METU Department of Mathematics

Math 112

Discrete Mathematics

GRAPHS

WALKS, TRAILS, PATHS AND CYCLES
CONNECTEDNESS
TREES
EULERIAN TOURS, HAMILTONIAN CYCLES
PLANARITY

(Week 14)

LECTURE NOTES

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WALKS, TRAILS, PATHS AND CYCLES

*In a graph G, a sequence of vertices $v_{i_1} v_{i_2} \cdots v_{i_k}$ such that v_{i_j} and $v_{i_{j+1}}$ are adjacent for $j = 1, \cdots, k-1$ is called a **walk of length** **k** from v_{i_1} to v_{i_k}. A walk is called a **trail** if no edge is repeated and if no vertex is repeated it is called a **path.** If the length of a walk is even (resp. odd), it is called an even (resp. odd) walk.*



walk *trail* *path*

*If $v_{i_1} = v_{i_k}$ then the walk is said to be **closed**, otherwise it is called **open**. A closed trail is called a **tour**. Although in a path, repetition of vertices is not allowed, by letting $v_{i_1} = v_{i_k}$ exceptionally, we obtain a closed path which is also called a **cycle** (or a **circuit**).*



closed walk *closed trail (tour)* *closed path (cycle/circuit)*

Theorem 1. *A graph is bipartite if and only if it contains no odd cycles.*

Sketch of proof. *Suppose that G is bipartite with (v_1, v_2) being the partition of the vertex set. Consider a cycle $v_1 v_2 \ldots v_k v_1$. We may assume that $v_1 \in V_1$. Since G is bipartite, two adjacent vertices cannot be both in V_1, thus it follows that $v_2 \in V_2$, $v_3 \in V_1$, $v_4 \in V_2 \cdots, v_k \in V_2$. It is seen that k is even and length of the cycle is even.*

Now assume that G has no odd cycles. Pick an arbitrary vertex v and put it in a set X and put all vertices adjacent to v in another set Y. If two vertices in Y are adjacent, together with v, these vertices constitute a triangle which is an odd cycle. So, no two vertices in Y can be adjacent. Now, take all vertices in G which are adjacent to some vertex in Y and put them in V_1. Continuing in this manner we obtain a partition (V_1, V_2) of the vertex set so that G is bipartite. ■

The **distance** between two vertices of a graph is the length of the shortest path joining these two vertices. If there is no such path, we write ∞ to mention this case.

CONNECTEDNESS

In a graph G, if there exists a path between any pair of vertices, G is said to be **connected**. A graph which is not connected is **disconnected**. In a disconnected graph, each connected maximal subgraph (that is a subgraph which is not contained in any other connected subgraph) is called a **component** of G. We denote the number of components by c(G).

Theorem 2. Let e be the number edges and n be the number of vertices of a graph G.

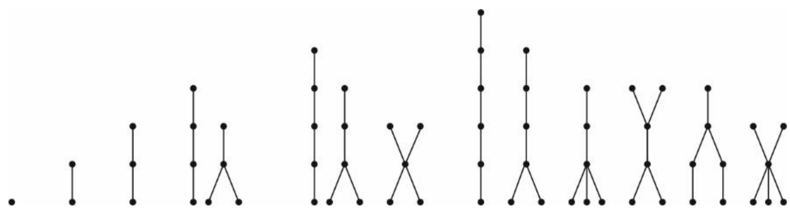
a) If $e < n - 1$, then G is disconnected.

b) If $e > \frac{(n-1)(n-2)}{2}$, then G is connected.

An edge of a graph is called a **bridge** if the removal of this edge disconnects the graph.

TREES

A graph which has no cycles is called **acyclic**. A **tree** is a connected acyclic graph; a union of trees is called a **forest**. Path graphs and star graphs are examples of trees. The following figure illustrates all trees with at most six vertices.



Theorem 3. *A graph is a tree if and only if there is exactly one path between every pair of its vertices.*

Proof. *Since any tree T is connected, there is at least one path between every pair of vertices in T. If there are two distinct paths between two vertices, then the union of these two paths contains a cycle which contradicts the fact that T is a tree. Hence there is exactly one path between every pair of vertices of a tree.*

Conversely, let T be a graph and in which there is a unique one path between every pair of vertices. It follows that T is connected. If T contains a cycle, say $v_1v_2\cdots v_k\cdots v_1$, *then there are two distinct paths between* v_1 *and any other vertex in the cycle, which is a contradiction. Now, T is connected and has no cycles, therefore it is a tree.* ■

There are many ways of defining trees, the following two theorems give two alternative definitions.

