Proceedings of the Edinburgh Mathematical Society (2011) **54**, 77–89
DOI:10.1017/S0013091509000583

FIXED-POINT FREE ACTION OF AN ABELIAN GROUP OF ODD NON-SQUAREFREE EXPONENT

GÜLİN ERCAN[1], İSMAİL Ş. GÜLOĞLU[2] AND ÖZNUR MUT SAĞDIÇOĞLU[1]

[1]*Department of Mathematics, Middle East Technical University, Ankara 06531, Turkey* (ercan@metu.edu.tr; oznur@uekae.tubitak.gov.tr)
[2]*Department of Mathematics, Doğuş University, Istanbul, Turkey* (iguloglu@dogus.edu.tr)

(Received 4 May 2009)

Abstract Let A be a finite group acting fixed-point freely on a finite (solvable) group G. A longstanding conjecture is that if $(|G|, |A|) = 1$, then the Fitting length of G is bounded by the length of the longest chain of subgroups of A. It is expected that the conjecture is true when the coprimeness condition is replaced by the assumption that A is nilpotent. We establish the conjecture without the coprimeness condition in the case where A is an abelian group whose order is a product of three odd primes and where the Sylow 2-subgroups of G are abelian.

Keywords: automorphisms of solvable groups; non-coprime action; fixed-point free action; Carter subgroup

2010 *Mathematics subject classification:* Primary 20D10
Secondary 20D15; 20D30; 20D45

1. Introduction

A well-known conjecture of Thompson states that if A is a finite group acting fixed-point freely on a finite solvable group G of order coprime to $|A|$, then the Fitting length $f(G)$ of G is bounded by $\ell(A)$, the length of the longest chain of subgroups of A. By an elegant result due to Bell and Hartley [**1**], it is known that any finite non-nilpotent group A can act fixed-point freely on a solvable group G of arbitrarily large Fitting length with $(|G|, |A|) \neq 1$. If A is nilpotent, it is expected that the conjecture is still true without the coprimeness condition. This question is settled for cyclic groups A of order pq [**2**] and pqr [**4**] for pairwise distinct primes p, q and r. We also establish the conjecture without the coprimeness condition in the case where A is abelian of squarefree exponent coprime to 6 and $|G|$ is odd [**5**]. The main theorems of [**2**] and [**4**] have the advantage of providing an answer in the absence of any restriction on the nature of the primes involved. In comparison with the former results mentioned above, the novelty of the present article consists in the possibility of allowing non-squarefree exponents. To be precise, the main result of this paper is the following theorem, which settles the conjecture without the coprimeness condition when A is an abelian group whose order is a product of three odd primes that are not necessarily distinct under some additional hypothesis on G.

Theorem C. *Let $G \lhd GA$, where A is an abelian group whose order is a product of three odd primes. Assume that Sylow 2-subgroups of G are abelian. If A acts fixed-point freely on G, then $f(G) \leqslant 3$.*

This statement is new only in the case $|A| = p^2 q$ for distinct primes p and q, since the case $|A| = pqr$ is treated in [**4**] and if $|A| = p^3$ for a prime p, the action is coprime and the result is well known. When $|A|$ is divisible by the primes 2 or 3, the study of such problems needs much more effort. Our Theorem C allows $|A|$ to be odd and divisible by 3, in contrast to the assumption in [**5**] that $|A|$ is coprime to 6.

If A is a finite nilpotent group acting fixed-point freely on a finite group G, then A is a self-normalizing nilpotent subgroup of the semidirect product GA of G by A: that is, it is a Carter subgroup of GA. This observation allows us to exploit the results and methods of Dade's fundamental paper [**3**]. However, by employing the ideas of both [**3**] and [**9**], we have been able to obtain Theorem A in [**5**]. In the present paper, we take this result further by slightly modifying the techniques used in [**5**]; namely, we obtain the following theorem.

Theorem A. *Let A be an abelian group acting fixed-point freely on a group G whose Sylow 2-subgroups are abelian. If A has a squarefree exponent coprime to 6, then $f(G) \leqslant \ell(A)$.*

Theorem B. *Let A be an abelian group acting fixed-point freely on a group G. If A has squarefree odd exponent coprime to all Fermat primes, then $f(G) \leqslant \ell(A)$.*

Theorem A is a generalization of Theorem A in [**5**] that replaces the hypothesis that $|G|$ is odd by the weaker one that the Sylow 2-subgroups of G are abelian. While this is a theorem that was designated as a *partial special case* in the survey article [**10**] by Turull, Theorem B is a *full special case* theorem in the sense of the same article.*

All the groups considered in this paper are finite and solvable. The notation and terminology are as in [**5**].

2. Proofs of Theorems A and B

The proofs of Theorems A and B depend on a few technical results presented below.

Theorem 2.1. *Let $S\langle\alpha\rangle$ be a group such that $S \lhd S\langle\alpha\rangle$, S is an s-group with $\Phi(\Phi(S)) = 1$, $\Phi(S) \leqslant Z(S)$ and $\langle\alpha\rangle$ is cyclic of order p for primes s and p. Assume that either $s = p \geqslant 5$ or $s \neq p$, p is odd. Assume further that if $s = 2$, either S is abelian or p is not a Fermat prime. Let V be a $kS\langle\alpha\rangle$-module, where k is a field of characteristic not dividing ps such that $[S, \alpha]^{p-1}$ acts non-trivially on each irreducible submodule of $V|_S$. Let Ω be an $S\langle\alpha\rangle$-stable subset of V^* that linearly spans V^* and set $V_0 = \bigcap\{\mathrm{Ker}\, f \mid$*

* We are indebted to the anonymous referee who suggested that we include Theorem B.

$f \in \Omega - C_\Omega(\alpha)\}$. *Then* $C_V(\alpha) \nsubseteq V_0$ *and* $(C_D(\alpha)$ *on* $C_V(\alpha)/C_{V_0}(\alpha)) \equiv_w (C_D(\alpha)$ *on* $V)$, *where*

$$D = \begin{cases} [S,\alpha]^{p-1} & \text{when } s = p, \\ S & \text{otherwise.} \end{cases}$$

Proof. One may observe that [**5**, Theorem 1] can be amended using the weakened conditions 'either $s = p \geqslant 5$ or $s \neq p$, p is any odd prime' and 'if $s = 2$, either S is abelian or p is not a Fermat prime'. □

Theorem 2.2. *Let* $S\langle\alpha\rangle$ *be a group such that* $S \lhd S\langle\alpha\rangle$, S *is an* s*-group,* $\langle\alpha\rangle$ *is cyclic of order* p *for distinct primes* s *and* p, $\Phi(\Phi(S)) = 1$ *and* $\Phi(S) \leqslant Z(S)$. *Assume that if* $s = 2$, *either* S *is abelian or* p *is not a Fermat prime. Let* V *be an irreducible* $kS\langle\alpha\rangle$*-module on which* $[S,\alpha]$ *acts non-trivially, where* k *is a field of characteristic different from* s. *Then* $[V,\alpha]^{p-1} \neq 0$ *and* $(C_S(\alpha)$ *on* $V) \equiv_w (C_S(\alpha)$ *on* $[V,\alpha]^{p-1})$.

