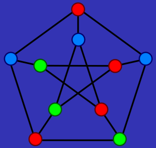
# Graph Theory, Part 1



Lecture Notes in Math 212 Discrete Mathematics

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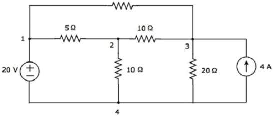
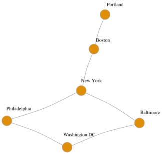
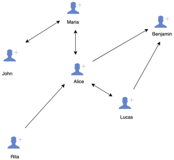
# The idea of graph

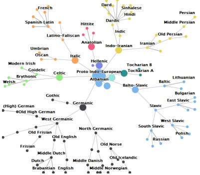




People often use schemes where some objects are marked by nodes connected with links. Examples: transport networks, electrical circuits, etc.

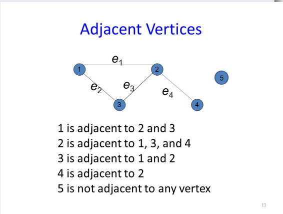




## Mathematically significant information in these examples is the **graph**

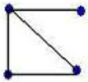
formed by a set of nodes, called vertices (usually marked as points on a plane) and a set of links called edges (drawn as lines connecting some vertices). Every edge connects two vertices called the endpoints of this edge. The endpoints are said to be incident to the edge. Vertices connected by an edge are called adjacent, or neighbors.



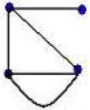
# Variants of graphs

## Simple graphs and other kinds

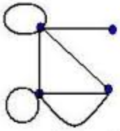
- If some pairs of points are connected by several edges, such kind of a graph is called multigraph.
- If in addition to multiple edge a graph is allowed to contain loop-edges connecting a vertex with itself, it is called pseudograph.
- If a graph has neither multiple edges nor loops, it is called simple graph. Later on we usually suppose that the graphs are simple.
- If in a graph every edge is assigned a direction, it is called directed graph or digraph.



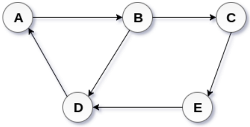
simple graph



multigraph



pseudograph



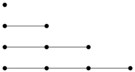

**Directed Graph**

# Important examples

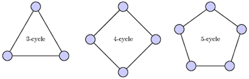
We suppose by definition that a graph contains at least one vertex, that is, its vertex set $V$ is non-empty. But the set of edges $E$ can be empty.

- If $E = \varnothing$, the graph is called empty or nullgraph.

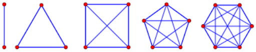

- An n-path graph $P_n$ has $n$ vertices $v_1, \ldots, v_n$ and edges connecting $v_i$ with $v_{i+1}$, $i = 1, \ldots, n-1$.



- An n-cycle graph $C_n$, the vertices are cyclically connected like in polygon: $v_i$ to $v_{i+1}$ and $v_n$ to $v_1$.

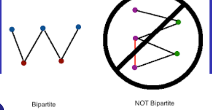



- A graph is said to be complete if every pair of its vertices is connected by an edge (in other words, all vertices are adjacent to each other). A complete graph with $n$ vertices will be denoted $K_n$.


Complete Graphs

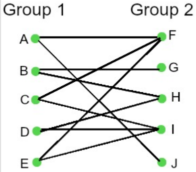
**Question:** Find the number of edges in $K_n$. **Answer:** $\binom{n}{2} = \frac{n(n-1)}{2}$, which is the number of pairs of points.

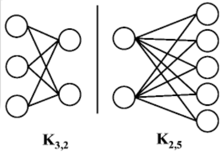
# Bipartite graphs




**A graph is called bipartite graph if its vertices**

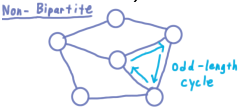
can be split in two groups (or can be colored in two colors), so that every edge have endpoints in different sets (of different colors).
In a complete bipartite graph every pair of vertices from different groups must be connected by an edge. Such graph is denoted $K_{m,n}$, where $m$ and $n$ are the numbers of vertices in the two groups.