Theorem 4. *The number of edges of a tree with n vertices is n - 1.*

Proof . *We prove the result by using induction on the number of vertices. The result is obviously true for* $n = 1,2$ *and* 3. *Assume that any tree with fewer vertices than* n_0 *has one more vertices than its edges. Let* T *be a tree with* n_0 *vertices. since there is only one path between any pair of vertices, deletion of any edge separates* T *into two components, say* T_1 *and* T_2. *Let the number of these components be* n_1 *and* n_2. *Since each component is a tree and* $n_1, n_2 < n_0$. *By induction hypothesis, the number of edges of these components are* $n_1 - 1$ *and* $n_2 - 1$. *It follows that* T *has* $(n_1 - 1) + (n_2 - 1) + 1 = n - 1$ *edges.* ■

Theorem 5. Any connected graph with n vertices and $n-1$ edges is a tree.

Proof. Let T be a connected graph with n vertices and $n-1$ edges. It suffices to show that T contains no cycles. Assume that T contains cycles. When we remove an edge from a cycle graph remains connected. Keep on removing one edge from a cycle till the resulting graph T' is a tree. As T' has n vertices, it has $n-1$ edges. But the number of edges in T is larger than the number of edges in T' which means $n-1 > n-1$. Then we conclude that T has no cycles and it is a tree. ■

Theorem 6. Each edge of a tree is a bridge.

Proof. A tree has no cycles and therefore removal of any edge G disconnects it. Hence each edge of a tree is a bridge. ■

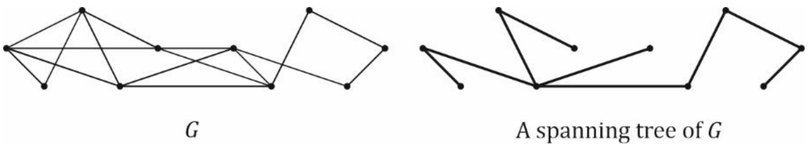
Theorem 7. A graph G with n vertices, $n-1$ edges and no cycles is connected.

Proof. Let T be a graph without cycles with n vertices and $n-1$ edges. Assume that T is disconnected. Without loss of generality we may assume that T has two components, say T_1 and T_2. Pick a vertex u in T_1 and a vertex v in T_2. Since there is no path between u and v, adding the edge uv does not create a cycle. Thus $T+e$ is a tree with n vertices n edges. This contradicts the fact that a tree with n vertices has $n-1$ edges. Hence T is connected. ■

Theorem 8. Any tree with at least two vertices has at least two leaves (end vertices).

Proof. Let the number of vertices in a given tree T be $n > 1$. So the number of edges in is $n-1$. Sum sum of local degrees is $2(n-1)$, twice the number of edges. Since a tree is connected it cannot have a vertex of degree 0, so each vertex contributes at least 1 to the above sum. Thus there must be at least two vertices of degree exactly 1. ■

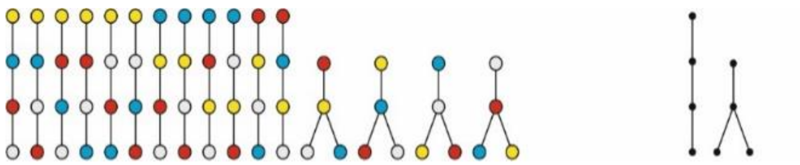
A tree is said to be a **spanning tree** of a connected graph G, if T is a subgraph of G and T contains all vertices of G. The following result shows the existence of spanning trees in connected graphs.



Theorem 9. Every connected graph has at least one spanning tree.

Proof. Let G be a connected graph. If G has no cycles, then it is its own spanning tree. If it has cycles, then by deleting an edge from some cycles, the graph remains connected and cycle free containing all the vertices of G. ■

The figure given below illustrates all labeled and unlabeled trees with four vertices.



16 labeled trees with 4 vertices

2 unlabeled trees with 4 vertices

A general result for the number of unlabeled graphs is not known yet. Some known results which cannot be expressed by means of simple formulas are obtained by using a fundamental enumeration theorem of Polya. Naturally, the number of labeled trees is much more larger than unlabeled ones. Some known figures are listed in the below table.

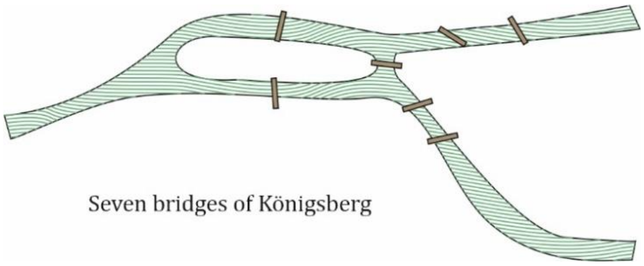
Number of Vertices	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of Unlabeled Graphs	1	1	1	2	3	6	11	23	47	106	235	551	1301	7741	19320

Contrary to the case for unlabeled graphs, for the number of labeled graphs we have the following theorem whose proof is attributed to Cayley.