Proof. One may observe that the conclusion of [**5**, Theorem 2] is true under the assumption that 'if $s = 2$, either S is abelian or p is not a Fermat prime'. □

Theorem 2.3. *Let* $G \lhd GA$, *let* $\langle a\rangle \trianglelefteq A$ *be of prime order* p *and let* $P_1, \ldots, P_t$ *be an* A*-Fitting chain of* G *such that* $[P_1, a] \neq 1$, P_i *is a* p_i*-group for a prime* p_i *and* $t \geqslant 3$. *Assume that* $p \geqslant 5$ *whenever* $p_i = p$ *for some* $i \in \{1, \ldots, t\}$. *Assume further that if* $p_i = 2$, *either* P_i *is abelian or* p *is not a Fermat prime. There are then sections* $D_{i_0}, \ldots, D_t$ *of* $P_{i_0}, \ldots, P_t$, *respectively, forming an* A*-Fitting chain of* G *such that* a *centralizes each* D_j *for* $j = i_0, \ldots, t$ *where*

$$i_0 = \begin{cases} 2 & \text{if } p_1 \neq p, \\ 3 & \text{if } p_1 = p. \end{cases}$$

Proof. It will be sufficient to demonstrate Claim 1 and Claim 2 appearing in the proof of [**5**, Theorem 3] with the hypothesis as revised above. One can observe that these claims can be restated and proven as by-products of Theorems 2.1 and 2.2, which are slightly altered versions of Theorems 1 and 2 in [**5**], by an analogous reasoning. □

One may easily check that Theorems A and B below can be proven by essentially the same reasoning as in the proof of Theorem A in [**5**], with the only difference being the replacement of Theorem 3 in [**5**] by Theorem 2.3.

Theorem A. *Let* A *be an abelian group acting fixed-point freely on a group* G *whose Sylow* 2*-subgroups are abelian. If* A *has a squarefree exponent coprime to* 6, *then* $f(G) \leqslant \ell(A)$.

Theorem B. *Let* A *be an abelian group acting fixed-point freely on a group* G. *If* A *has squarefree odd exponent coprime to all Fermat primes, then* $f(G) \leqslant \ell(A)$.

3. Concerning the presence of the factor 3

In this section we present two results that allow $|A|$ to be divisible by 3 in Theorem C. The following result, which is essentially due to Dade [**3**], is offered here for the sake of completeness in a formulation and with a notation that is suitable for our purposes.

Theorem 3.1. *Let $S\langle\alpha\rangle$ be a 3-group where $S \lhd S\langle\alpha\rangle$, $\Phi(S) \leqslant Z(S)$, $\Phi(\Phi(S)) = 1$ and $\langle\alpha\rangle$ is cyclic of order 3. Assume that there exists an α-invariant t-group T acting on S for some prime $t \neq 3$, $[S,T] = S$, $[\Phi(S),T] = 1$ and $\tilde{S}$ is completely reducible as a $T\langle\alpha\rangle$-module. Let V be an irreducible $kS\langle\alpha\rangle$-module for a field k with characteristic different from 3 and let Ω be an $S\langle\alpha\rangle$-stable subset of V^* which linearly spans V^*. Set $V_0 = \bigcap\{\operatorname{Ker} f \mid f \in \Omega - C_\Omega(\alpha)\}$ and $D = [[S,\alpha]^2, C_T(\alpha)]$. Assume that ($S$ on V) is weakly T-invariant. If D is non-trivial on V, then $C_V(\alpha) \nsubseteq V_0$ and*

$$\operatorname{Ker}(C_T(\alpha) \textit{ on } D/\operatorname{Ker}(D \textit{ on } C_V(\alpha))) = \operatorname{Ker}(C_T(\alpha) \textit{ on } D/\operatorname{Ker}(D \textit{ on } C_{V/V_0}(\alpha))).$$

Proof. It is easy to observe that $D \leqslant C_S(\alpha)$. We also have that $\operatorname{Ker}(S \text{ on } V)$ is T-invariant as (S on V) is weakly T-invariant. It follows by [**5**, Lemma 1] that $\operatorname{Ker}(D \text{ on } C_V(\alpha)) = \operatorname{Ker}(D \text{ on } V)$ and hence $\operatorname{Ker}(D \text{ on } C_V(\alpha))$ is $C_T(\alpha)$-invariant.

We use induction on $|ST| + \dim_k V$.

(1) *S acts faithfully on V.*

Set $\bar{S} = S/\operatorname{Ker}(S \text{ on } V)$. We observe that $[[\bar{S},\alpha]^2, C_T(\alpha)] = \bar{D}$. If $|\bar{S}| < |S|$, an induction argument applied to $(V, \bar{S}, T, \langle\alpha\rangle)$ shows that $C_V(\alpha) \nsubseteq V_0$ and

$$\operatorname{Ker}(C_T(\alpha) \text{ on } \bar{D}/\operatorname{Ker}(\bar{D} \text{ on } C_V(\alpha))) = \operatorname{Ker}(C_T(\alpha) \text{ on } \bar{D}/\operatorname{Ker}(\bar{D} \text{ on } C_{V/V_0}(\alpha))).$$

It follows that the conclusion of Theorem 3.1 holds. Hence $|\bar{S}| = |S|$: that is, S is faithful on V.

(2) $[Z(S),\alpha,\alpha] = 1$.

This is obtained by the same argument as in the proof of Claim 2 appearing in the proof of [**5**, Theorem 1], just by replacing C there with D.

(3) *We may assume that $T\langle\alpha\rangle$ acts irreducibly on $\tilde{S}$.*

As $\tilde{S}$ is a completely reducible $T\langle\alpha\rangle$-module, there is a collection $\{W_1,\ldots,W_l\}$ of irreducible $T\langle\alpha\rangle$-submodules such that $\tilde{S} = \bigoplus_{i=1}^{l} W_i$. For each i, $[W_i,T]$ is either W_i or trivial. Let $[W_i,T] = W_i$ for $i = 1,\ldots,s$ and $[W_i,T] = 0$ for $i = s+1,\ldots,l$. Set $\hat{D} = D\Phi(S)/\Phi(S)$. Then $\hat{D} = \bigoplus_{i=1}^{s}[[W_i,\alpha]^2, C_T(\alpha)]$. Since $t \neq 3$, $\hat{D} = [\hat{D}, C_T(\alpha)]$: that is, $C_{\hat{D}}(C_T(\alpha)) = 1$.

Let $W_i = Y_i/\Phi(S)$ for $i = 1,\ldots,s$. Y_i is a normal $T\langle\alpha\rangle$-invariant subgroup of S. Now $C_{W_i}(T) = C_{Y_i}(T)\Phi(S)/\Phi(S) = 1$, giving $\Phi(S) = C_{Y_i}(T)$. Set $S_i = [Y_i,T]$ and $D_i = [[S_i,\alpha]^2, C_T(\alpha)]$ for $i = 1,\ldots,s$. Now $S_i \lhd S_iT\langle\alpha\rangle$, $[S_i,T] = S_i$, $\Phi(S_i) \leqslant \Phi(Y_i) \leqslant \Phi(S)$. Furthermore, $\Phi(S) \cap S_i = \Phi(S_i)$, since $\Phi(S) \cap S_i \leqslant C_{S_i}(T) \leqslant \Phi(S_i)$. Then

$$S_i/\Phi(S_i) = S_i/(\Phi(S) \cap S_i) \cong S_i\Phi(S)/\Phi(S) = [Y_i,T]\Phi(S)/\Phi(S) = W_i$$

and so $S_i/\Phi(S_i)$ is an irreducible $T\langle\alpha\rangle$-module.

