METU, Fall 2011, Math 523.

Homework 5

1. Let R be an integral domain. For nonzero ideals $I, J \subset R$, define the relation

$$I \sim J \iff \alpha I = \beta J \text{ for some } \alpha, \beta \in R.$$

   - Prove that $\sim$ is an equaivalence relation.
   - If I is principal then describe the corresponding equivalence class.
   - Given an ideal I, suppose that there is an ideal J such that IJ is principal. Does this property make the set of ideal classes a group? If so what is the group operation?

2. Let R be an integral domain. Prove that the followings are equivalent.

   - Every ideal is finiteley generated.
   - Every ascending chain of ideals stabilizes (**A**scending **C**hain **C**ondition).
   - Every non-empty set $\mathcal{S}$ of ideals has a maximal member.

3. Let K be a number field of degree n over $\mathbf{Q}$. Show that every non-zero ideal $\mathfrak{a} \subset \mathcal{O}_K$ is a free abelian group of rank n.

4. Prove that a Dedekind domain is a unique factorization domain if and only if it is a principal ideal domain.

5. Let $K = \mathbf{Q}(\alpha)$ where $\alpha = \sqrt[3]{2}$.

   - Consider the ideal $(5) \subset \mathcal{O}_K$. Verify that $(5) = (5, \alpha + 2)(5, \alpha^2 + 3\alpha - 1)$.
   - Set $\mathfrak{p} = (5, \alpha^2 + 3\alpha - 1)$. Show that there is an endomomorphism

$$\mathbf{Z}[x]/(5, x^2 + 3x - 1) \twoheadrightarrow \mathcal{O}_K/\mathfrak{p}.$$

   - What can you say about $\mathfrak{p}$ and $\mathcal{O}_K/\mathfrak{p}$?