M E T U
Department of Mathematics

Group	Algebraic Number Theory **Midterm 2**	List No.

Code : *Math 523* Acad. Year : *2011* Semester : *Fall* Instructor : *Küçüksakallı* Date : *Dec 6, 2011* Time : *10:40* Duration : *110 minutes*	Name : Last Name : Signature : 7 QUESTIONS ON 4 PAGES 60 TOTAL POINTS

1	2	3	4	5	6	7		

1. (12pts) Let K and L be distinct **quadratic** number fields.

- Show that $K = \mathbf{Q}(\sqrt{m})$ for some squarefree integer $m \in \mathbf{Z}$. (Hint: Pick $\alpha \in K \setminus \mathbf{Q}$ and construct $\sqrt{m}$ in terms of α.)

- Give an example of K and L such that $\mathcal{O}_{KL} \neq \mathcal{O}_K\mathcal{O}_L$.

- Is the extension $KL/\mathbf{Q}$ normal? If so, what is the Galois group?

2. (8pts) Let R be a Dedekind domain. If R is a UFD, then show that it is a PID.

3. (8pts) Let α be a complex number such that $\alpha^3 = -(\alpha+1)$. Set $K = \mathbf{Q}(\alpha)$. Give an integral basis for $\mathcal{O}_K$ and compute the discriminant d_K. Find the ideal prime decomposition of the ideal $(31) \subset \mathcal{O}_K$. (Hint: $3^3 = -(3+1) \pmod{31}$.)

4. (8pts) Explain briefly why the ring of Gaussian integers $\mathbf{Z}[i]$ is a Dedekind domain and find the ideal prime decomposition of $\mathfrak{a} = (30, 21 + 3i)$.

5. (8pts) Let $\alpha = \sqrt[3]{19}$ and $\beta = 6/(\sqrt[3]{19} - 1)$ be elements of $L = \mathbf{Q}(\sqrt[3]{19})$. You are given that $\{1, \alpha, \beta\}$ is an integral basis for L. Show that $[\mathcal{O}_L : \mathbf{Z}[\alpha]] = 3$ and $[\mathcal{O}_L : \mathbf{Z}[\beta]]$ is not divisible by 3. Find the ideal prime decomposition of ideals $(2), (3)$ and (5) in L.

6. (6pts) Let L/K be a normal extension of number fields. Let $\mathfrak{P}$ be a prime of $\mathcal{O}_L$ lying over $\mathfrak{p}$ of $\mathcal{O}_K$.

- Give the definitions of decomposition and inertia groups:
  - $D(\mathfrak{P}|\mathfrak{p}) =$
  - $I(\mathfrak{P}|\mathfrak{p}) =$
- Explain briefly why D/I is cyclic.

7. (10pts) Let $L = \mathbf{Q}(\sqrt[3]{5}, \sqrt{-3})$.

- Show that L is the splitting field of $x^3 - 5$ over $\mathbf{Q}$.
- Prove that $L/\mathbf{Q}$ is a normal extension with Galois group isomorphic to S_3.
- Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_L$ lying over $(5) \subset \mathbf{Z}$.
  - Compute the ramification index $e(\mathfrak{P}|(5))$ and the residual degree $f(\mathfrak{P}|(5))$.
  - Determine the cardinality of the groups $D(\mathfrak{P}|(5))$ and $I(\mathfrak{P}|(5))$.