- How many edges are in $K_{mn}$ ? **Answer:** $mn$.
- Are path graphs $P_n$ and cycle graphs $C_n$ bipartite ? **Answer.** $P_n$ is bipartite for any $n$: one group of vertices are $v_1, v_3, \dots$ (odd indices) and another groups is $v_2, v_4, \dots$ (even indices). $C_n$ is bipartite for even $n$ and not for odd $n$ (since the colors of vertices alternate).

A bipartite graph cannot contain triangles and more generally, odd-length cycles.



# Subgraphs

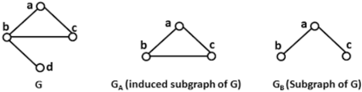

Graph Subgraph

Graphs are denoted as pairs $G = (V, E)$, where $V$ and $E$ are the sets of vertices and edges (recall that we always assume that $V \neq \varnothing$).

**A subgraph of graph $G = (V, E)$ is a graph $G_1 = (V_1, E_1)$ such that**

$V_1 \subset V$ and $E_1 \subset E$. Warning: one needs to check that $V_1 \neq \emptyset$ and that every edge in $E_1$ has its endpoints in $V_1$, otherwise $G_1$ is not a graph.

- $G_1$ is called spanning subgraph if $V_1 = V$.
- $G_1$ is called subgraph induced by subset of vertices $V_1$ if it includes all the edges whose endpoints belong to $V_1$.




**Exercises: prove that**

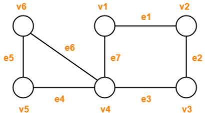
- if a subgraph $G_1$ of graph $G$ is spanning and induced, then $G_1 = G$;
- a subgraph of a bipartite graph is also bipartite.

# Paths and cycles in a graph

A path or a cycle in a graph can be represented by subgraphs which are path graphs and cycle graphs respectively. Formal definitions:

**A walk in a graph is a sequence of consecutive vertices**

linked by edges: $v_1, e_1, v_2, e_2 \ldots, e_{n-1}, v_n$, where edge $e_i$ connects $v_i$ with $v_{i+1}$. A walk is called closed if $v_n = v_1$.




**For simple graphs** the edges are determined by their endpoints and a walk can be denoted just $v_1, \ldots, v_n$.

- A trail is a kind of a walk without repetitions of edges. A closed trail is called a cirquit.
- A path is a kind of a trail (walk) without repetitions of vertices. A closed path is called a cycle.

| | Vertices | Edges | |
|---|---|---|---|
| Walks | Repetition allowed | Repetition allowed | |
| Trails | Repetition allowed | No repetition of edges | |
| Paths | No repetition of vertices except possibly starting and terminal vertices | No repetition of edges | |
| Circuits | Repetition allowed | No repetition of edges | A nontrivial closed trail |
| Cycles | No repetition of vertices except starting and terminal vertices | No repetition of edges | A nontrivial closed trail without repetition of vertices except starting and terminal vertices |

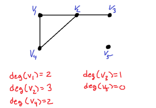
# Degree (valency) of a vertex

## Degree or valency of a vertex $v$ in a graph

is the number of edges incident to this vertex, notation deg(v). In a simple graph it is the same as the number of vertices adjacent to $v$.

**Exercise:** why for multigraph it is not the same ?

- vertices of degree 0 are called isolated
- vertices of degree 1 are called pendant.




## Hand-shaking theorem

The sum of degrees of all vertices in a graph $G = (V, E)$ (possibly multigraph) equals to the double number of its edges: $\sum_{v \in V} deg(v) = 2|E|$.

**Proof.** If an edge has endpoints $v$ and $w$, then it contributes 2 to the sum $\sum_{v \in V} deg(v)$: 1 in summand $deg(v)$ and 1 in $deg(w)$.

