M E T U
Department of Mathematics

Group	Algebraic Number Theory Midterm 1		List No.
Code : *Math 523* Acad. Year : *2011* Semester : *Fall* Instructor : *Küçüksakallı* Date : *Nov 1, 2011* Time : *10:40* Duration : *110 minutes*	Last Name : Name : Department : Signature :	Student No. : Section :	
	6 QUESTIONS ON 4 PAGES 60 TOTAL POINTS		

1	2	3	4	5	6		

1. (12pts) Let K and L be number fields. Determine for each of the following statements if it is true or not.

- **(4pts)** $K \cap L = \mathbf{Q}$ if and only if $\gcd(d_K, d_L) = 1$.

- **(4pts)** If $K = \mathbf{Q}[\alpha]$ for some algebraic integer α, then $\mathbf{Z}[\alpha]$ is an additive subgroup of $\mathcal{O}_K$ with finite index.

- **(4pts)** $\mathcal{O}_{KL} = \mathcal{O}_K \mathcal{O}_L$.

2. (8pts) Let $R = \mathbf{Z}[x]/(2, x^3 + x + 1)$ and $S = \mathbf{Z}[t]/(2, t^4 + t^3 + t^2 + t + 1)$. Prove that R and S are both fields. Is it possible to write a ring homomorphism $\phi : R \to S$ or $\psi : S \to R$?

3. (8pts) Let $\zeta_7 = e^{2\pi i/7}$ and consider $L = \mathbf{Q}(\zeta_7)$, the 7-th cyclotomic field. Let K be its unique subfield such that $[L : K] = 3$. Does there exist an element $\alpha \in L \setminus K$ such that $\alpha^3 \in K$?

4. (8pts) Let $f(x) = x^3 + ax + b$ be an irreducible polynomial over $\mathbf{Z}$. If α is a root of $f(x)$ and $K = \mathbf{Q}(\alpha)$, then calculate $T^K_{\mathbf{Q}}(\alpha^i)$ for $i \in \{0, 1, 2, 3\}$ and $N^K_{\mathbf{Q}}(\alpha - j)$ for $j \in \{0, 1, 2\}$.

5. (8pts) Let p be an odd prime. Show that $\mathrm{disc}(1, \zeta_p, \ldots, \zeta_p^{p-2}) = \pm p^{p-2}$ where plus sign holds if and only if $p \equiv 1 \pmod 4$. Show that $\mathbf{Q}(\zeta_p)$ has a unique subfield K such that $[K : \mathbf{Q}] = 2$. Find an element α such that $K = \mathbf{Q}(\alpha)$.

6. **(16pts)** Let K be a quadratic number field. (In other words $[K : \mathbf{Q}] = 2$.)

- Show that $K = \mathbf{Q}(\sqrt{m})$ for some squarefree integer $m \in \mathbf{Z}$.

- Let $\mathcal{O}_K$ be the set of algebraic integers in K. Prove that

$$\mathcal{O}_K = \begin{cases} \{a + b\sqrt{m} : a, b \in \mathbf{Z}\} & if \quad m \equiv 2, 3 \pmod 4, \\ \{(a + b\sqrt{m})/2 : a, b \in \mathbf{Z},\ a \equiv b \pmod 2\} & if \quad m \equiv 1 \pmod 4. \end{cases}$$

- Find an integral basis for $\mathcal{O}_K$ and compute the discriminant d_K.

- Define $w = (\sqrt{d_K} + d_K)/2$. Show that $\mathcal{O}_K = \mathbf{Z}[w]$.