M E T U
Department of Mathematics

Group	Algebraic Number Theory **Final**	List No.

Code : *Math 523* Acad. Year : *2011* Semester : *Fall* Instructor : *Küçüksakallı* Date : *10/01/2011* Time : *10:01* Duration : *180 minutes*	Name : Last Name : Signature : 8 QUESTIONS ON 6 PAGES 80 TOTAL POINTS

1	2	3	4	5	6	7		

1. (6pts) If $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ are two integral bases for some number field K then show that $\mathrm{disc}(\alpha_1, \ldots, \alpha_n) = \mathrm{disc}(\beta_1, \ldots, \beta_n)$.

2. (6pts) Find a 6×6 matrix M with coefficients from $\mathbf{Z}$ such that the minimal polynomial of $\alpha = \sqrt[3]{5} + \zeta_3$ over $\mathbf{Q}$ is given by the determinant of $xI - M$.

3. (14pts) Let $f(x) = x^3 + x - 3$.

- Show that $f(x)$ is irreducible over $\mathbf{Q}$.

- Show that $f(x) = 0$ has a unique real solution $\alpha > 1.2$.

- Let $K = \mathbf{Q}(\alpha)$. Find an integral basis for $\mathcal{O}_K$ and evaluate d_K.

- Show that $\mathcal{O}_K^\times = \{\pm u^k : k \in \mathbf{Z}\}$ for some $u > 1$.

- Show that $\epsilon = \alpha - 1$ is a unit in $\mathcal{O}_K$.

- Using the fact $u^3 > |d_K|/4 - 7$, determine u in terms α.

4. (14pts) Let $K = \mathbf{Q}(\sqrt{-23})$.

- Show that $\alpha = (\sqrt{-23}+1)/2$ is an algebraic integer.

- Compute $\operatorname{disc}(1,\alpha)$ and show that $\{1,\alpha\}$ is an integral basis for $\mathcal{O}_K$.

- Show that the Minkowski's constant M_K is less than 5.

- Find the ideal prime decomposition of ideals generated by 2 and 3 in $\mathcal{O}_K$.

- Find $\mathcal{O}_K$-ideals $\mathfrak{p}_2$ and $\mathfrak{p}_3$ of norms 2 and 3 respectively so that $[\mathfrak{p}_2] = [\mathfrak{p}_3]$ in $\mathrm{Cl}(K)$.

- Show that $\mathfrak{p}_2$ is not principal whereas $\mathfrak{p}_2^3$ is principal.

- What is the class number h_K?

5. (14pts) Let $K = \mathbf{Q}(\sqrt[3]{5}, \zeta_3)$. Recall that $K/\mathbf{Q}$ is normal and $\mathrm{Gal}(K/\mathbf{Q}) \cong S_3$. You can also use the fact that $\mathcal{O}_{\mathbf{Q}(\sqrt[3]{5})} = \mathbf{Z}[\sqrt[3]{5}]$.

- Determine all primes in $\mathbf{Z}$ which ramify in K.

- Let $\mathfrak{P} \subset \mathcal{O}_K$ be a prime ideal lying over 7. Find $N(\mathfrak{P})$.

- Show that $(\frac{K/\mathbf{Q}}{\mathfrak{P}})(\zeta_3) = \zeta_3$.

- Evaluate $N^K_{\mathbf{Q}}(4 - \zeta_3)$. If $4 - \zeta_3 \in \mathfrak{P}$, then find $(\frac{K/\mathbf{Q}}{\mathfrak{P}})(\sqrt[3]{5})$.

- Let $f(x) = \min(\sqrt[3]{5} + \zeta_3, \mathbf{Q})$, a polynomial of degree six in $\mathbf{Z}[x]$. Does there exist a prime $p > 5$ such that $f(x)$ is irreducible modulo p.

6. (6pts) Let K be a number field and let $\mathfrak{a} \subset \mathcal{O}_K$ be a nonzero ideal. Prove that $N(\mathfrak{a})$ divides $N_{\mathbf{Q}}^{K}(\alpha)$ for all $\alpha \in \mathfrak{a}$.

7. (6pts) Let $K = \mathbf{Q}(i)$ and let p be an odd prime. Show that p splits in K if and only if

$$\left(\frac{d_K}{p}\right) = 1.$$

Let $\mathfrak{p} = (\pi)$ be a prime ideal of $\mathcal{O}_K$ lying over p generated by $\pi \in \mathcal{O}_K$. What is $N_{\mathbf{Q}}^{K}(\pi)$? Prove that an odd prime $p = x^2 + y^2$ if and only if $p \equiv 1 (\mathrm{mod}\ 4)$.

8. **(14pts)** Let $K = \mathbf{Q}(\zeta_p)$ be the p-th cyclotomic field for some odd prime p and let $K^+ = K \cap \mathbf{R}$ be its maximal real subfield.

- Give the multiplicative $\mathbf{Z}$-module structure of both $\mathcal{O}_K^\times$ and $\mathcal{O}_{K^+}^\times$ using Dirichlet's unit theorem.

- Let ϵ be a unit in $\mathcal{O}_K$. Show that each conjugate of $\epsilon/\bar{\epsilon}$ has absolute value 1.

- If all conjugates of an algebraic integer have absolute value 1 then show that it must a be root of unity.

- Prove that $\epsilon/\bar{\epsilon} = \zeta_p^a$ for some integer $a \in \mathbf{Z}$.

- Show that $\epsilon = \zeta_p^b \eta$ for some integer $b \in \mathbf{Z}$ and real unit $\eta \in \mathcal{O}_{K^+}$. Does this contradict Dirichlet's unit theorem?