## Corollary: in a graph there are even number of vertices of odd degree.

**Proof:** because the sum of all the degrees is even.

# Exercises

## How many edges ?

Graph $G$ has 10 vertices: four of degree 4, two of degree 5 and four of degree 6. How many edges are there in $G$ ?
**Solution.** By the hands-shaking theorem, $2|E| = 4 \cdot 4 + 2 \cdot 5 + 4 \cdot 6 = 50$. Thus, $G$ has $|E| = 25$ edges.

## $G$ is not bipartite !

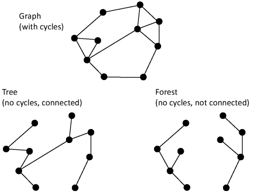
Prove that the above graph $G$ is not bipartite.
**Solution.** If $G$ is bipartite, then it is a subgraph of one of the complete bipartite graphs $K_{m,n}$, $m + n = 10$. Hence, $|E|$ cannot exceed $mn$, which is the number of edges in $K_{m,n}$. But $1 \cdot 9, 2 \cdot 8, 3 \cdot 7, 4 \cdot 6 < 25$. The only possibility is $K_{5,5}$, which has the same number of edges as $G$. Then, $G$ can be a subgraph of $K_{5,5}$ only if they coincide. But $K_{5,5}$ has all vertices of degree 5 and $G$ hasn't. So, $G \neq K_{5,5}$ and thus, $G$ cannot be bipartite.

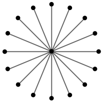
# Trees

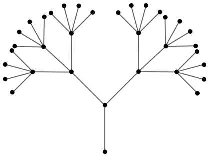
A graph is said to be connected if any pair of its vertices can be connected by a path (equivalently, by a trail or a walk).

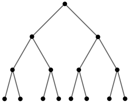
**A tree is a connected graph that does not contain cycles**

A graph without cycles, but not connected is called forest. It contains several connected components, which are trees.



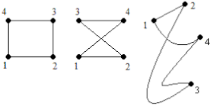




- In a tree $v = e + 1$, where $v$, $e$ are the numbers of vertices and edges.
- In a tree every pair of edges can be linked by a unique path.
- Pendant vertices (of degree 1) in a tree are called leaves.
- If in a tree $> 1$ vertices, it has at least two leaves. **Exercise:** prove it!

# Isomorphism of graphs

**Informally speaking**, isomorphic graphs differ just by the way of their presentation in the plane: **vertices and edges may change their position, but cannot appear or disappear.**

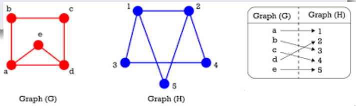



First, assume that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are simple graphs.

## An isomorphism between graphs $G_1$ and $G_2$

is a bijective correspondence between their vertices, $f : V_1 \to V_2$, such that a pair of vertices $v, w \in V_1$ are adjacent in $G_1$ if and only if their images $f(v)$, $f(w)$ are adjacent in $G_2$.

Bijection $f$ is defined by a list of correspondence between the sets of vertices $V_1$ and $V_2$

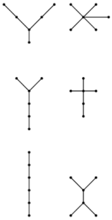


Graph (G) Graph (H)

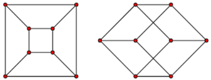
## In the definition of isomorphism for multigraphs $G_1$ and $G_2$

we require in addition to $f$ a bijection between edges $\tilde{f} : E_1 \to E_2$, such that that a vertex $v \in V_1$ and edge $e \in E_1$ are incident if and only if $f(v)$ and $\tilde{f}(e)$ are incident.

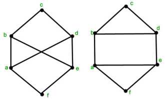
# Examples of isomorphism



**Example:** six non-isomorphic trees with six vertices. In all examples except two, non-isomorphism is proved by counting the number of vertices of different degrees. How to prove that two exceptional examples are not isomorphic ? Note that in one of them the vertices of degree 2 are adjacent, and in the other are not.



**Exercise:** show isomorphism of two graphs by labeling their corresponding vertices with $1, \dots, 8$




**How to prove non-isomorphism of these two graphs ?**
Note that one of them is bipartite and the other is not !

**Exercise:** prove or disprove isomorphisms of the following graphs.