Discrete (Finite) probability, Part 1

Lecture Notes in Math 212 Discrete Mathematics

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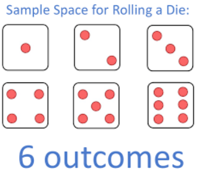
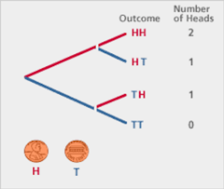
Language of Probability

An experiment

a procedure resulting in several possible outcome: like *rolling dice*, or *tossing (flipping) a coin*, or *serving cards* in a card game.





The Sample Space

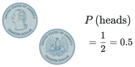
is the set of outcomes after some experiment: like $\{1, 2, 3, 4, 5, 6\}$ after rolling a die, or {heads, tails} after tossing a coin, or $\{HH, HT, TH, TT\}$ after tossing a coin 2 times (here, for example, HT means that you get heads after the first tossing and tails after the second one).

Events and their Probability (Chances)

Event is any subset of the sample space.

If S is the sample set and $E \subset S$ an event, then the probability of E is $p(E) = \frac{|E|}{|S|}$.

For instance, the probability of each outcome of an experiment (one-element subset of S) is $\frac{1}{|S|}$.



Outcome of die roll	**1**	**2**	**3**	**4**	**5**	**6**
Probability	1/6	1/6	1/6	1/6	1/6	1/6

Examples

•Probability of "heads" after flipping a coin is $\frac{1}{2}$.
•Probability of "6" after rolling a die is $\frac{1}{6}$.
• Probability of even outcome (2, 4, or 6) after rolling a die is $\frac{3}{6} = \frac{1}{2}$.

Random Choice

Random ball in an urn

An urn contains four blue and five red balls. What is the probability that a ball chosen at random is blue.

Solution: *An experiment here is choosing a ball. An outcome is one of the balls which is chosen. The sample space S contain $4+5$ outcomes (one for each ball that can be chosen). The event E contains 4 outcomes (since there are 4 blue balls). Hence, the probability is* $p(E) = \frac{|E|}{|S|} = \frac{4}{9}$.

Random number

What is a probability that an integer selected at random from the set $\{1, 2, \ldots, 100\}$ is divisible by either 2 or 5 ?

Solution: Here $S = \{1, \ldots, 100\}$ and $E = \{2, 4, 5, 6, 8, 10, \ldots, 100\}$ includes $[\frac{100}{2}] + [\frac{100}{5}] - [\frac{100}{10}] = 50 + 20 - 10 = 60$ numbers (by inclusion and exclusion formula). So, $p(E) = \frac{60}{100} = \frac{3}{5}$.

Rolling dice

The sum is 7

What is the probability that when two dice are rolled, the sum of the numbers obtained is 7 ?

Solution: *The sample space* S *contains* $6 \times 6 = 36$ *outcomes, which are pairs* (i, j) *of numbers* $i, j \in \{1, 2, 3, 4, 5, 6\}$ *on the dice. The sum is* 7 *in six cases,* $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 2)$, *and* $(6, 1)$. *These 6 pairs are elements of the event (set)* $E \subset S$, *and* $p(E) = \frac{6}{36} = \frac{1}{6}$.

At least one 6 after a die is rolled 3 times

Find the probability to get at least one 6 after a die is rolled 3 times.

Solution: *Now, set* S *contains* $6^3 = 216$ *outcomes: triples* (i, j, k) *of* $i, j, k \in \{1, 2, 3, 4, 5, 6\}$. *With* $i = 6$ *there are* $6^2 = 36$ *triples, and similar with* $j = 6$, *or with* $k = 6$. *The inclusion-exclusion formula gives* $3 \cdot 36 - \binom{3}{2}6 + 1 = 108 - 18 + 1 = 91$ *triples with at least one* 6. *So, the answer is* $\frac{91}{216}$.

Sampling of balls in an urn

Sampling without or with replacement

An urn contains 10 balls numerated $0, 1, 2, \ldots, 9$ and 3 of them are taken randomly one-by one.
(a) What is the probability that the chosen balls are 3, 5, 8 and they are chosen in this order ?
(b) What is the probability to take this collection of balls in any order ?
(c) How the answers to questions (a) and (b) will change if a ball is returned to the urn before the next one is selected ?

