Probabilistic Design of Multi-Dimensional Spatially-Coupled Codes

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ISIT 2024
Athens, Greece

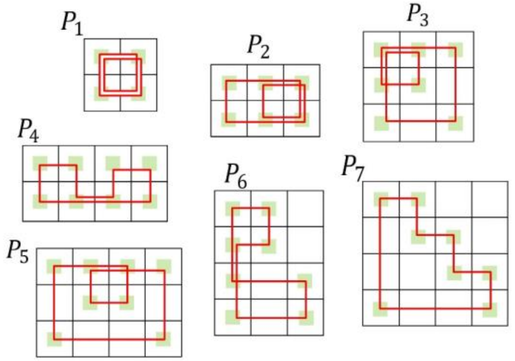
Presentation Outline

- **Motivation and technical vision**

- **Preliminaries**
  - Spatially-Coupled (SC) codes
  - Multi-Dimensional SC (MD-SC) construction

- **Novel framework for probabilistic MD-SC code design**
  - MD Solution form for short cycles
  - Expected number of short cycles
  - Multi-Dimensional gradient-descent distributor (MD-GRADE)
  - Finite-length algorithmic optimizer (FL-AO)
  - Comparison with previous MD-SC schemes

- **Conclusion and ongoing research**



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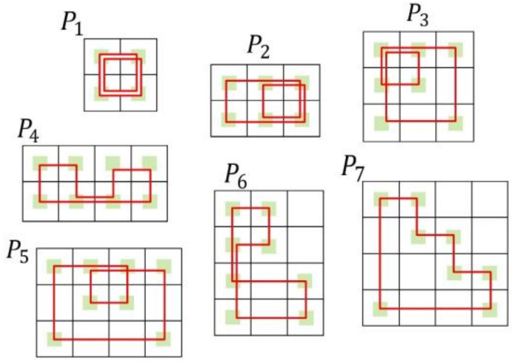
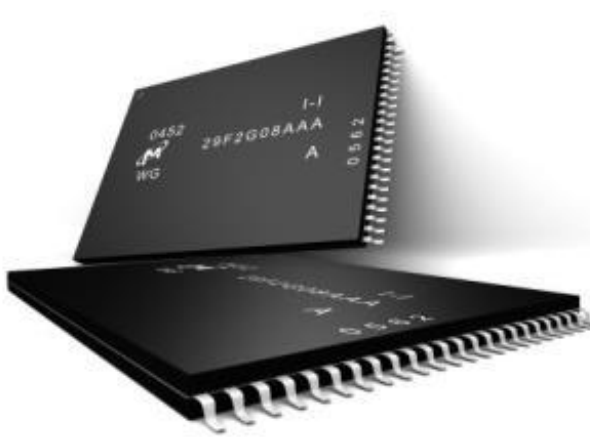


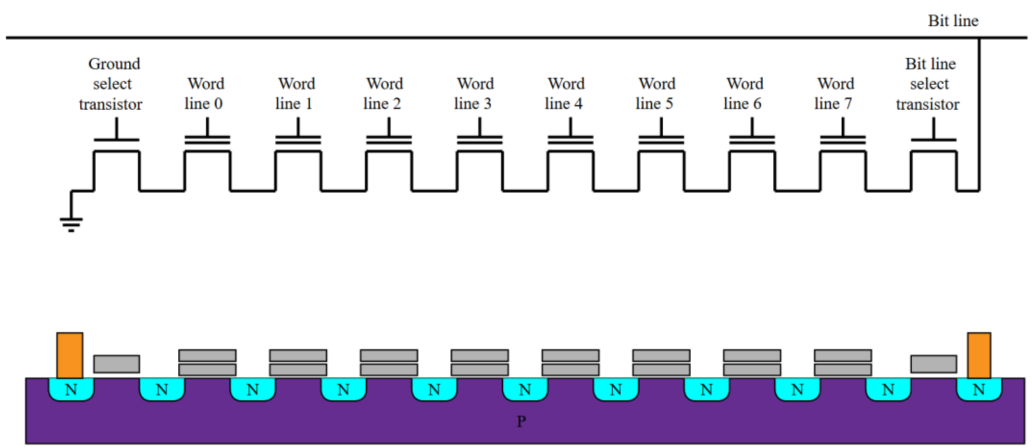
- **Conclusion and ongoing research**

Motivation of the Research

- **We are in the era of big data.**
  - Modern data centers now have storage capacities in the exabyte range (10^{18} bytes) or higher.
  - SSDs and HDDs are nearing storage densities of 10 terabits per square inch!