Assume that $D_i \neq 1$ for $i = 1, \dots, u$ and apply induction to $(V, S_i, T, \langle\alpha\rangle)$ for $i = 1, \dots, u$. It follows that $C_V(\alpha) \nsubseteq V_0$ and

$$\operatorname{Ker}(C_T(\alpha) \text{ on } D_i / \operatorname{Ker}(D_i \text{ on } C_V(\alpha))) = \operatorname{Ker}(C_T(\alpha) \text{ on } D_i / \operatorname{Ker}(D_i \text{ on } C_{V/V_0}(\alpha))).$$

$D_i\Phi(S)/\Phi(S) \cong [[W_i, \alpha]^2, C_T(\alpha)]$, giving $\prod_{i=1}^{s} D_i\Phi(S)/\Phi(S) \cong \hat{D}$.

Let $x \in \operatorname{Ker}(C_T(\alpha) \text{ on } D/\operatorname{Ker}(D \text{ on } C_{V/V_0}(\alpha)))$ and observe that

$$[\hat{D}, x] = \prod_{i=1}^{s}[D_i, x]\Phi(S)/\Phi(S) \leqslant \prod_{i=1}^{s}\operatorname{Ker}(D_i \text{ on } C_V(\alpha))\Phi(S)/\Phi(S).$$

Then

$$[D, x] \leqslant \prod_{i=1}^{s}\operatorname{Ker}(D_i \text{ on } C_V(\alpha))\Phi(S) \leqslant \operatorname{Ker}(D \text{ on } C_V(\alpha))\Phi(S).$$

This shows that

$$\operatorname{Ker}(C_T(\alpha) \text{ on } D/\operatorname{Ker}(D \text{ on } C_{V/V_0}(\alpha))) = \operatorname{Ker}(C_T(\alpha) \text{ on } D/\operatorname{Ker}(D \text{ on } C_V(\alpha))),$$

which is not the case.

(4) *S is a non-abelian special group such that $D \cap \Phi(S) = 1$, $V|_{C_{\Phi(S)}(\alpha)}$ is homogeneous and $|C_{\Phi(S)}(\alpha)| = 3 = |\Phi(S)/[\Phi(S), \alpha]|$.*

This is [**3**, Lemma 6.23] by letting $B = S$ and $P = \langle\alpha\rangle$.

(5) *Let $1 \neq \delta \in D$ and let $C = C_{\Phi(S)}(\alpha)\langle\delta\rangle$. For each $1 \neq \tau \in C$, there is an irreducible component of $V|_C$ on which both δ and τ act non-trivially.*

This is [**3**, Lemma 6.24] by letting $B = S$ and $P = \langle\alpha\rangle$.

(6) *For each $1 \neq \delta \in D$, there exists an element $\tilde{\tau} \in \tilde{S}$ such that (i) $[\tilde{\tau}, \alpha, \alpha] = \tilde{\delta}$ and (ii) $[[\tau, \alpha]^i, [\tau, \alpha]^j] \in C_{\Phi(S)}(\alpha)$ for any $i, j = 0, 1, 2$.*

Since $\delta \in [S, \alpha]^2$, there exists $\tilde{\tau} \in \tilde{S}$ such that (i) holds. If $[\Phi(S), \alpha] = 1$, then (ii) trivially holds. Hence we may assume that $[\Phi(S), \alpha] \neq 1$.

Lemmas 6.30 and 6.32 in [**3**] with $B = S$ and $P = \langle\alpha\rangle$ imply that $\tilde{S}$ is a free $\mathbb{Z}_3\langle\alpha\rangle$-module and that the function $h : \tilde{S} \times \tilde{S} \to \Phi(S)/[\Phi(S), \alpha]$ defined by $h(\tilde{x}, \tilde{y}) = [x, [y, \alpha]][y, [x, \alpha]][\Phi(S), \alpha]$ is a symmetric bilinear map with radical $C_{\tilde{S}}(\alpha)$. Also, the subspace $[\tilde{S}, \alpha]$ is h-isotropic.

Repeating the argument in the proof of [**3**, Lemma 6.34], we may obtain $\tilde{\tau} \in \tilde{S}$ such that (i) and (ii) hold.

(7) *The theorem follows.*

It will be sufficient to show that each non-trivial element of D acts non-trivially on $C_{V/V_0}(\alpha)$. Let $1 \neq \delta \in D$. We may choose $\tilde{\tau} \in \tilde{S}$ satisfying (6). Let S_1 be the inverse image in S of the $\mathbb{Z}_3\langle\alpha\rangle$-module $\langle\tilde{\tau}, [\tilde{\tau}, \alpha], [\tilde{\tau}, \alpha, \alpha] = \tilde{\delta}\rangle$ in $\tilde{S}$. Then S_1 is a non-trivial α-invariant subgroup of S containing δ. One can observe that $\Phi(S) \times \langle\delta\rangle \leqslant Z(S_1)$. (See the proof of the inequality 6.27 in [**3**, Lemma 6.25] by letting $B = S$ and $B_1 = S_1$.)

Now $[\tau, \alpha, \alpha] \equiv \delta (\bmod \Phi(S))$, where $\tilde{\tau} = \tau\Phi(S)$. Recall that $C = \langle C_{\Phi(S)}(\alpha), \delta\rangle$. It can be observed that there exists $\sigma \in S_1$ such that $1 \neq [\sigma, \alpha^i, \alpha] = [\sigma, \alpha, \alpha]^i \in C$ for $i = 1, 2$. (See the proof of the equation 6.28 in [**3**, Lemma 6.25] by letting $B = S$ and $B_1 = S_1$.)

By (5), there is an irreducible component W of $V|_C$ such that neither $[\sigma, \alpha, \alpha]$ nor δ acts trivially on W. As V is completely reducible as an $S_1\langle\alpha\rangle$-module, there is a collection $\{V_1, \ldots, V_l\}$ of irreducible $S_1\langle\alpha\rangle$-modules V_i, $i = 1, \ldots, l$, such that W is contained in one of them, say V_1. Since $\langle\delta\rangle \leqslant Z(S_1\langle\alpha\rangle)$, $V_1|_{\langle\delta\rangle}$ is homogeneous. Now $[S_1, \alpha]^2$ acts non-trivially on V_1, because $[\sigma, \alpha, \alpha] \in [S_1, \alpha]^2$. That is, V_1 is an ample $S_1\langle\alpha\rangle$-module. It follows that $C_{V_1}(\alpha) \neq 0$ by [**5**, Lemma 1] applied to $S_1\langle\alpha\rangle$ on V_1.