Theorem 10 **(Cayley, 1889).** There are n^{n-2} labeled trees with n vertices.

EULERIAN TOURS, HAMILTONIAN CYCLES

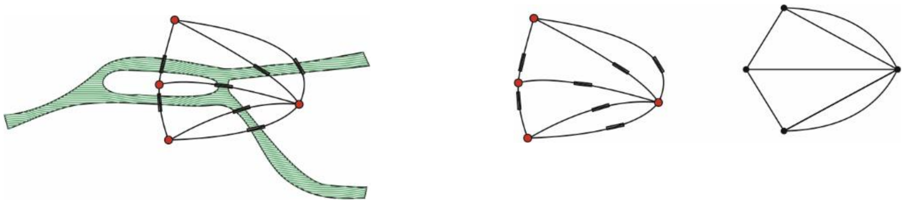
The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River together with two large islands which were connected to each other



and the other parts of the city. The people wondered if it is possible to walk around the city by crossing each bridge exactly once. The walk need not start and end at the same part.

Euler[1] approached this problem by representing the areas of land separated by the river with vertices and represented the bridges with the edges joining the vertices. The resulting figure is the graph which represents Königsberg's 7 bridges.

The problem now is to draw the above figure without retracing any line and without picking the pencil up off the paper. In terms of graphs, to find a trail that contains all the edges of the graph.



All vertices in the above picture are odd. Pick any of these, say the one with degree 3. The first time we reach this vertex by an edge, we can leave by another edge. But the next time we arrive, there is no edge by which we can leave. So this vertex cannot be an intermediate vertex; it can only be the first or the last vertex of the walk. Thus, an intermediate vertex cannot be odd so it is impossible to draw the above graph in one pencil stroke without retracing.

[1] Leonhard Euler (1707-1782), Swiss mathematician.

A trail which contains all the edges of a graph is called an **Eulerian trail** and a closed such trail is called an **Eulerian tour**. A graph which admits an Eulerian tour is called an **Eulerian graph** and a graph which admits an Eulerian trail is said to be **traceable**. Above discussion shows that a for graph to be traceable, two of the vertices must be odd and all the vertices must be even. For the graph be Eulerian, all the vertices must be even. This necessary condition is also sufficient:

Theorem 11. A connected graph is Eulerian if and only if every vertex is even.

Proof. Since the tour enters a vertex through some edge and leaves by another edge, necessity of the condition is obvious. To show the sufficiency, start with a vertex v and begin making a tour. Keep going, never using the same edge twice, until it is not possible to go further. Since every vertex is even, the end point of the tour is v. If all the edges are used, proof is completed. Otherwise each component of the subgraph consisting of unused edges is a graph whose all vertices are even. Apply the same procedure to a component to obtain a second tour. If this tour starts in a vertex of the first tour, the two tours can be combined to produce a new tour. Continuing in this manner one obtains a tour which includes all the edges. ■

Corollary. A connected graph is traceable if and only if it has exactly two odd vertices. In this case end points of the trail are the odd vertices.

Theorem 12. A strongly regular connected digraph is Eulerian if and only if the in-valency of each vertex is equal to the out-valency.

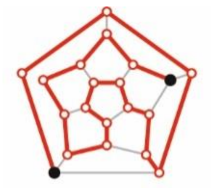
Example. Among all platonic graphs, only the octahedron is Eulerian.

K_n is Eulerian if and only if n is odd $(n \geq 3)$

$K_{p,q}$ is Eulerian if and only if p and q are both even $(p, q \geq 2)$.

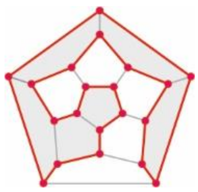
***Example**. In the following figure, three graphs on left are all Eulerian. Three graphs on right are traceable graphs (odd vertices emphasized).*

*A path is called a **Hamilton**[2] **path** if it contains all the vertices. Below figure illustrates a Hamilton path in dodecahedron.*



A Hamiltonian path in dodecahedron graph

*A **Hamilton cycle** is a cycle, if exists, which contains all the vertices. A graph which admits a Hamilton cycle is called an **Hamiltonian graph**. In the mid 19-th century, Hamilton tried to popularize the exercise of finding such a cycle in the dodecahedron. Below figure shows that dodecahedron is indeed a Hamiltonian graph.*

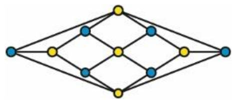


A Hamiltonian cycle in dodecahedron graph

We have seen that it can be easily checked whether a graph is Eulerian or not. Moreover, it is easy to construct an Eulerian trail (or tour) if there exists any. In contrast to this, no trivial necessary and sufficient condition for a graph to be Hamiltonian is known; the problem of deciding whether an arbitrary graph admits a Hamiltonian cycle is hard[3]*. We only have some necessity and some sufficiency conditions.*

[2] Named after Sir William Rowan Hamilton (1805-1865), Irish mathematician.
[3] In fact, the problem has been proved to be NP —complete hard problem.

