Pigeonhole Principle

Lecture Notes in Math 212 Discrete Mathematics

Sergey Finashin

METU, Depart. of Math

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Pigeonhole Principle, or Dirichlet Box Principle

General Meme

If 10 pigeons are located in 9 pigeonholes, then there is a pigeonhole with more than one pigeon.

Meme for reverse Pigeonhole Principle

If 9 pigeons are located in 10 pigeonholes, then at least one pigeonhole will be empty.

In language of functions

• If a set A has more elements than a set B, then a map $F : A \to B$ **cannot be injective** (one-to-one). That is, some elements a_1, a_2 from A will have the same image.

• If a set A has fewer elements than a set B, then a map $F : A \to B$ **cannot be surjective** (onto). That is, some element b from B has empty preimage.

Simplest Application

Examples

• In a class of 13 students, at least two must be born in the same month. *Here, the 13 students are "pigeons" and the 12 months are "pigeonholes".*

• If 102 students took an exam with maximal score 100 points, then at least two students will have the same score.
The students are "pigeons", the numbers of points are "pigeonholes".

• In a letter with 30 words at least two words begin with the same letter. *The words are "pigeons" and the 26 letters are "pigeonholes".*

• Among 100 integers $a_1, \dots, a_{100}$ one can find two a_i, a_j, $i \neq j$, whose difference is divisible by 97.
Integers $a_1, \dots, a_{100}$ are "pigeons", residues mod 97 are "pigeonholes".

• A drawer contains 10 pairs of socks of different colors and you pick some randomly. What minimum number guarantees a pair of one color?
The 10 colors are "pigeonholes", so we need to pick 11 to guarantee.

Some little tricks

Example

For any choice of six digits in the set $S = \{1, 2 \ldots, 9\}$ one can find two chosen digits giving in sum 10.

Solution: *Pigeons here are digits and pigeonholes are 5 subsets* $\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}$*. Among six chosen digits two will be in the same subset, and thus, give in sum* 10*.*

Example

If there are $n > 1$ people who can shake hands with one another, then there is always two persons who will shake hands with the same number of people.

Solution: *Each person shakes hands from* 0 *to* $n - 1$ *people, totally* n *possibilities. But* 0 *means that someone shakes hands to nobody, while* $n - 1$ *means shaking hands to everybody. So, both* 0 *and* $n - 1$ *cannot happen at the same time and this leaves* n *people to* $n - 1$ *possibilities.*

Divisibility of numbers $111\dots1$ (written with only 1s)

Example

Show that for every integer n, some multiple of n has only 0s and 1s in its decimal presentation.

Solution: *Consider the* $n+1$ *integers* $1,\ 11,\ 111, \dots, \underbrace{1\dots1}_{n+1}$. *There are* n *possible remainders after division by* n. *So, by the pigeonhole principle, two of these numbers have the same remainder. Then* n *divides the difference of the larger and the smaller one* $\underbrace{1\dots1}_{r} - \underbrace{1\dots1}_{k} = \underbrace{1\dots1}_{r\text{-}k}\underbrace{0\dots0}_{k}$

Corollary

If n is odd and not divisible by 5, then it has a multiple looking as $1\dots1$.

Since n *is relatively prime to* 10, *we can drop zeros in the above example.*

A problem with the same idea of solution

Example

Prove that for any odd $n \in \mathbb{N}$ some of its multiple looks like $2^m - 1$ for some $m \in \mathbb{N}$.

Solution: *Consider $n+1$ integers $2^1 - 1$, $2^2 - 1 \dots$, $2^n - 1$, $2^{n+1} - 1$. By pigeonhole principle two of them have the same remainder upon division by n: $2^r - 1 = an + r$, $2^k - 1 = bn + r$, where $r > k$. Then*

$$(2^r - 1) - (2^k - 1) = 2^r - 2^k = 2^k(2^{r-k} - 1) = (a - b)n$$

Since n is odd, we have $gcd(n, 2^k) = 1$ and we conclude that $2^m - 1$ for $m = r - k$ is divisible by n.