Solution: (a) *The number of consecutive selections of 3 balls is* $10 \cdot 9 \cdot 8 = 720$ *and* $3, 5, 8$ *is just one of these selections (one element of the Sample space). So, the probability is* $\frac{1}{720}$.
(b) *The Event now contains 6 elements:* $E = \{(358), (538), \ldots, (853)\}$ *(six permutations of* 3, 5, 8*) and so,* $p(E) = \frac{6}{720} = \frac{1}{120}$.
(c) *The sample space* S *now contains* $10^3 = 1000$ *elements (since repetitions in triples are allowed). The answers to (a) and (b) will be then* $\frac{1}{1000} = 0.001$ *and* $\frac{6}{1000} = 0.006$.

Poker combinations of cards

Four cards of one kind

Find the probability that a hand of five cards contains four of one kind.

Solution: *The sample space contains* $\binom{52}{5}$ *elements (total number of 5-card selections). Four of one kind appears in* $\binom{13}{1} \cdot \binom{48}{1}$ *of the selections. Here* 13 *is the number of kinds and* 48 *is the number of choices for the fifth card (after 4 of one kind are chosen). So, the answer is* $\frac{13 \cdot 48}{\binom{52}{5}}$.

A full house

Find the probability of a full house combination, that is three of one kind and two of another. How much it is bigger than for four of one kind ?

Solution: *Three of one kind can be selected in* $\binom{13}{1} \cdot \binom{4}{3} = 52$ *ways, and two of another kind in* $\binom{12}{1} \cdot \binom{4}{2} = 72$ *ways. Totally,* $52 \cdot 72$ *way to get a full house, which gives probability* $\frac{52 \cdot 72}{\binom{52}{5}}$. *It is* $\frac{52 \cdot 72}{13 \cdot 48} = 6$ *times bigger than in the previous example.*

Further terminology and properties

For any subset $E \subset S$ we have $|E| \leq |S|$, so we can conclude that

$$0 \leq p(E) = \frac{|E|}{|S|} \leq 1, \text{ and } \begin{cases} p(E) = 0 \text{ only if } E = \varnothing \\ p(E) = 1 \text{ only if } E = S \end{cases}$$

The event $\overline{E} = S - E$ is called *the complementary event* to E.

Its probability is $p(\overline{E}) = \frac{|S-E|}{|S|} = \frac{|S|-|E|}{|S|} = 1 - p(E)$.

Events A and B are called *disjoint* if $A \cap B = \varnothing$. In this case

$p(A \cup B) = p(A) + p(B)$, because $|A \cup B| = |A| + |B|$ if $A \cap B = \varnothing$.

Events A and B are *overlapping* if $A \cap B \neq \varnothing$. In this case

$p(A \cup B) = p(A) + p(B) - p(A \cap B)$. This is because in the case of $A \cap B \neq \varnothing$, we have $|A \cup B| = |A| + |B| - |A \cap B|$.

Estimates of probabilities using complementary events

Four people choose at random a digit $0, 1, \dots, 9$. What is more likely

that all 4 chosen digits are different, or that at least two of them coincide ?

Solution: Four digits can be chosen in $10^4 = 10000$ ways. Among these choices, $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ selections give distinct digits. Hence, probability to choose four distinct numbers $p = 0.504$ is slightly more than probability $1 - p = 0.496$ to obtain coincidence of at least two digits.

Two distinct integers $1 \leq a, b \leq 100$ are chosen at random

What is the probability that one of them is not greater than 10 ? Estimate whether the probability is less or more than 20% ?

Solution: A pair of distinct numbers can be chosen in $100 \cdot 99 = 9900$ ways. Both numbers are greater than 10 in $90 \cdot 89$ cases. Hence, the probability that both numbers are greater than 10 is $p = \frac{90 \cdot 89}{9900} = \frac{89}{110}$. The probability of the complementary event that one of the numbers is ≤ 10 is $1 - p = \frac{21}{110}$. It is less than 20% because $\frac{21}{110} = \frac{210}{1100} < \frac{220}{1100} = \frac{20}{100}$. (By definition, $20\% = \frac{20}{100}$.)