Assume that $C_{V_1}(\alpha) \subseteq V_0$. Set $\Omega_1 = \Omega|_{V_1}$. Ω_1 is an $S_1\langle\alpha\rangle$-stable subset of V_1^* and there exists $f \in \Omega_1$ such that $f(V_1 \cap V_0) \neq 0$: that is, $f \in \Omega_1 - C_{\Omega_1}(\alpha)$. By [**5**, Lemma 2], we have $C_{S_1}(f) \cap C = \operatorname{Ker}(C \text{ on } V_1)$ because $C \leqslant Z(S_1)$. As $[\sigma, \alpha, \alpha] \notin \operatorname{Ker}(C \text{ on } W)$, we also have $[\sigma, \alpha, \alpha] \notin \operatorname{Ker}(C \text{ on } V_1) = C_{S_1}(f) \cap C$. Now Lemma 3 in [**5**] applied to the action of $S_1\langle\alpha\rangle$ on V_1 with Ω_1 leads to a contradiction. Therefore, we have obtained that $C_{V_1}(\alpha) \nsubseteq V_0$.

We also observe that δ acts non-trivially on $C_{V_1/V_1\cap V_0}(\alpha)$, since $V_1|_{\langle\delta\rangle}$ is homogeneous and δ acts non-trivially on W. So δ acts non-trivially on $C_{V/V_0}(\alpha)$: that is, $\operatorname{Ker}(D \text{ on } C_{V/V_0}(\alpha)) = 1$. It follows that $\operatorname{Ker}(D \text{ on } C_V(\alpha)) = 1$ and so

$$\begin{aligned}\operatorname{Ker}(C_T(\alpha) \text{ on } D) &= \operatorname{Ker}(C_T(\alpha) \text{ on } D/\operatorname{Ker}(D \text{ on } C_V(\alpha)))\\ &= \operatorname{Ker}(C_T(\alpha) \text{ on } D/\operatorname{Ker}(D \text{ on } C_{V/V_0}(\alpha))).\end{aligned}$$

This forces that $C_V(\alpha) \subseteq V_0$, which means that $\operatorname{Ker}(D \text{ on } C_{V/V_0}(\alpha)) = D = 1$, which is a contradiction. □

Theorem 3.2. *Let $G \lhd GA$ and $\langle a\rangle \trianglelefteq A$ be of prime order p for an odd prime p. Suppose that $P_1, \ldots, P_t$ is an A-Fitting chain of G such that $[P_1, a] \neq 1$, P_i is a p_i-group for a prime p_i with $p_1 \neq p$, $i = 1, \ldots, t$, and P_i is abelian when $p_i = 2$. If $p = 3$, suppose that $p_{j+2} \neq p_j$ for any $j \in \{2, \ldots, t-2\}$ with $p_j = p$. There are then A-invariant non-trivial sections $D_2, \ldots, D_t$ of $P_2, \ldots, P_t$, respectively, each of which is centralized by a such that D_i normalizes D_{i+1} with $\operatorname{Ker}(D_i \text{ on } D_{i+1}) = 1$ for $i = 2, \ldots, t-1$.*

Proof. By Theorem 2.3, we may assume that $p = 3$ and $p \in \{p_1, \ldots, p_t\}$. Let q be a prime different from p_t, let $p_{t+1} = q$ and let P_{t+1} stand for the regular $\mathbb{Z}_q[P_tP_{t-1}A]$-module. We shall add P_{t+1} to the given chain and define subspaces E_i of P_i for each $i = 1, \ldots, t+1$ as follows: $E_1 = P_1$, $E_i = [X_i, E_{i-1}]$ for $i = 2, \ldots, t+1$, where $X_i/\Phi(P_i)$ is the sum of all ample irreducible $E_{i-1}\langle a\rangle$-submodules of $\tilde{P}_i$. It is easy to observe that for each $i = 2, \ldots, t+1$, the E_i are all $E_{i-1}A$-invariant subgroups of P_i and $\tilde{E}_i$ is a direct sum of ample irreducible $E_{i-1}\langle a\rangle$-submodules.

We now define subgroups F_i of E_i for $i = 1, \ldots, t+1$ as follows:

$$\begin{aligned}&F_1 = \{1\},\\ &F_i = C_{E_i}(a) && \text{if } p_i \neq p \text{ and } i \geqslant 2,\\ &F_2 = C_{[E_2,a]^2}(a) && \text{if } p_2 = p = 3,\\ &F_i = [[E_i, a]^2, F_{i-1}] && \text{if } p_i = p = 3 \text{ and } i \geqslant 3.\end{aligned}$$

It can also be easily seen that for each $i = 2, \dots, t+1$, F_i is $F_{i-1}A$-invariant and is centralized by a.

We next define the sections D_i by $D_i = F_i/\operatorname{Ker}(F_i \text{ on } \tilde{E}_{i+1})$ if $p_i \neq p$ for $i = 2, \dots, t$ and $D_i = F_i/\operatorname{Ker}(F_i \text{ on } \tilde{D}_{i+1})$ if $p_i = p$ for $i = 2, \dots, t-1$, and $D_t = F_t$ if $p_t = p$.

Claim A. Assume that $i \geqslant 2$ and that $p_i \neq p \neq p_{i-1}$. If $E_i \neq 1$, then D_i is a non-trivial F_{i-1}-invariant section such that $(F_{i-1} \text{ on } \tilde{E}_i) \equiv_w (F_{i-1} \text{ on } \tilde{D}_i)$.

Notice that Claim A can be proved by imitating the proof of Claim 1 in [**5**, Theorem 3] by replacing every occurrence of [**5**, Theorem 1] with Theorem 2.1 of this paper.

We shall proceed by proving the following claim.

Claim B. Assume that $i \geqslant 3$ and that $p_i = p = 3$. If $D_{i-1} \neq 1$, then $\operatorname{Ker}(F_i \text{ on } \tilde{E}_{i+1}) = 1$, $D_i \neq 1$. $\operatorname{Ker}(F_{i-1} \text{ on } \tilde{E}_i) = \operatorname{Ker}(F_{i-1} \text{ on } \tilde{D}_i)$.

By our hypothesis, $p_{i+2} \neq 3$. As $D_{i-1} = F_{i-1}/\operatorname{Ker}(F_{i-1} \text{ on } \tilde{E}_i) \neq 1$, we see that $E_i \neq 1$. Theorem 2.2 applies to the action of $E_{i-1}\langle a\rangle$ on $\tilde{E}_i$ and gives that $F_i = [[E_i, a]^2, F_{i-1}] \neq 1$. Now, $\operatorname{Ker}(F_{i-1} \text{ on } \tilde{D}_i) = \operatorname{Ker}(F_{i-1} \text{ on } D_i) = \operatorname{Ker}(F_{i-1} \text{ on } F_i/\operatorname{Ker}(F_i \text{ on } \tilde{D}_{i+1}))$.

We know that $\tilde{E}_{i+1} = \bigoplus_{j=1}^{l} W_j$, where $W_j = U_j/\Phi(E_{i+1})$ are irreducible ample $E_i\langle a\rangle$-submodules for $j = 1, \dots, l$. Since $\tilde{P}_{i+2}|_{E_{i+1}}$ is completely reducible and E_{i+1} is faithful on $\tilde{P}_{i+2}$, there exists at least one irreducible component of $\tilde{P}_{i+2}|_{E_{i+1}}$ on which U_j acts non-trivially. Let $\mathfrak{N}_j$ denote the set of all such components of $\tilde{P}_{i+2}|_{E_{i+1}}$. There are two cases: either (I) there is at least one N in $\mathfrak{N}_j$ on which $\Phi(E_{i+1})$ acts trivially; or (II) there is no N in $\mathfrak{N}_j$ on which $\Phi(E_{i+1})$ acts trivially.

