}_3, \dots, \hat{P}_n$ forms also an $F'H$-tower. It also follows by the hypothesis that the action of F' on $C_{P_1}(H)$ is also Frobenius.

Let now $\langle y \rangle$ be a maximal subgroup of F' of prime power order. As $[C_{\hat{P}_1}(H), x] \neq 1$ for any $x \in F'$ of prime order, by Theorem 3.3 in [1] we observe that the sequence $C_{[\hat{P}_2, C_{\hat{P}_1}(H)]}(y), C_{\hat{P}_3}(y), \dots, C_{\hat{P}_n}(y)$ forms an $F'H$-tower. Notice that the group $F/\langle y \rangle$ has derived subgroup of prime order. By Theorem A in [2] applied to the action of the Frobenius-like group $FH/\langle y \rangle$ on $C_G(y)$ we get $f(C_G(y)) = f(C_G(\langle y \rangle H)) = n-1$. This forces that $C_{[\hat{P}_2, C_{\hat{P}_1}(H)]}(H), C_{\hat{P}_3}(H), \dots, C_{\hat{P}_n}(H)$ forms a tower and hence $f(C_G(H)) \geqslant n-1$. Notice that $[C_{\hat{P}_2}(H), C_{\hat{P}_1}(H)] = [C_{\hat{P}_2}(H), C_{\hat{P}_1}(H), C_{\hat{P}_1}(H)] \leqslant [C_{[\hat{P}_2, C_{\hat{P}_1}(H)]}(H), C_{\hat{P}_1}(H)]$ and $[C_{[\hat{P}_2, C_{\hat{P}_1}(H)]}(H), C_{\hat{P}_1}(H)] \leqslant [C_{\hat{P}_2}(H), C_{\hat{P}_1}(H)]$. That is $[C_{[\hat{P}_2, C_{\hat{P}_1}(H)]}(H), C_{\hat{P}_1}(H)] = [C_{\hat{P}_2}(H), C_{\hat{P}_1}(H)]$. As a consequence we see that $[C_{\hat{P}_2}(H), C_{\hat{P}_1}(H)], C_{\hat{P}_3}(H), \dots, C_{\hat{P}_n}(H)$ is a tower.

On the other hand, as $[\hat{P}_1, x] = \hat{P}_1$ for any $x \in F'$, Proposition applied to the action of P_1FH on $V = P_2/\Phi(P_2)$ yields that

$$Ker(C_{P_1}(H) \, on \, C_V(H)) = Ker(C_{P_1}(H) \, on \, V) = 1.$$

This forces that $C_{\hat{P}_1}(H), [C_{\hat{P}_2}(H), C_{\hat{P}_1}(H)], C_{\hat{P}_3}(H), \dots, C_{\hat{P}_n}(H)$ forms a tower and hence $f(C_G(H)) = f(G) = n$ as desired.

References

[1] G. Ercan, On a Fitting length conjecture without the coprimeness condition, *Monatsh. Math.* **167** (2) (2012) 175187.

[2] G. Ercan and İ. Ş. Güloğlu, Action of a Frobenius-like group with fixed-point-free kernel, *J. Group Theory* **17** (5) (2014) 863-873.

[3] G. Ercan, İ. Ş. Güloğlu, and E. I. Khukhro, Rank and Order of a Finite Group admitting a Frobenius-like Group of Automorphisms, *Algebra and Logic* **53** (3) (2014) 258-265.
[4] G. Ercan, İ. Ş. Güloğlu, and E. I. Khukhro, Frobenius-like groups as groups of automorphisms, *Turkish J. Math.* **38** (6) (2014) 965-976.
[5] İ. Ş. Güloğlu and G. Ercan, Action of a Frobenius-like group, *J. Algebra* **402** (2014) 533-543.
[6] E. Shult, On groups admitting fixed point free abelian operator groups, *Illinois J. Math.* **9** (1965) 701-720.
[7] A.Turull, Fitting Height of Groups and of Fixed Points, *J. Algebra* **86** (1984) 555-556.

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