**METU, Fall 2011, Math 523.**

# Homework 3

1. Let $K$ be a quadratic extension of $\mathbf{Q}$.
   - Show that $K = \mathbf{Q}(\sqrt{m})$ for some integer $m \in \mathbf{Z}$.
   - Let $m \neq 0, 1$ be a squarefree integer. Show that the quadratics fields $\mathbf{Q}(\sqrt{m})$ are pairwise distinct.
2. Find a $6 \times 6$ matrix $M$ with coefficients from $\mathbf{Z}$ such that the minimal polynomial of $\alpha = \sqrt[3]{2} + \sqrt{5}$ over $\mathbf{Q}$ is given by the determinant of $xI - M$.
3. Show that $f(x) = x^3 + 5x + 1$ is irreducible. Let $\alpha$ be a root of $f(x)$ and let $K = \mathbf{Q}(\alpha)$.
   - Calculate $T_{\mathbf{Q}}^{K}(\alpha^i)$ for $i \in \{0, 1, 2, 3\}$.
   - Calculate $N_{\mathbf{Q}}^{K}(\alpha - j)$ for $j \in \{0, 1, 2\}$.
4. Set $\alpha = \sqrt[4]{2}$ and $K = \mathbf{Q}(\alpha)$. Use the trace map $T_{\mathbf{Q}}^{K} : K \to \mathbf{Q}$ to show that $\sqrt{3}$ is not an element of $K$.
5. Consider the fifth cyclotomic field $K = \mathbf{Q}(\zeta_5)$. It is easy to see that $\{1, \zeta_5, \zeta_5^2, \zeta_5^3\}$ is a basis for $K$ over $\mathbf{Q}$.
   - Show that $\mathrm{disc}(1, \zeta_5, \zeta_5^2, \zeta_5^3) = 5^3$.
   - Is it true that $\sqrt{5} \in K$? If it is true, than write $\sqrt{5}$ in the basis given above?