In the latter case, for each irreducible component N of $\tilde{E}_{i+2}|_{E_{i+1}}$, $N \in \mathfrak{N}_j$ if and only if $\Phi(E_{i+1})$ acts non-trivially on N. (This is obvious when we replace i by $i+1$ in [**5**, Lemma 4].) Thus U_j is trivial on each irreducible component N of $\tilde{P}_{i+2}|_{E_{i+1}}$ lying outside $\tilde{E}_{i+2}$, because otherwise $N \in \mathfrak{N}_j$, implying that $\Phi(E_{i+1})$ and hence $[E_{i+1}, a]$ is non-trivial on N, which is a contradiction. It follows that $1 = \operatorname{Ker}(U_j \text{ on } \tilde{P}_{i+2}) = \operatorname{Ker}(U_j \text{ on } \tilde{E}_{i+2})$ when (II) holds.

Now suppose that $\operatorname{Ker}(U_j \text{ on } \tilde{E}_{i+2}) = 1$ for each $j = 1, \dots, s$ and $\operatorname{Ker}(U_j \text{ on } \tilde{E}_{i+2}) \neq 1$ for each $j = s+1, \dots, l$. One may observe that for each $j = s+1, \dots, l$,

$$\begin{aligned}\Omega_j = \{f \in W_j^* \mid {} & \text{there exists } N \text{ in } \mathfrak{N}_j \text{ on which } \Phi(E_{i+1}) \text{ acts trivially} \\ & \text{and } \operatorname{Ker}(U_j \text{ on } N)/\Phi(E_{i+1}) \subseteq \operatorname{Ker} f\}\end{aligned}$$

is an $E_i\langle a\rangle$-invariant subset of W_j^* such that $\langle \Omega_j\rangle = W_j^*$. For $j = 1, \dots, s$, let Ω_j denote the whole of W_j^*. Now for each $j = 1, \dots, l$, we appeal to Theorem 3.1 by letting $V = W_j$, $S = E_i$, $T = E_{i-1}$ and $\Omega = \Omega_j$ and get $C_{W_j}(a) \nsubseteq (W_j)_0$ and

$$\operatorname{Ker}(F_{i-1} \text{ on } F_i/\operatorname{Ker}(F_i \text{ on } C_{W_j}(a))) = \operatorname{Ker}(F_{i-1} \text{ on } F_i/\operatorname{Ker}(F_i \text{ on } C_{W_j/(W_j)_0}(a))).$$

We may observe that for each $j = 1, \dots, l$, $K_j\Phi(E_{i+1})/\Phi(E_{i+1}) \subseteq (W_j)_0$, where $K_j = \operatorname{Ker}(U_j \text{ on } \tilde{E}_{i+2})$. If not, then $j \in \{s+1, \dots, l\}$ and there exist $x \in K_j$, $f \in \Omega_j - C_{\Omega_j}(a)$ such that $f(x\Phi(E_{i+1})) \neq 0$. By the definition of Ω_j, we can find an irreducible submodule N of $\tilde{P}_{i+2}|_{E_{i+1}}$ on which U_j is non-trivial, $\Phi(E_{i+1})$ is trivial and $\operatorname{Ker}(U_j \text{ on } N)/\Phi(E_{i+1}) \subseteq$

$\operatorname{Ker} f$. Then $x \notin \operatorname{Ker}(U_j \text{ on } N)$. As $x \in \operatorname{Ker}(U_j \text{ on } \tilde{E}_{i+2})$, N lies outside $\tilde{E}_{i+2}|_{E_{i+1}}$: that is, $[E_{i+1}, a]$ acts trivially on N. Thus $[U_j, a]$ is trivial on N and so $f \in C_{\Omega_j}(a)$, which is a contradiction.

Hence $L_j = \operatorname{Ker}(C_{U_j}(a) \text{ on } \tilde{E}_{i+2}) \subseteq Y_j$ where $(W_j)_0 = Y_j/\Phi(E_{i+1})$ for $j = 1, \ldots, l$, implying that $L_j = \operatorname{Ker}(C_{U_j}(a) \text{ on } \tilde{E}_{i+2}) = \operatorname{Ker}(C_{Y_j}(a) \text{ on } \tilde{E}_{i+2})$ for $j = 1, \ldots, l$. We observe that

$$C_{W_j}(a)/C_{(W_j)_0}(a) \cong C_{U_j}(a)\Phi(E_{i+1})/C_{Y_j}(a)\Phi(E_{i+1})$$

and so $\operatorname{Ker}(F_i \text{ on } C_{W_j}(a)/C_{(W_j)_0}(a)) \supseteq \operatorname{Ker}(F_i \text{ on } C_{U_j}(a)/L_j)$ as $[\Phi(E_{i+1}), F_i] = 1$. Set $(\tilde{E}_{i+1})_0 = \bigoplus_{j=1}^{l} (W_j)_0$. Notice that

$$C_{\tilde{E}_{i+1}}(a) = \bigoplus_{j=1}^{l} C_{W_j}(a) = \bigoplus_{j=1}^{l} C_{U_j}(a)\Phi(E_{i+1})/\Phi(E_{i+1})$$

and

$$C_{(\tilde{E}_{i+1})_0}(a) = \bigoplus_{j=1}^{l} C_{(W_j)_0}(a)$$

and

$$\tilde{E}_{i+1}/(\tilde{E}_{i+1})_0 \cong \bigoplus_{j=1}^{l} W_j/(W_j)_0.$$

We observe that

$$\begin{aligned}
\operatorname{Ker}(F_{i-1} \text{ on } D_i) &= \operatorname{Ker}(F_{i-1} \text{ on } F_i/\operatorname{Ker}(F_i \text{ on } D_{i+1})) \\
&\subseteq \operatorname{Ker}(F_{i-1} \text{ on } F_i/\operatorname{Ker}(F_i \text{ on } C_{\tilde{E}_{i+1}/(\tilde{E}_{i+1})_0}(a))) \\
&\subseteq \operatorname{Ker}(F_{i-1} \text{ on } F_i/\operatorname{Ker}(F_i \text{ on } C_{W_j/(W_j)_0}(a))) \\
&= \operatorname{Ker}(F_{i-1} \text{ on } F_i/\operatorname{Ker}(F_i \text{ on } C_{W_j}(a)))
\end{aligned}$$

for each $j = 1, \ldots, l$. Thus we have

$$\operatorname{Ker}(F_{i-1} \text{ on } D_i) \subseteq \operatorname{Ker}(F_{i-1} \text{ on } F_i/\operatorname{Ker}(F_i \text{ on } \tilde{F}_{i+1})).$$

