NILPOTENT LENGTH OF A FINITE SOLVABLE GROUP WITH A FROBENIUS GROUP OF AUTOMORPHISMS

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Abstract We prove that a finite solvable group G admitting a Frobenius group FH of automorphisms of coprime order with kernel F and complement H such that $[G,F]=G$ and $C_{C_G(F)}(h)=1$ for all nonidentity elements $h\in H$, is of nilpotent length equal to the nilpotent length of the subgroup of fixed points of H.

1. **Introduction**

Let A be a finite group that acts on the finite solvable group G by automorphisms. There have been a lot of research to obtain information about certain group theoretical invariants of G in terms of the action of A on G. A particular major problem is to bound the nilpotent length $f(G)$ of G in terms of information about the structure of A alone when $C_G(A)=1$, that is, the action of A is fixed point free. One of the recent results in this framework is [2] in which Khukhro handled the case where $A=FH$ is a Frobenius group with kernel F and complement H. He proved that the nilpotent lengths of G and $C_G(H)$ are the same if $C_G(F)=1$ and $(|G|,|H|)=1$ and later in [3], he removed the coprimeness assumption of the theorem in [2]. In the present paper, we keep the coprimeness condition but weaken the fixed point freeness of F on G slightly, and obtain the same conclusion about the nilpotent length of G. More precisely, we prove the following:

Theorem *Let G be a finite solvable group admitting a Frobenius group of automorphisms FH of coprime order with kernel F and complement H such that $C_{C_G(F)}(h)=1$ for all nonidentity elements $h\in H$. Then $f([G,F])=f(C_{[G,F]}(H))$ and $f(G)\leq f([G,F])+1$.*

We obtained the following proposition which is crucial in proving the theorem above and is of independent interest, too.

Proposition *Let Q be a normal q-subgroup of a group having a complement FH which is a Frobenius group with kernel F and complement H such that $C_{C_Q(F)}(h)=1$ for all nonidentity elements $h\in H$. Assume further that $|FH|$ is not divisible by q and Q is of class at most 2. Let V be a $kQFH$-module where k is a field with characteristic not dividing $|QFH|$. Then we have*

$$Ker(C_{[Q,F]}(H) \text{ on } C_V(H)) = Ker(C_{[Q,F]}(H) \text{ on } V).$$

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Throughout the article all groups are finite. The notation and terminology are mostly standard.

2. **Proof of the Proposition**

In this section we establish the key result in proving the main theorem of this paper.

Proposition 2.1. *Let Q be a normal q-subgroup of a group having a complement FH which is a Frobenius group with kernel F and complement H such that $C_{C_Q(F)}(h) = 1$ for all nonidentity elements $h \in H$. Assume further that $|FH|$ is not divisible by q and Q is of class at most 2. Let V be a $kQFH$-module where k is a field with characteristic not dividing $|QFH|$. Then we have*

$$Ker(C_{[Q,F]}(H) \text{ on } C_V(H)) = Ker(C_{[Q,F]}(H) \text{ on } V).$$

Proof. Suppose the proposition is false and choose a counterexample with minimum $dim_k V + |QFH|$. We split the proof into a sequence of steps. To simplify the notation we set $K = Ker(C_Q(H) \text{ on } C_V(H))$.

Claim 1. We may assume that k is a splitting field for all subgroups of QFH.

Proof. We consider the QFH-module $\bar{V} = V \otimes_k \bar{k}$ where $\bar{k}$ is the algebraic closure of k. Notice that $dim_k V = dim_{\bar{k}} \bar{V}$ and $C_{\bar{V}}(H) = C_V(H) \otimes_k \bar{k}$. Therefore once the proposition has been proven for the group QFH on $\bar{V}$, it becomes true for QFH on V also. □

Claim 2. We have $Q = [Q, F]$ and hence $C_Q(F) \leq Q' \leq Z(Q)$.

Proof. We may assume that $[Q, F]$ acts nontrivially on V. If $[Q, F] \neq Q$, then the proposition holds by induction for the group $[Q, F]FH$ on V. Since $[Q, F, F] = [Q, F]$ due to the coprime action of F on Q, the conclusion of the proposition is true. This contradiction shows that $[Q, F] = Q$ and hence $C_Q(F) \leq Q'$. □

Claim 3. V is an irreducible QFH-module on which Q acts faithfully.

