METU, Fall 2011, Math 523.

Homework 8

1. Let K be a number field. If $e(\mathfrak{p}|(p)) = 1$ for each prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ lying over each prime $p \in \mathbf{Z}$, then show that $K = \mathbf{Q}$.

2. For squarefree integers $-10 \leq m \leq 10$, compute the class number of $K = \mathbf{Q}(\sqrt{m})$.

3. Let $K = \mathbf{Q}(\sqrt{-3})$ and $L = \mathbf{Q}(\sqrt{-6})$. Prove that $h_{KL} \neq h_K h_L$.

4. There are 9 imaginary quadratic fields $K = \mathbf{Q}(\sqrt{m})$ with class number one (this is a well known fact due to Heegner and Stark). The set of squarefree integers m giving such fields is
$$M = \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}.$$
   - Verify that the imaginary quadratic field $K = \mathbf{Q}(\sqrt{m})$ has class number one, for each integer $m \in M$.
   - (*Exercise 5.10. Marcus*) Let $-2000 < m < 0$ be a squarefree negative integer such that $K = \mathbf{Q}(\sqrt{m})$ has class number one. Show that $m \in M$.

5. Let α be a root of $x^3 - 17$ and let $K = \mathbf{Q}(\alpha)$. Recall that $\mathcal{O}_K$ has an integral basis $\{1, \alpha, \beta\}$ where $\beta = (\alpha^2 - \alpha + 1)/3$. Show that $h_K = 1$.