By [**5**, Lemma 1] applied to the action $E_i\langle a\rangle$ on $\tilde{E}_{i+1}$, we see that $\operatorname{Ker}(F_i \text{ on } \tilde{F}_{i+1}) = \operatorname{Ker}(F_i \text{ on } \tilde{E}_{i+1})$. Notice that $\operatorname{Ker}([E_i, a]^2 \text{ on } \tilde{E}_{i+1}) = \operatorname{Ker}([E_i, a]^2 \text{ on } \tilde{P}_{i+1}) = 1$ and so $\operatorname{Ker}(F_i \text{ on } \tilde{F}_{i+1}) = 1$. Consequently, $\operatorname{Ker}(F_{i-1} \text{ on } D_i) \subseteq \operatorname{Ker}(F_{i-1} \text{ on } F_i)$. As the reversed inclusion also holds, we have

$$\operatorname{Ker}(F_{i-1} \text{ on } \tilde{D}_i) = \operatorname{Ker}(F_{i-1} \text{ on } \tilde{F}_i) = \operatorname{Ker}(F_{i-1} \text{ on } [\tilde{E}_i, a]^2).$$

Applying Theorem 2.2 to $E_{i-1}\langle a\rangle$ on E_i gives $\operatorname{Ker}(F_{i-1} \text{ on } \tilde{E}_i) = \operatorname{Ker}(F_{i-1} \text{ on } [\tilde{E}_i, a]^2)$ and so $\operatorname{Ker}(F_{i-1} \text{ on } \tilde{E}_i) = \operatorname{Ker}(F_{i-1} \text{ on } \tilde{D}_i)$. We observe that $D_i \neq 1$ as $1 \neq D_{i-1} = F_{i-1}/\operatorname{Ker}(F_{i-1} \text{ on } \tilde{E}_i) = F_{i-1}/\operatorname{Ker}(F_{i-1} \text{ on } \tilde{D}_i)$, completing the proof of Claim B.

We are now ready to complete the proof of Theorem 3.2. Firstly, we shall observe that $D_2 \neq 1$, for if $p_2 = p$, then the fact that $D_2 \neq 1$ is a consequence of [**3**, Proposition 7.13] by observing that the critical index j appearing in that proposition can actually be taken as 1, which means that $E_2 \neq 1$ in our case. And if $p_2 \neq p$, Claim A applied to the action of E_1 on E_2 gives that $D_2 \neq 1$.

Suppose that $D_{i-1} \neq 1$ for some $i \geqslant 3$. If $p_{i-1} = p$, then $D_{i-1} = F_{i-1}/\operatorname{Ker}(F_{i-1} \text{ on } \tilde{D}_i)$ and so $D_i \neq 1$ holds. Assume that $p_{i-1} \neq p$. Appealing to Claims A and B, respectively, when $p_i \neq p$ and $p_i = p$, we see that $D_i \neq 1$ always holds.

Therefore, we have obtained a chain of non-trivial A-invariant sections $D_2, \ldots, D_t$ that are all centralized by a and $\operatorname{Ker}(D_{i-1} \text{ on } D_i) = 1$ for each $i = 3, \ldots, t$. □

4. Some technical results pertaining to the proof of Theorem C

Lemma 4.1. *Let $S \lhd SA$, where A is an abelian group and S is an s-group for some prime s which is coprime to $|A|$. Assume that S is abelian when $s = 2$. Let V be an irreducible kSA-module, where k is a splitting field for all subgroups of SA and is of characteristic not dividing $|SA|$. Suppose that S acts non-trivially and A acts fixed-point freely on V. Then there is a non-trivial subgroup D of A such that $[S, D]$ acts trivially on V.*

Proof. If S is abelian, D can be taken as the stabilizer of an S-homogeneous component of V, and the result follows. Therefore, we may assume that S is a non-abelian group of odd order. By induction on $|S|$, we may also assume that S is faithful on V. Now the result follows directly from [**8**, Theorem 4.1]. □

Lemma 4.2. *Let A be an abelian group whose order is a product of two odd primes and let G be a group on which A acts. If P_i, $i = 1, \ldots t$, are A-invariant sections of G such that P_i is a p_i-group, P_i is abelian when $p_i = 2$, $C_{P_i}(A) = 1$ for $i = 1, \ldots, t$ and P_i normalizes P_{i+1} with $\operatorname{Ker}(P_i$ on $P_{i+1}) = 1$ for $i = 1, \ldots, t-1$, then $t \leqslant 2$.*

Proof. We may assume that $t = 3$ and that P_1 is an elementary abelian group on which A acts irreducibly. If every element $a \in A$ of prime order centralizes P_1, then A must be cyclic of order p^2 for an odd prime p. Let $A = \langle \alpha \rangle$ and let $z = \alpha^p$. Now $[P_2, z] \neq 1$, because otherwise $A/\langle z \rangle$ acts fixed-point freely on P_2P_1. This leads to a contradiction by Lemma 4.1. Thus there exists $a \in A$ of prime order p such that $[P_1, a] \neq 1$. Now $[P_1, a] = P_1$ and so $p_1 \neq p$. We first consider the case $p_2 = p$. Theorem 2.2 applied to the action of $P_1\langle a \rangle$ on P_2 then tells us that $[P_2, a]^{p-1} \neq 1$. This enables us to apply [**5**, Lemma 1] to the action of $P_2\langle a \rangle$ on P_3. It follows that $[C_{P_3}(a), C_{[P_2,a]^{p-1}}(a)] \neq 1$, which is impossible as $A/\langle a \rangle$ is fixed-point free on $C_{P_3P_2}(a)$. Hence $p_2 \neq p$. Notice that $C_{P_2}(a) \neq 1$. Applying Theorems 2.1 and 2.2 to the action of $P_2\langle a \rangle$ on P_3, respectively, when $p_3 \neq p$ and $p_3 = p$, we see that $[C_{P_3}(a), C_{P_2}(a)] \neq 1$, which is not the case. □

Lemma 4.3. *Let $A = \langle a \rangle$ be a cyclic group of order p^n for some prime p, and let G be a group acted on by A. Suppose that $S \lhd GA$ is an s-group and that T is an A-invariant t-subgroup of G for distinct primes s and t that are both different from p such that*

$[S,T] \neq 1$ and $[T,z] = T$, where $z = a^{p^{n-1}}$. If $\Phi(T/T_0) \neq 1$, where $T_0 = C_T(S)$, then assume that there is an A-invariant h-subgroup H of G for a prime h different from p such that $H \leqslant C_G(\Phi(T/T_0))$, $H/C_H(T/T_0)$ is elementary abelian and $[T/T_0, H] = T/T_0$. Let V be a kGA-module on which S acts non-trivially and let k be a field of characteristic not dividing $|STHA|$. Then $[C_V(A), C_S(A)] \neq 1$.

Proof. We use induction on $|SA| + \dim_k V$. Set $\bar{S} = S/\operatorname{Ker}(S \text{ on } V)$. Let $\bar{S}_1$ be a minimal $T\langle\alpha\rangle$-invariant subgroup of $\bar{S}$ on which T acts non-trivially. Then $[\bar{S}_1, T] = \bar{S}_1$, $\bar{S}_1/\Phi(\bar{S}_1)$ is an irreducible $T\langle\alpha\rangle$-module, $[\Phi(\bar{S}_1), T] = 1$ and $\bar{S}_1$ is a special group. If $|\bar{S}_1| < |S|$, an induction argument gives that $[C_V(A), C_{\bar{S}_1}(A)] = [C_V(A), \overline{C_{S_1}(A)}] \neq 1$: that is, $[C_V(A), C_S(A)] \neq 1$. Thus $\bar{S}_1 = S$.