Example. The graph given in the figure is not Hamiltonian. To see this, notice that the graph is bipartite and has an odd number of vertices. But a bipartite graph does not admit an odd cycle, hence the given graph does not admit a cycle containing all the vertices.



In general if $p+q$ is odd, then $K_{p,q}$ (and of course any subgraph of it) is not Hamiltonian.

PLANARITY

A graph can be represented in plane in arbitrarily many different ways. If it is possible to represent a graph in plane such that no two edges intersect (except at vertices), we say that the graph is **planar** (or **embeddable in plane**). Such a representation of a planar graph is called a **plane graph** (or an **embedding** of the graph in plane).

Example. Figure below shows three representations of K_4 in plane. The first one is not an embedding whereas the others are embeddings (plane graphs).

Not an embedding

Plane Graphs (Embeddings)

Example. Path graphs, circuit graphs, wheel graphs, all trees and all platonic graphs are planar.

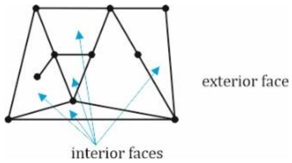
When we deal with planarity, parallel edges and loops are immaterial. In fact, a multi-graph is planar if and only if its underlying simple graph is planar. As well, vertices of degree 1 or 2 do not have any effect on a graph for being planar or not.

Theorem 13. *For any graph G the following are equivalent.*

a) G is planar,

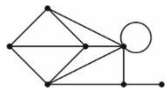
b) Underlying simple graph of G is planar,

c) Any graph which is homeomorphic to G is planar.



*A plane graph partitions the rest of the plane into a number of bounded regions together with an unbounded region. Boundary of each edge is a cycle of the graph. Each of these regions is called a **face** of the plane graph. The unbounded face is called the **exterior face**; other faces are called **interior faces**. The set and the number of faces of a plane graph G are denoted respectively by $F(G)$ and f (or $f(G)$).*

Example. *A plane graph on 7 vertices with 13 edges and 8 faces.*



Theorem 14. *For any plane graph G,*

a) two distinct faces of G are disjoint; their boundaries can intersect only on edges and vertices,

b) G has a unique exterior face,

c) a bridge belongs to the boundary of one face,

d) any edge which is not a bridge belongs to the boundary of two faces,

e) *each cycle of G surrounds at least one internal face,*

f) *G has no interior face if and only if it is acyclic (that is, it has no cycles).*

Corollary. *If a plane graph has two distinct faces with the same boundary, then the graph is a cycle.*

Theorem 15 **(Euler's Polyhedral Formula).** *Let G be a connected plane graph with v vertices, e edges and f faces. Then $v + f = e + 2$.*

Proof. *We proceed by induction on f, the number of faces. If $f = 1$, the only face is the unbounded face and the graph have no cycles. Such a graph is connected if and only if it has $v - 1$ edges. Then it follows that $v + f - e = v + 1 - (v - 1) = 2$ and the theorem holds. Now assume that the claim is true for all connected plane graphs with less than $f_0 \geq 2$ faces and let G be a gaph with f_0 faces. Pick some edge u of G which is not a bridge. Then, $G - u$ is connected with $v' = v$ vertices, $e' = e - 1$ edges and $f' = f_0 - 1$ faces. By the induction hypothesis $v' + f' = e' + 2$ which implies that $v + f_0 - 1 = e - 1 + 2$ or $v + f_0 = e + 2$.* ■

Let G be a connected planar graph on v vertices with e edges and also let G_1 and G_2 be two different plane representations of G. Let the number of faces of G_1 and G_2 be respectively, f_1 and f_2. Since each of these plane graphs has v vertices and e edges, Euler's polyhedral theorem says that $v + f_1 = e + 2$ and $v + f_2 = e + 2$ which implies that $f_1 = f_2$. This observation implies that, no matter how a planar graph is embedded in the plane, the number of faces will always be the same. We call this common number of faces as the number of faces of the graph itself.

