METU, Fall 2011, Math 523.

Homework 1

(due October 5)

1. Let R be a ring (i.e. a commutative ring with identity). Show that

   - Every maximal ideal of R is a prime ideal.
   - An ideal $I \subset R$ is maximal $\Leftrightarrow$ R/I is a field.
   - An ideal $I \subset R$ is prime $\Leftrightarrow$ R/I is an integral domain.

2. We know that $R[x]$ is a principal ideal domain if R is a field (because the division algorithm works). What about the converse? If $R[x]$ is a principal ideal domain, what can you say about R? Is it necessarily a field?

3. Set $w = (\sqrt{-23} + 1)/2$ and consider the ring $\mathcal{O} = \mathbf{Z}[w]$. Let $\mathfrak{a} \subset \mathcal{O}$ be an ideal generated by 2 and w. Determine for each $n \in \{1, 2, 3\}$ if $\mathfrak{a}^n$ is principal or not?

4. Let R be a principal ideal domain. If $\mathfrak{p} \subset R$ is a prime ideal then prove that the quotient $R/\mathfrak{p}$ is also a principal ideal domain. Can you drop the primeness condition on the ideal $\mathfrak{p}$ and still have the same property?

5. The product of two ideals I and J consists of all finite sums of products ij where $i \in I$ and $j \in J$. Prove that
$$I \cdot J = I \cap J$$
   if I and J are relatively prime. Is the above equality still true if we drop the condition of being relatively prime on ideals I and J?