We may also assume that $G = STH$. Notice that $[T,z] = T$ acts non-trivially on each irreducible component of $\tilde{S}|_T$. It is easy to see that

$$\Omega = \{f \in (\tilde{S})^* \mid \text{there exists an irreducible component } N \text{ of } V|_S \\ \text{such that } \operatorname{Ker}(S \text{ on } N)\Phi(S)/\Phi(S) \subseteq \operatorname{Ker} f\}$$

is a GA-invariant subset that linearly spans the dual space $(\tilde{S})^*$. Applying [**9**, Theorem 2.1.A] to the action of $(T/T_0)A$ on $\tilde{S}$ with Ω, we see that $C_{\tilde{S}}(A) \nsubseteq \bigcap\{\operatorname{Ker} f \mid f \in \Omega - C_\Omega(z)\}$. This gives an $\tilde{x} \in C_{\tilde{S}}(A)$ and an $f \in \Omega - C_\Omega(z)$ such that $f(\tilde{x}) \neq 0$. Now $\tilde{x} \notin \operatorname{Ker}(\tilde{S} \text{ on } N)$ for some irreducible component N of $V|_S$ by the definition of Ω.

On the other hand, one more application of [**9**, Theorem 2.1.A] to the action of GA on V gives that $C_S(A) = C_{C_S(A)}(C_V(A))$ is contained in the kernel of each irreducible component of $V|_S$ on which $[S,z]$ acts non-trivially. It follows that $[S,z]$ is trivial on N: that is, $[S,z] \subseteq \operatorname{Ker} f$ and so $f \in C_\Omega(z)$, which is a contradiction. $\square$

Lemma 4.4. *Let $G = ST$, where $S \lhd G$, S is an s-group with $\Phi(S) \leqslant Z(S)$, $\Phi(\Phi(S)) = 1$, T/T_0 is a t-group for distinct primes s and t with $[\Phi(S), T] = 1$, $\Phi(T/T_0) \leqslant Z(T/T_0)$, $\Phi(\Phi(T/T_0)) = 1$, where $T_0 = C_T(S)$. Assume that Sylow 2-subgroups of G are abelian. Let α be an automorphism of G of order p for an odd prime p which leaves T invariant with $[T/T_0, \alpha] = T/T_0$. Let V be a $kG\langle\alpha\rangle$-module on which S acts faithfully for a field k of characteristic not dividing sp. If $[C_V(\alpha), C_S(\alpha)] = 1$, then $[S,T] = 1$.*

Proof. We first observe that $s \neq p$. If this is not the case, appealing to Theorem 2.2 with the action of $T/T_0\langle\alpha\rangle$ on $\tilde{S}$, we get $[\tilde{S}, \alpha]^{p-1} \neq 0$. Now [**5**, Lemma 1] applies to $S\langle\alpha\rangle$ on V and gives that $[C_V(\alpha), C_S(\alpha)] \neq 1$. Thus we have obtained that $s \neq p$. We may consider V, S, T/T_0 as an α-Fitting chain and by Theorem 2.3 we get sections D_2, D_3 centralized by α, of S and V, respectively, such that D_2 acts non-trivially on D_3. This contradiction completes the proof. $\square$

Lemma 4.5. *Let SA be a group where $S \lhd SA$, S is a q-group for an odd prime q, $\Phi(S) \leqslant Z(S)$, and A is cyclic of order pq for some prime p. Suppose that $[S, A_q]^{q-1} \nleqslant \Phi(S)$ and $[S, A_p] = S$, where A_p and A_q denote the Sylow p- and q-subgroups of A, respectively. Let V be a $\mathbb{C}SA$-module on which $[S, A_q]^{q-1}$ acts non-trivially. Then $C_V(A) \neq 0$.*

Proof. Assume the contrary. Set $\bar{S} = S/\operatorname{Ker}(S \text{ on } V)$. By [**5**, Lemma 1] applied to the action of $\bar{S}A_q$ on V, we see that $C_V(A_q) \neq 0$ and $\operatorname{Ker}(C_{\bar{D}}(A_q) \text{ on } C_V(A_q)) = \operatorname{Ker}(C_{\bar{D}}(A_q) \text{ on } V)$, where $D = [S, A_q]^{q-1}$. This gives

$$\operatorname{Ker}([C_{\bar{D}}(A_q), A_p] \text{ on } C_V(A_q)) = \operatorname{Ker}([C_{\bar{D}}(A_q), A_p] \text{ on } V)$$

also. If $[C_{\bar{D}}(A_q), A_p] \neq 1$, then we apply Lemma 4.1 to the action of $[C_{\bar{D}}(A_q), A_p]A_p$ on $C_V(A_q)$ and get a contradiction. Hence $[C_{\bar{D}}(A_q), A_p] = 1$, forcing $[\bar{D}, A_p] = 1$ by Thompson's $A \times B$ Lemma. But then $\bar{D} \leqslant \Phi(\bar{S}) = \overline{\Phi(S)}$, which is not the case. $\square$

5. Proof of Theorem C

Theorem C. *Let $G \lhd GA$, where A is an abelian group whose order is a product of three odd primes. Assume that Sylow 2-subgroups of G are abelian. If A acts fixed-point freely on G, then $f(G) \leqslant 3$.*

Proof. By Lemmas 8.1 and 8.2 in [**3**], we may assume the existence of an A-Fitting chain P_1, P_2, P_3, P_4 of length 4 in G, where $P_i = S_i/T_i$ are A-invariant sections of G satisfying the following.

(a) P_i is a p_i-group for some prime p_i, $\Phi(P_i) \leqslant Z(P_i)$, $\Phi(\Phi(P_i)) = 1$, and P_i has exponent p_i when p_i is odd, for $i = 1, 2, 3, 4$.

(b) $p_i \neq p_{i+1}$ for $i = 1, 2, 3$.

(c) S_i normalizes P_j for i, j with $i < j$, $T_i = \operatorname{Ker}(S_i \text{ on } P_{i+1})$ for $i = 1, 2, 3$ and $T_4 = 1$.

(d) $[\Phi(P_{i+1}), S_i] = 1$ for $i = 1, 2, 3$.

(e) $P_4 \leqslant F(G)$ and P_4 is irreducible as an $S_3S_2S_1A$-module.

We may also assume that P_1 is an elementary abelian group on which A acts irreducibly. Since A is nilpotent, it is a Carter subgroup of any semidirect product of it with a section of G. Thus A acts fixed-point freely on any section of this chain. As P_4 is an HA-module for $H = P_3S_2S_1$, we may assume that $(|P_4|, |HA|) = 1$ by the Fong–Swan Theorem.