Corollary. *Let G be a graph with v vertices, e edges, f faces and c components. Then*

$$v + f = e + c + 1.$$

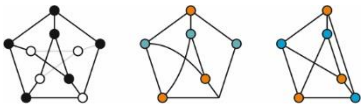
One of the most significant studies in the early history of planar graphs was the attempt to characterize which graphs are planar and which are not. As with many graph theory problems, roots can be found in recreational mathematics, as well as a number of applications to electric circuit boards. Two planarity problems, one by Mobius and another by Dudeney, would prove to be especially significant in leading up to Kuratowski's characterization of planar graphs in the late 1920's. Kuratowski's Theorem would inspire and enlighten many other individuals to discover other characterizations of planar graphs.

One of the earliest questions concerning planarity was presented by Mobius during a lecture in about the year 1840, where Mobius presented the following problem:

> *"There was once a king with five sons. In his will he stated that after his death the sons should divide the kingdom into five regions so that the boundary of each region should have a frontier line in common with each of the other four regions. Can the terms of the will be satisfied?" (Wilson 1999, 516-517).*
>
> *In 1917, another significant puzzle about planarity first appeared in a book entitled Amusements in Mathematics by Henry Ernest Dudeney of England: "There are some half-dozen puzzles, as old as the hills, that are perpetually cropping up, and there is hardly a month in the year that does not bring inquiries as to their solution. Occasionally, one of these, that one had thought was a distinct volcano, bursts into eruption in a surprising manner. I have*

received an extraordinary number of letters respecting the ancient puzzle that I have called 'Water, Gas, and Electricity.' It is much older than electric lightning or even gas, but the new dress brings it up to date. The puzzle is to lay on water, gas, and electricity, from W, G, and E, to each of the three houses A, B, and C, without any pipe crossing one another. Take your pencil and draw lines showing how this should be done. You will soon find yourself landed in difficulties." (Dudeney 1958, 73).

***Theorem 16* (Kuratowski, 1930).** *A graph is planar if and only if it contains no subgraph homeomorphic to* K_5 *or* $K_{3,3}$.

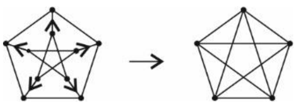


Example. *Any complete graph on 5 or more vertices is nonplanar.*

Example. *In the figure, a subgraph of Petersen graph which is homeomorphic to* $K_{3,3}$ *is shown. Thus, Petersen graph is not planar.*

In terms of contractions, a useful extension of Kuratowski's theorem is given by Wagner:

***Theorem 17* (Wagner, 1937).** *A graph is planar if and only if it contains no subgraph contractible to* K_5 *or* $K_{3,3}$.



Example. *Considering the contractions along the emphasized edges in the figure, we see that the Petersen graph is contractible to* K_5. *Once more we observe that Petersen graph is nonplanar.*

EXERCISES

1. *For any even integer* $n \geq 4$ *show that there exists a* $3-$*regular simple graph.*
2. *Let* G *be a graph with order* 9 *such that the degree of each vertex is either 5 or 6. Prove that there are either at least 5 vertices of degree 6 or at least 6 vertices of degree 5.*
3. *Let* n *and* r *be positive integers such that* $n \cdot r$ *is even. Show that there exists a* $r-$*regular graph of order* n.
4. *Show that a sequence* $\rho_1, \rho_2, \ldots, \rho_n$ *of non-negative integers is valency sequence of a (not necessarily simple) graph if and only if their sum is even.*

5. Show that there exists no simple graph whose valency sequence is 2, 3, 3, 4, 5, 6, 7 or 1, 3, 3, 4, 5, 6, 6.

6. For any graph show that $\delta \leq 2e/v \leq \Delta$ where δ is the minimum local degree, Δ is the maximum local degree of the graph.

7. Prove that any simple graph has at least two vertices with the same valency.

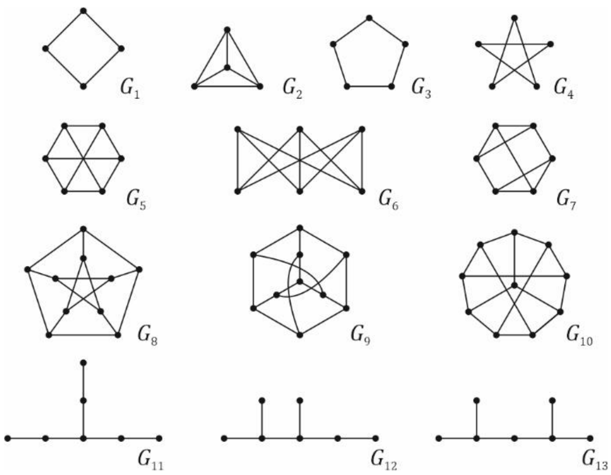
8. Alex and Leo are a couple, and they organize a party together with 4 other couples. There are a number of greetings but, naturally, nobody says hello to their own partner. At the end of the party Alex asks everyone how many people did they greet, receiving nine different answers. How many people did Alex greet and how many people did Leo greet?

