METU, Fall 2011, Math 523.

Homework 9

1. (*Exercise 5.40. Marcus*) Let $K = \mathbf{Q}(\sqrt{m})$ be a real quadratic field where m is a squarefree integer.

   (a) Assume that $m \equiv 1 \pmod 4$. Let a be smallest positive integer such that

   $$ma^2 \pm 4 = b^2$$

   for some positive integer b. Explain why such integers a and b exist and prove that $u = (a\sqrt{m} + b)/2$ is the fundamental unit.

   (b) Describe a similar procedure for other cases $m \equiv 2, 3 \pmod 4$.

   (c) For each squarefree integer $2 \leq m \leq 17$, find u such that $\mathcal{O}_K^\times = \{\pm u^k : k \in \mathbf{Z}\}$.

2. Let K be a cubic extension of $\mathbf{Q}$ with only one real embedding. Let u be the fundamental unit in $\mathcal{O}_K^\times$.

   (a) (*Exercise 5.35. Marcus*) Show that $u^3 > |d_K|/4 - 7$.

   (b) If α is a real root of $x^3 - 2$ and $K = \mathbf{Q}(\alpha)$, then show that $u = \alpha^2 + \alpha + 1$.

   (c) If α is a real root of $x^3 - 4x + 4$ and $K = \mathbf{Q}(\alpha)$, then show that $u^2 = -\alpha + 1$.

3. (*The simplest cubic fields*) Let a be an integer such that $p = a^2 + 3a + 9$ is a prime number. Set $f(x) = x^3 - ax^2 - (a + 3)x - 1$. Let ρ be a root of f and $K = \mathbf{Q}(\rho)$.

   (a) Show that f is irreducible and compute d_K.

   (b) Is the extension $K/\mathbf{Q}$ normal? How many real embeddings are there?

   (c) Prove that $u = \rho$ and $v = \rho + 1$ are units in $\mathcal{O}_K$.

   (d) Find generators for the unit group $\mathcal{O}_K^\times$.

4. Let $K = \mathbf{Q}(\zeta_p)$ be the p-th cyclotomic field and let u be a unit in $\mathcal{O}_K$. Show that u is the product of a real unit and a root of unity. Does this contradict to Dirichlet's Unit Theorem?

5. Let $m \geq 3$ be an integer and $K = \mathbf{Q}(\zeta_m)$. If k is an integer relatively prime to m then show that $\epsilon_k = \sin(k\pi/m)/\sin(\pi/m)$ is a real unit in $\mathcal{O}_K^\times$.