In the last two examples we used the following fact.

Theorem

If $n|ab$ and n is relatively prime to a (notation: $gcd(n, a) = 1$), then $n|b$.

Divisibility of consecutive sums

Example

In every sequence of $n \in \mathbb{N}$ integers $a_1, \dots, a_n$ one can find several consecutive ones whose sum $a_i + a_{i+1} + \dots + a_j$ is divisible by n.

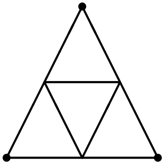
Solution: *Consider n integers formed by consecutive summation:* a_1, $a_1 + a_2, \dots a_1 + \dots a_n$. *If any of them is divisible by n we are done. Otherwise, there are* $n-1$ *remainders* $1, 2, \dots, n-1$ *that may happen upon division by n. By pigeonhole principle, for some pair of integers the remainders are equal, so their difference is a multiple of n. Such difference is also a consecutive sum* $(a_1 + \dots + a_j) - (a_1 + \dots + a_i) = a_{i+1} + \dots + a_j$ *if* $i < j$.

Pigeonhole Principle in Geometry

Example

One has chosen 5 points inside an equilateral triangle with side 1. Prove that between some pair of chosen points the distance is $\leq \frac{1}{2}$.

Solution Connect pairwise the midpoints of the sides of the triangle. This subdivide the triangle into 4 equilateral triangles with the side length $\frac{1}{2}$. Among the 5 given points (pigeons) two must appear in one of these triangles (pigeonholes). Finally, observe that the maximal distance between two points in a small triangle is $\frac{1}{2}$.



One more tricky example of pigeonholes

Example

Show that among any 101 positive integers not exceeding 200 there must be an integer that divides one of the other integers.

Solution: *Set* $S = \{1, \dots, 200\}$ *is partitioned into* 100 *subsets numerated by odd numbers* $1, 3, \dots, 199$. $A_1 = \{1, 2, 4, 8, \dots, 128\}$, $A_3 = \{3, 6, 12, \dots, 192\}$, $A_5 = \{5, 10, 20, 40, 80, 160\}, \dots$, $A_{199} = \{199\}$. *Each subset* A_n *start with an odd number* n *and contains its multiples* $2n$, $4n, \dots$ *obtained by multiplication by powers of* 2, *so that the product does not exceed* 200. *If we pick 101 integers from* S, *then two of them appear in one subset. And it is left to notice that among two numbers in one subset the lesser one divides the greater ones.*

Generalized Pigeonhole Principle

Assume that m objects are distributed to n boxes. Then

Estimate from below

if $m > nk$, then some box contains $\geq k+1$ objects

Estimate from above

if $m < nk$, then some box contains $\leq k-1$ objects

Proof by contrapositive (by contradiction)

If each box contained $\leq k$ objects, there would be $\leq nk$ objects totally.

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Examples

- Among 100 people one can always find 9 born in the same month.
- How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards are of the same suit ?

The "boxes" are 4 suits, so the minimum is $n = 9$ cards, because $\frac{n}{4} > 2$ is needed to get 3 cards of the same color.

Problems solved in consecutive days

Example

During 30 days a student solves at least one problem every day from a list of 45 problems. Show that there must be a period of several consecutive days during which he solves exactly 14 problems.

Solution *Let* a_i *be the number of problems solved during first* i *days. This gives an increasing sequence* $0 < a_1 < a_1 < \cdots < a_{30} \leq 45$*. Then we have* $14 < a_1 + 14 < a_2 + 14 < \cdots < a_{30} + 14 \leq 59$*. Altogether we have* $30 + 30 = 60$ *positive integers less than* 60*. By Pigeonhole principle there must be two equal among them. But the integers* a_i*,* $i = 1, \ldots, 30$ *are all distinct, and* $a_i + 14$*,* $i = 1, \ldots, 30$ *are distinct too. So, we must have* $a_j = a_i + 14$ *for some* i *and* j*. Then* $a_j - a_i = 14$ *problems were solved from day* $i + 1$ *to day* j*.*