9. Determine all simple graphs with valency sequence 1, 2, 2, 3, 3, 3 up to isomorphism.

10. Determine all simple graphs of order 4 up to isomorphism.

11. Determine all simple regular graphs of order 6 up to isomorphism.

12. Classify the graphs in the given figure by isomorphism type:



13. Show that $n-$cube is bipartite.

14. Show that if G is a simple graph with $e > n^2/4$ where e and n are respectively the number of edges and vertices of G, then G cannot be bipartite.

15. Each side and each diagonal of a regular hexagon in plane is colored either in blue or red. Show that there exists at least one triangle whose all sides are of the same color.

16. Show that if there is a walk from u to v in G, then there is also a path from v to u in G.

17. Show that if G is simple and $\delta \geq k$, then G has a path of length k.

18. Show that if G is connected and all vertices are even, then for any vertex v, the number of components of $G - v$ is at most $\frac{1}{2}d(v)$.

19. Show that if G is disconnected then $\bar{G}$ is connected.

20. Draw all unlabeled trees with seven vertices and eight vertices.

21. If a tree has an even number of edges, then show that it contains at least one vertex of even degree.

22. Show that a tree has Δ pendant vertices (a vertex with local degree 1). (Δ:maximum local degree).

23. Show that a path is its own spanning tree.

24. Prove that every tree is a bipartite graph.

25. Let G be a graph with exactly two connected components, both being Eulerian. Which is the minimum number of edges that need to be added to G to obtain an Eulerian graph?

26. Prove that a connected graph in which each vertex has even degree is bridgeless.

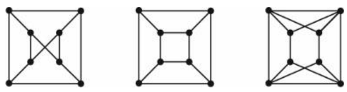
27. A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the 27 unit subcubes. If he starts at one corner and always moves on to an uneaten subcube, can he finish at the center of the cube?

28. Let G be a 3-regular plane graph of order 24. Find the number of faces of G.

29. Find a simple graph G with degree sequence [4, 4, 3, 3, 3, 3] such that

    a) G is planar,

    b) G is nonplanar.



30. Determine which of the following graphs are planar.

31. Let e be an edge of $K_{3,3}$. Show that $K_{3,3} - e$ is planar. Determine whether $K_{3,3} - e$ is maximal.

32. Let e be an edge of K_5. Show that $K_5 - e$ is planar. Determine whether $K_5 - e$ is maximal.

Solutions

1. Connect all n vertices like a cycle. Then, since there are even number of vertices, match each vertex with the vertex on the exact opposite side. A 3-regular simple graph with n vertices is obtained.

2. Assume the case where both conditions are not satisfied. Then there are 4 vertices with order 6 and 5 vertices of order 5. Since there are odd number of vertices of odd degree this is not graphical. Hence, the statement in the question is true.

3. *If n is even, construct a graph with the vertex sets* $\left\{v_i : i \in \mathbb{Z}_{\frac{n}{2}}\right\}$ *and* $\left\{w_i : i \in \mathbb{Z}_{\frac{n}{2}}\right\}$. *Then connect each* v_i *to* $w_{i+1}, \dots, w_{i+r}$. *A r-regular graph is obtained. If r is even, construct a graph with the vertex set* $\{v_i : i \in \mathbb{Z}_n\}$. *Connect every vertex* v_i *to* $v_{i-\frac{r}{2}}, \dots, v_{i-1}, v_{i+1}, \dots, v_{i+\frac{r}{2}}$. *Again, a r-regular graph is obtained.*

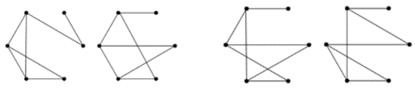
4. *We know that for simple graphs* $\sum_{i=1}^{n} \delta_i = 2e$. *Then their sum is even. If the sum of the degrees is even, make loops around all vertices until all vertices are required to be adjacent to 1 more vertex or no vertex at all. Then there must be even number of vertices left with one degree opening, match these vertices as pairs and a graph is obtained.*

5. *There are seven vertices in the first one so it cannot have a vertex of degree 7 if it is simple. For the second graph, there are two vertices of degree 6 so they must be adjacent to all other vertices but there is a vertex of degree 1 so it cannot be adjacent to both vertices of degree 6.*