Proof. Since V is completely reducible as a QFH-module, $V = \bigoplus_{i=1}^{s} W_i$ for irreducible QFH-modules W_i. Suppose $s > 1$. Then we have

$$Ker(C_Q(H) \text{ on } C_{W_i}(H)) = Ker(C_Q(H) \text{ on } W_i)$$

for each W_i on which Q acts nontrivially by induction. This equality holds obviously also for each W_i on which Q acts trivially. Hence

$$Ker(C_Q(H) \text{ on } V) = \bigcap_{i=1}^{s} Ker(C_Q(H) \text{ on } C_{W_i}(H)) = K.$$

which is nothing but the claim of the theorem. Therefore we can regard V as an irreducible QFH-module.

We set next $\overline{Q} = Q/Ker(Q \text{ on } V)$ and consider the action of the group $\overline{Q}FH$ on V assuming $Ker(Q \text{ on } V) \neq 1$. An induction argument gives

$$Ker(C_{\overline{Q}}(H) \text{ on } C_V(H)) = Ker(C_{\overline{Q}}(H) \text{ on } V).$$

This leads to a contradiction as $\overline{C_Q(H)} = C_{\overline{Q}}(H)$ due to the coprime action of H on Q. Thus we may assume that Q acts faithfully on V. □

It should be noted that we need only to prove $K = 1$ due to the faithful action of Q on V. So we assume this to be false.

Claim 4. Let Ω denote the set of Q-homogeneous components of V, and let Ω_1 be an F-orbit on Ω. Set $H_1 = Stab_H(\Omega_1)$. Then H_1 is a nontrivial subgroup of H stabilizing exactly one element W of Ω_1 and all the remaining orbits of H_1 on Ω_1 are of length $|H_1|$. Furthermore K acts trivially on each member of Ω_1 except W.

Proof. Suppose that $H_1 = 1$. Pick an element W from Ω_1. Clearly, we have $Stab_H(W) \leq H_1 = 1$ and hence the sum $X = \sum_{h \in H} W^h$ is direct. It is straightforward to verify that $C_X(H) = \left\{\sum_{h\in H} v^h \mid v \in W\right\}$. By definition, K acts trivially on $C_X(H)$. Note also that K normalizes each W^h as $K \leq Q$. It follows now that K is trivial on X. Notice that the action of H on the set of F-orbits on Ω is transitive, and hence K is trivial on the whole of V contrary to Claim 3. Thus $H_1 \neq 1$.

Let now $S = Stab_{FH_1}(W)$ and $F_1 = F \cap S$. Then $|F : F_1| = |\Omega_1| = |FH_1 : S|$ and so $|S : F_1| = |H_1|$. Notice next that as $(|F_1|, |H_1|) = 1$ there exists a complement, say S_1, of F_1 in S with $|H_1| = |S_1|$ by Schur-Zassenhaus theorem. Therefore by passing, if necessary, to a conjugate of W in Ω_1, we may assume that $S = F_1H_1$, that is, W is H_1-invariant.

It remains to show that W is the only member of Ω_1 which is stabilized by H_1, and all the remaining orbits are of length $|H_1|$: Let $x \in F$ and $1 \neq h \in H_1$ such that $(W^x)^h = W^x$ holds. Then $[h, x^{-1}] \in F_1$ and so $F_1x = F_1x^h = (F_1x)^h$ implying the existence of an element $g \in F_1x \cap C_F(h)$ by Theorem 3.27 in [1]. Now the Frobenius action of H on F gives that $C_F(h) = 1$ and so $x \in F_1$. This means that $Stab_{H_1}(W^x) = 1$ for each $x \in F - F_1$. Then, as a consequence of the argument in the first paragraph, K acts trivially on W^x for every $x \in F - F_1$. □

Claim 5. F acts transitively on Ω and hence we have $H = H_1$.

Proof. The group H acts transitively on $\{\Omega_i \mid i = 1, 2, ..., s\}$, the collection of F-orbits on Ω. Let now $V_i = \bigoplus_{W\in\Omega_i} W$ for $i = 1, 2, ..., s$. Suppose that $H_1 = Stab_H(\Omega_1)$ is a proper subgroup of H. Equivalently, $s > 1$. By induction the proposition holds for the group QFH_1 on V_1, that is,

$$Ker(C_Q(H_1) \text{ on } C_{V_1}(H_1)) = Ker(C_Q(H_1) \text{ on } V_1).$$

In particular, we have $Ker(C_Q(H) \text{ on } C_{V_1}(H_1)) = Ker(C_Q(H) \text{ on } V_1)$. On the other hand we observe that $C_V(H) = \{u^{x_1} + u^{x_2} + ... + u^{x_s} \mid u \in C_{V_1}(H_1)\}$ where $x_1, ..., x_s$ is a complete set of right coset representatives of H_1 in H. By definition, K acts trivially on $C_V(H)$ and normalizes each V_i. Then K is trivial on $C_{V_1}(H_1)$ and hence on V_1. As K is normalized by H we see that K is trivial on each V_i and hence on V contrary to Claim 3. Therefore $H_1 = H$ and F acts transitively on Ω as desired. □

From now on the unique H-invariant element of Ω the existence of which is established by Claim 4 and Claim 5 will be denoted by W.

