METU, Fall 2011, Math 523.

Homework 2

(due October 12)

1. Let L/K be a finite field extension in complex numbers and let α be a nonzero element in L. If $f(x) \in K[x]$ is a monic polynomial of smallest degree such that $f(\alpha) = 0$ then show that $f(x)$ is unique and irreducible.

2. Let $f(x)$ be a polynomial of degree n over K (not necessarily irreducible) and let L be the splitting field for f over K. By definition $L = K(\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbf{C}$ are roots of $f(x)$.
   - Explain briefly why L/K is normal.
   - Show that $[L : K]$ divides $n!$.

3. Let $\zeta_7 = e^{2\pi i/7}$ and consider $L = \mathbf{Q}(\zeta_7)$, the 7-th cyclotomic field. Let K be its unique subfield such that $[L : K] = 3$. Does there exist an element $\alpha \in L \setminus K$ such that $\alpha^3 \in K$? (Hint: You may use the fact that $\mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q}) \cong (\mathbf{Z}/m\mathbf{Z})^\times$.)

4. Let $K = \mathbf{Q}(\alpha, \beta)$ where $\alpha^4 - 2 = 0$ and $\beta^2 + 1 = 0$.
   - Show that $[K : \mathbf{Q}] = 8$.
   - Show that $K/\mathbf{Q}$ is normal and determine $\mathrm{Gal}(K/\mathbf{Q})$.
   - List all subfields of K using the Galois correspondence.

5. Construct a field $\mathbf{F}_{16} \cong R/\mathfrak{m}$ with 16 elements by choosing a ring R and a maximal ideal $\mathfrak{m} \subset R$ suitably.
   - Determine an element $r \in R$ such that the corresponding element in $\mathbf{F}_{16}$ generates the multiplicative group $\mathbf{F}_{16}^\times$.
   - Similarly construct a field $\mathbf{F}_8$ with 8 elements. Find an embedding φ from $\mathbf{F}_8$ into $\mathbf{F}_{16}$. How many different embeddings are there?
   - Is there a nontrivial automorphism of $\mathbf{F}_{16}$ fixing the subfield $\varphi(\mathbf{F}_8)$ pointwise?