If $[P_1, a] \neq 1$ for some element a in A of prime order, then $[P_1, a] = P_1$ and so $p_1 \neq p$. Now Theorem 3.2 gives an A-chain D_1, D_2, D_3 whose terms are all centralized by a such that $\operatorname{Ker}(D_i \text{ on } D_{i+1}) = 1$ for $i = 1, 2$. As $A/\langle a\rangle$ acts fixed-point freely on D_i for each $i = 1, 2, 3$, we get a contradiction by Lemma 4.2. Thus $[P_1, a] = 1$ for every element a in A of prime order. This forces A to be cyclic of order p^2q, for two distinct odd primes p and q. Let $A = A_p \times A_q$, where $|A_p| = p^2$ and $|A_q| = q$, and let z denote the unique element of order p in A_p. We have $[P_1, \langle z\rangle A_q] = 1$ and so $p_1 \neq p$, because otherwise $C_{P_1}(A_q) = 1$. As $p_1 \neq p$, we may assume that $[S_1, z] = 1$.

As A is cyclic, we may assume that the S_i are p_i-groups for $i = 1, 2, 3$. Passing to an irreducible A-tower in the sense of [**9**], we may also assume that P_i is a special group on the Frattini factor group of which $(\prod_{i>j} S_j)A$ acts irreducibly for each $i = 1, 2, 3, 4$ and $[S_{i+1}, S_i] = S_{i+1}$ for $i = 1, 2, 3$.

As A is nilpotent, HA is a relative M-group with respect to H by [**7**, 6.22]: that is, there is a subgroup HB of HA and an irreducible HB-module U such that $U^{HA} = P_4$ and $U|_H$ is irreducible. If $P_4|_H$ is irreducible, by [**6**, Theorem] we see that $C_H(A) \neq 1$. Thus $B \neq A$. On the other hand, by Mackey's Theorem, $P_4|_A \cong U|_B \otimes k(A/B)$, and so $C_{P_4}(B) \cong C_U(B) \otimes k(A/B)$ and $C_{P_4}(A) \cong C_U(B) \otimes C_{k(A/B)}(A/B)$. It follows that $C_U(B) = 1$ and so $C_{P_4}(B) = 1$. This argument shows the existence of a non-trivial proper subgroup B of A such that $C_{P_4}(B) = 1$.

If $|B| = q$, then we set $Y = P_4 C_H(A_p)$. As $C_Y(B) = 1$, Y must be nilpotent and so $C_H(A_p) = 1$. This forces $f(H) \leqslant 2$, which is not the case. Therefore, $|B|$ is either pq or a divisor of p^2.

We shall observe that $\tilde{P}_3$ is centralized by neither z nor A_q: assume that $[\tilde{P}_3, a] = 1$ for an element a in A of prime order. Then both P_2 and P_1 are centralized by a. Now P_1, P_2, P_3 are A-invariant sections on each of which $A/\langle a\rangle$ acts fixed-point freely, which is impossible by Lemma 4.2.

We first assume that $|B|$ divides p^2. Then $C_{P_4}(A_p) = 1$. Also, if P_3 is a p'-group, Lemma 4.1 applies to the action of P_3A_p on P_4 and gives that $[P_3, z] = 1$, which is not the case. Hence P_3 is a p-group. It follows that $[P_2, z] = P_2$, because otherwise we would have $[\tilde{P}_3, z] = 1$. Lemma 4.1 applied to the action of S_2A_p on P_4 leads to a contradiction.

Thus we have $B = \langle z\rangle A_q$. If P_3 is a p-group, then A_q is fixed-point free on $C_{P_4P_3}(z)$ implying that $C_{P_4}(z)$ and $C_{P_3}(z)$ commute. As $[\tilde{P}_3, z] \neq 1$, we have $[P_2, z] \neq 1$. Appealing to Lemma 4.4 with the action of $P_3[S_2, z]\langle z\rangle$ on P_4, we get $[C_{P_4}(z), C_{P_3}(z)] \neq 1$, which is not the case. Thus $p_3 \neq p$.

If, in addition, $p_3 \neq q$, then we consider an irreducible $[P_3, z]B$-submodule Y of P_4 on which $[P_3, z]$ acts non-trivially. It follows by Lemma 4.1 that $[P_3, z, A_q]$ is trivial on Y. We now have $[P_3, z, A_q] = 1$ by the faithful action of P_3 on P_4, forcing that $[P_3, z] \neq P_3$. Hence $[P_2, z] \neq 1$. If $p_2 \neq p$, we apply Lemma 4.3 by letting $V = P_4$, $S = P_3$, $T = [S_2, z]$ and $H = S_1$ and get $[C_{P_4}(A_p), C_{P_3}(A_p)] \neq 1$, which is a contradiction as A_q acts fixed-point freely on $C_{P_4P_3}(A_p)$. Thus we have $p_2 = p$ and so $[P_2, z] = 1$, which is not the case. Therefore, P_3 must be a q-group.

Notice that $[P_2, A_q] \neq 1$, because otherwise $[\tilde{P}_3, A_q] = 1$, which is not the case. Applying Theorem 2.2 to the action of P_2A_q on $\tilde{P}_3$, we see that $[P_3, A_q]^{q-1} \nleqslant \Phi(P_3)$. If $[P_3, z] = P_3$, then Lemma 4.5 applies to the action of P_3B on an irreducible P_3B-submodule of P_4 on which $[P_3, A_q]^{q-1}$ acts non-trivially, and it tells us that $C_{P_4}(B) \neq 1$, which is a contradiction. Hence $[P_3, z] \neq P_3$. This forces P_2 to be a p'-group and $[P_2, z] = P_2$, as $[P_1, z] = 1$. Now Lemma 4.3 applied to the action of $P_3S_2A_p$ on P_4 supplies that $C_{P_4P_3}(A_p)$ is not nilpotent, which is a contradiction that completes the proof. □

References

1. S. D. Bell and B. Hartley, A note on fixed-point-free actions of finite groups, *Q. J. Math.* **41**(162) (1990), 127–130.
2. K. N. Cheng, Finite groups admitting fixed point free automorphisms of order pq, Groups—St Andrews 1985, *Proc. Edinb. Math. Soc.* **30**(1) (1987), 51–56.

3. E. C. Dade, Carter subgroups and Fitting heights of finite solvable groups, *Illinois J. Math.* **13** (1969), 449–514.
4. G. Ercan and İ. Ş. Güloğlu, Finite groups admitting fixed-point free automorphisms of order pqr, *J. Group Theory* **7**(4) (2004), 437–446.
5. G. Ercan and İ. Ş. Güloğlu, Fixed point free action on groups of odd order, *J. Alg.* **320** (2008), 426–436.
6. I. M. Isaacs, Fixed points and characters in groups with noncoprime operator groups, *Can. J. Math.* **20** (1968), 1315–1320.
7. I. M. Isaacs, *Character theory of finite groups* (Dover Publications, New York, 1994).
8. E. Shult, On groups admitting fixed point free abelian operator groups, *Illinois J. Math.* **9** (1965), 1315–1320.
9. A. Turull, Fitting height of groups and of fixed points, *J. Alg.* **86** (1984), 555–566.
10. A. Turull, *Character theory and length problems: finite and locally finite groups*, NATO Science Series C: Mathematical and Physical Sciences, Volume 471, pp. 377–400 (Kluwer, Dordrecht, 1995).