Claim 6. $C_Q(F) = 1$.

Proof. Due to the coprime action of H on $C_Q(F)$ and the fact that $C_Q(FH) = 1$, we have $C_Q(F) = [C_Q(F), H]$. Since $Z(Q/C_Q(W))$ acts by scalars on the homogeneous Q-module W, $Z(Q/C_Q(W))$ and H commute. In particular as $C_Q(F) \leq$

$Z(Q)$ and $C_Q(F) = [C_Q(F), H]$ we see that $C_Q(F) \leq [Z(Q), H] \leq C_Q(W)$. Then

$$C_Q(F) \leq \bigcap_{x \in F} C_Q(W)^x = C_Q(V) = 1,$$

as desired, since F acts transitively on Ω by Claim 5. □

Claim 7. Final Contradiction.

Proof. Since $1 \neq K \trianglelefteq C_Q(H)$, the group $L = K \cap Z(C_Q(H))$ is nontrivial. Pick $1 \neq z \in L$ and consider the group $Q_0 = \langle z^F \rangle$. As $C_Q(F) = 1$ by Claim 6, we have $[Q_0, F] = Q_0$. If $Q_0 \neq Q$, the proposition holds by induction for the group Q_0FH on V, that is,

$$Ker(C_{Q_0}(H) \text{ on } C_V(H)) = Ker(C_{Q_0}(H) \text{ on } V) = 1.$$

This leads to a contradiction since $z \in Ker(C_{Q_0}(H) \text{ on } C_V(H))$. Therefore $Q = Q_0$. Note that $Q = [Q, H]\, C_Q(H)$ as $(|Q|, |H|) = 1$. We have

$$[Q, L, H] \leq [Q', H] \leq [Z(Q), H] \leq C_Q(W)$$

and also $[L, H, Q] = 1$ as $[L, H] = 1$. It follows now by the three subgroup lemma that $[H, Q, L] \leq C_Q(W)$. On the other hand $[C_Q(H), L] = 1$ by the definition of L. Thus $LC_Q(W)/C_Q(W) \leq Z(Q/C_Q(W))$ and hence $z^f \in zC_Q(W)$ for any $f \in F_1$ due to the scalar action of $Z(Q/C_Q(W))$ on W. Recall that K acts trivially on $W^{g^{-1}}$ and hence $z^g \in C_Q(W)$ for any $g \in F - F_1$ by Claim 4. So we have $Q = \langle z \rangle C_Q(W)$ implying that $Q' \leq C_Q(W)$. This forces that

$$Q' \leq \textstyle\bigcap_{x \in F} C_Q(W)^x = C_Q(V) = 1,$$

as F acts transitively on Ω by Claim 5, that is, Q is abelian.

We consider now $\prod_{f \in F} z^f$. It is a well defined element of Q which lies in $C_Q(F) = 1$. Thus we have

$$1 = \prod_{f \in F} z^f = (\prod_{f \in F_1} z^f)(\prod_{f \in F - F_1} z^f) \in (\prod_{f \in F_1} z^f) C_Q(W) = z^{|F_1|} C_Q(W)$$

leading to the contradiction $z \in C_Q(W)$ as $|F_1|$ is coprime to $|z|$. This completes the proof of Proposition 2.1. □

Remark Our proof uses an inductive argument in which we need to know that every subgroup of FH containing F satisfies the same hypothesis. The assumption $C_{C_G(F)}(h) = 1$ for all nonidentity elements $h \in H$ is valid for any subgroup of FH containing F so that induction becomes possible. This property is heavily used in Claim 5. It is very natural to ask whether the proposition is true under the weaker condition $C_Q(FH) = 1$. We don't yet know the answer.

3. **The Main Theorem**

In this section we prove our main result which gives a bound for the nilpotent length of solvable groups admitting a coprime Frobenius group of automorphisms under some additional hypothesis.