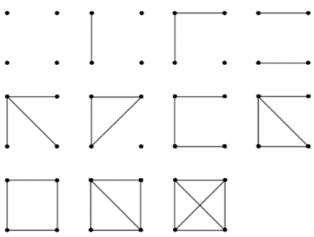
6. *If* $\delta > \frac{2e}{v}$, *then* $\delta v > 2e$, *which implies that* $\min(\delta_i)\, v > 2e$ *but we have* $\delta_1 + \delta_2 + \cdots + \delta_v = 2e$ *which is a contradiction, so* $\delta \leq \frac{2e}{v}$. *Similarly, assume* $\Delta < \frac{2e}{v}$, *then* $\Delta v < 2e$ *which implies that* $\max(\Delta_i)\, v < 2e$ *but we know that* $\Delta_1 + \Delta_2 + \cdots + \Delta_v = 2e$ *so we have a contradiction and* $\Delta \geq \frac{2e}{v}$.

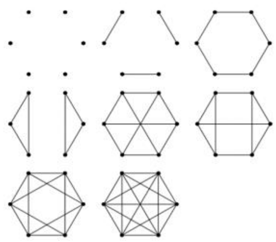
7. *For each vertex v there are n-1 many options for the value* δ_v. *So, by Pigeonhole Principle, since there are n many vertices, two vertices must have the same degree.*

8. *Since Alex got all different answers and there are 9 possible answers which are 0, 1, 2, 3, ..., 8, there must be a person who greeted 8 people. The pair of that person must have greeted no one since if he had greeted someone then he must've greeted the person who greeted 8 people considering there is also someone who greeted no one. Following a similar logic, a person who greeted n people is paired with a person who greeted 8-n people. Since all other people other than Alex and Leo greeted different number of people, Alex and Leo must have greeted 4 people each.*

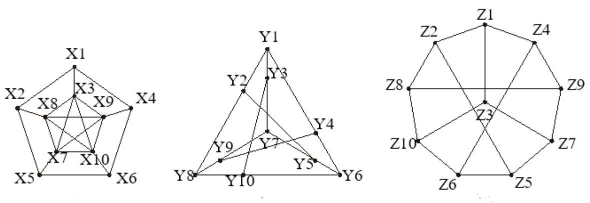
9. 

10.



11. 

12. G_1 and G_2 both have 4 vertices but they have different number of edges, hence, they are not isomorphic. G_3 and G_4 are both cycles of length 5, so they are isomorphic graphs. G_5 and G_6 are both isomorphic to the complete bipartite graph $K_{3,3}$ so they are isomorphic but G_7 is not the same as $K_{3,3}$ so it is not isomorphic to any.



Considering this labeling, G_8, G_9 and G_{10} are isomorphic graphs. Lastly, G_{11}, G_{12} and G_{13} are all non-isomorphic since the first one contains 1 vertex with order 3, the others contain two and the second one contains 2 vertices of order 3 adjacent to each other, where the 2 vertices of order 3 in the last graph are not adjacent.

13. Any vertex of n-cube is represented by an n-tuple e.g. $(0,1,1,0,1,1,0,\ldots)$. Any two vertices are adjacent if and only if they differ at one coordinate of the tuple. So, color vertices with even sum of elements in tuple as blue and vertices with odd ones as yellow. This provides that n-cube is bipartite.

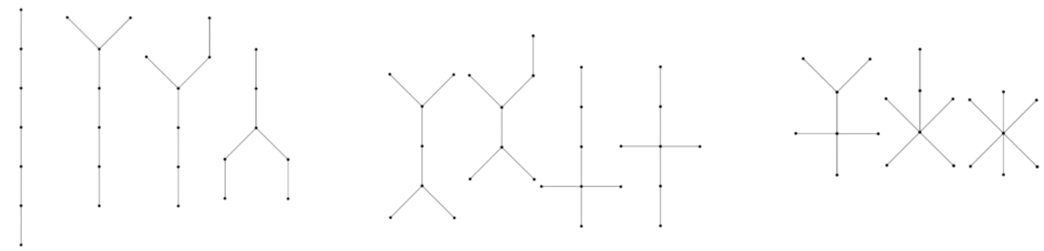
14. A bipartite graph $K_{p,q}$ has at most pq vertices. By Calculus this value reaches its maximum when $p = q$ which is when $pq = \frac{n^2}{4}$. Hence, if $e > \frac{n^2}{4}$, the graph is not bipartite.

15. Label the vertices of the hexagon as A, B, C, D, E and F. Consider the vertex A. It has 5 edges that are incident to it. Hence, at least 3 of them are colored blue or red. Without loss of generality assume that B, C and D are colored red. If any of edges BC, CD or BD is colored red we get a triangle. If not, all of those edges are blue we get a triangle out of those. Hence, there has to be a triangle of same color.