Theorem 3.1. *Let G be a finite solvable group admitting a Frobenius group of automorphisms FH of coprime order with kernel F and complement H such that $C_{C_G(F)}(h) = 1$ for all nonidentity elements $h \in H$. Then $f([G,F]) = f(C_{[G,F]}(H))$ and $f(G) \leq f([G,F]) + 1$.*

Proof. $C_G(F)$ is nilpotent by a well known result of Thompson as any element of prime order in H acts fixed point freely on $C_G(F)$. Note also that $G/[G,F]$ is covered by the image of $C_G(F)$ due to the coprime action of F on G and so $f(G) \leq f([G,F]) + 1$. Therefore we may assume $G = [G,F]$ and prove that $f(G) = f(C_G(H))$. Let $f(G) = n$. We proceed by induction on the order of G. The theorem is trivially true when $G = 1$. We assume now that the theorem is true for every group satisfying the hypothesis and of order smaller than the order of G. As $G = [G,F]$ and $(|G|, |FH|) = 1$, there exists an irreducible FH-tower $\hat{P}_1, ..., \hat{P}_n$ in the sense of [4] where

(a) $\hat{P}_i$ *is an FH-invariant p_i-subgroup, p_i is a prime, $p_i \neq p_{i+1}$, for $i = 1, ..., n-1$;*

(b) $\hat{P}_i \leq N_G(\hat{P}_j)$ *whenever $i \leq j$;*

(c) $P_n = \hat{P}_n$ *and* $P_i = \hat{P}_i / C_{\hat{P}_i}(P_{i+1})$ *for $i = 1, ..., n-1$ and $P_i \neq 1$ for $i = 1, ..., n$;*

(d) $\Phi(\Phi(P_i)) = 1$, $\Phi(P_i) \leq Z(P_i)$, *and* $exp(P_i) = p_i$ *when p_i is odd for $i = 1, ..., n$;*

(e) $[\Phi(P_{i+1}), P_i] = 1$ *and* $[P_{i+1}, P_i] = P_{i+1}$ *for $i = 1, ..., n-1$;*

(f) $(\Pi_{j<i}\hat{P}_j)FH$ *acts irreducibly on $P_i/\Phi(P_i)$ for $i = 1, ..., n$;*

(g) $P_1 = [P_1, F]$.

Set now $X = \prod_{i=1}^{n} \hat{P}_i$. As $P_1 = [P_1, F]$ by *(g)*, we observe that $X = [X, F]$ and so F is not contained in $Ker(FH \text{ on } X)$. Therefore $FH/Ker(FH \text{ on } X)$ is a Frobenius group of automorphisms of the group X. If X is proper in G, by induction we have $f(X) = f(C_X(H))$ and so the theorem follows. Hence $X = G$. Lemma 1.3 in [2] shows that $C_G(H) \neq 1$, that is $f(C_G(H) \geq 1$. Therefore the theorem is true if $G = F(G)$. We set next $\overline{G} = G/F(G)$. As $\overline{G}$ is a nontrivial group such that $\overline{G} = \left[\overline{G}, F\right]$, it follows by induction that $f(\overline{G}) = n - 1 = f(C_{\overline{G}}(H))$. That is, $Y = [C_{\hat{P}_{n-1}}(H), ..., C_{\hat{P}_1}(H)] \nleq F(G) \cap \hat{P}_{n-1} = C_{\hat{P}_{n-1}}(\hat{P}_n)$.

Note that $C_{[\hat{P}_{n-1},F]}(H) = C_{\hat{P}_{n-1}}(H)$ as $C_{\hat{P}_{n-1}}(FH) = 1$. Also $[\hat{P}_{n-1}, F] \neq 1$ because otherwise F centralizes P_i for each $i \leq n-1$ contradicting the fact that $P_1 = [P_1, F]$. By Proposition 2.1 applied to the action of the group $\hat{P}_{n-1}FH$ on the module $\hat{P}_n/\Phi(\hat{P}_n)$ we get

$$Ker(C_{\hat{P}_{n-1}}(H) \text{ on } C_{\hat{P}_n/\Phi(\hat{P}_n)}(H)) = Ker(C_{\hat{P}_{n-1}}(H) \text{ on } \hat{P}_n/\Phi(\hat{P}_n)).$$

It follows now that Y does not centralize $C_{\hat{P}_n}(H)$ and hence $f(C_G(H) = n = f(G)$. This completes the proof. □

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