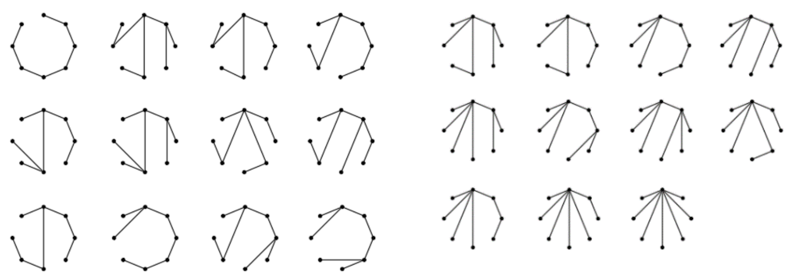
16. Assume there is a walk $v_1v_2v_3\ldots v_n$. If there are repeated vertices on this walk delete those that are not on the first place and a path is left.

17. Assume not. Then there are two longest paths of length k so that these two paths do not share a common vertex. Since the graph is connected there must be at least one vertex not in any of these paths but connected to both paths. Then taking the largest parts of these paths that are divided by the connection point to that vertex, a new path can be constructed of length greater than k which is a contradiction. Thus, these paths must share a common point.

18. Assume that all vertices have even degree and consider the vertex v. Any connected component after the vertex v is removed must be connected to the vertex v by at least two edges since otherwise, there will be odd sum of degrees of vertices of the connected component which is a contradiction. Hence, there can be at most $\frac{d(v)}{2}$ connected components after v is removed.
19. Let u and v be two vertices of $\bar{G}$. If these two vertices are not adjacent in G then they are adjacent in $\bar{G}$. If these two vertices are adjacent in G, since G is disconnected there are multiple connected components, so that these two vertices are not adjacent to another vertex w which is in another component. This means that in $\bar{G}$, u and v share the path uwv. Hence, $\bar{G}$ is connected.
20. Trees with seven vertices



Trees with eight vertices



21. Assume that there are n edges which is an even number. Then the tree has $n+1$ edges which is an odd number. If all vertices had odd degree then sum of degrees of vertices would have been odd which is a contradiction. So, the tree must have a vertex of even degree.
22. Consider the vertex with the highest degree. If that vertex is removed then the graph is split into Δ connected components where each component is itself a tree. Since trees have two pendants at least each component has at least one pendant. Considering that one of those pendants in each tree was adjacent to the removed vertex, there are at least Δ other pendants that were not adjacent to the removed vertex.
23. A path graph is itself a tree, hence, path graph is a tree that is a subgraph of itself and covers all its vertices. Thus, the spanning tree of a path graph is itself.

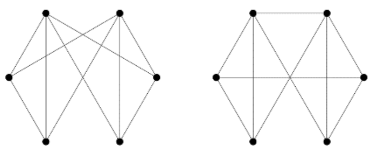
24. A tree does not contain any cycles. Since a tree does not contain any odd cycle it is bipartite.

25. If there are three vertices v_1, v_2, v_3 where $v_1 \in C_1$, $v_2 \in C_2$ and $v_1v_3, v_2v_3 \notin E(V)$, then adding the edges v_1v_3, v_2v_3 and v_1v_2 will create a Eulerian circuit so adding three edges is enough. Adding three edges is necessary to keep all degrees of vertices even. If one of these components is complete then for any vertices $v_1, v_2 \in C_1, v_3, v_4 \in C_2$, the edges $v_1v_3, v_1v_4, v_2v_3, v_2v_4$ are required so at least four edges are needed.

26. A connected graph with only even vertices contains a Eulerian circuit which implies that for any two pair of vertices, there are at least two paths joining them. Hence, no edge can be a bridge.

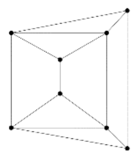
27. Consider the graph of 3-cube. If a corner has a tuple representation with the sum of its elements even, then the vertex at the center must have that sum odd. If the mouse starts from corner and end in center while passing through all other vertices just once, it would have made 26 moves in which case the sum of the tuple of corner and the sum of the tuple in the middle must have the same parity. Since this is not the case, the rat cannot end up at the center vertex.

28. Since the graph is planar, using Euler's Formula, the number of faces is 14.

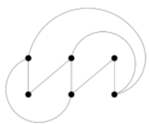


29. The first graph has a planar representation but the second one is not planar.

30. For the first graph contracting the two vertices at the top right side and the two vertices at the bottom right side, $K_{3,3}$ is obtained so it is not planar. The second graph is planar as its planar representation is given. For the last graph, a planar representation is given as



31. The following graph shows that $K_{3,3} - e$ is planar. However, more edges can be added without making the graph non planar. So, it is not maximal.



32. The following graph shows that $K_5 - e$ is planar. Also, there is only one edge that