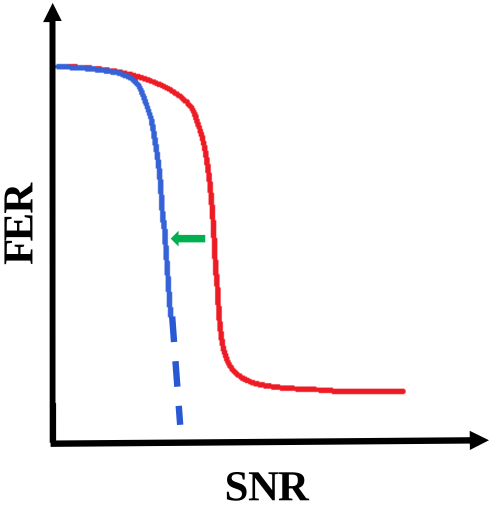
- **These high storage densities increase and intensify error sources in modern storage devices.**
  - Flash: Inter-cell interference (ICI) and wear-out.





Technical Vision

- **Modern storage devices, including both Flash and magnetic recording devices, must operate at very low error rates.**
  - Effective ECC techniques are **essential** for allowing storage engineers to confidently use these high-density devices.
  - Graph-based codes provide excellent performance in this regard!



Our mission is to develop effective ECC techniques that exploit the characteristics of the channels in underlying storage devices.

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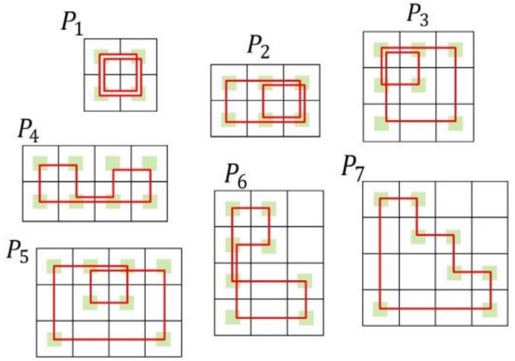


Reminder on SC Codes

- **Parameters:** γ, κ, z, m, L m: memory, L: coupling length
- **Ingredients:**
  - A matrix $\mathbf{K}$ of size $\gamma \times \kappa$ whose entries are in $\{0,1,..,m\}$.
  - A matrix $\mathbf{L}$ of size $\gamma \times \kappa$ whose entries are in $\{0,1,..,z-1\}$.
- **Output:** A parity-check matrix $\mathbf{H}_{\mathrm{sc}}$ of size $(\gamma z(m+L)) \times (\kappa z L)$

- An underlying block code is partitioned into a number of component matrices:

$$\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_m$$

such that

$$\mathbf{H} = \sum_{i=0}^{m} \mathbf{H}_i,$$

and L copies of replicas are coupled together to make a chain of coupled block codes.

$$\mathbf{H}_{\mathrm{SC}} \triangleq \begin{bmatrix} \mathbf{H}_0 & \mathbf{0} & & & \mathbf{0} \\ \mathbf{H}_1 & \mathbf{H}_0 & & & \vdots \\ \vdots & \mathbf{H}_1 & \ddots & & \vdots \\ \mathbf{H}_m & \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_m & \ddots & \ddots & \mathbf{H}_0 \\ \vdots & \mathbf{0} & & \ddots & \mathbf{H}_1 \\ \vdots & \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & & & \mathbf{H}_m \end{bmatrix}$$

What is MD Construction?

- **Coupling copies of an SC code notably mitigates channel non-uniformity.**
- **Parameters:** $\gamma, \kappa, z, m, L, M$ $\quad M$: number of auxiliary matrices

- **The MD-SC code is obtained from M copies of $\mathbf{H}_{\mathrm{SC}}$ on the diagonal by**
  - relocating some of its circulants from each replica of every copy of $\mathbf{H}_{\mathrm{SC}}$ to the corresponding locations in $\mathbf{X}_l$ copies, and then by
  - coupling them in a sliding manner as shown below.
- **For convention, $\mathbf{X}_0 \triangleq \mathbf{H}'_{\mathrm{SC}}$.**

$$\mathbf{H}_{\mathrm{MD}} \triangleq \begin{bmatrix} \mathbf{H}'_{\mathrm{SC}} & \mathbf{X}_{M-1} & \mathbf{X}_{M-2} & \cdots & \mathbf{X}_2 & \mathbf{X}_1 \\ \mathbf{X}_1 & \mathbf{H}'_{\mathrm{SC}} & \mathbf{X}_{M-1} & \cdots & \mathbf{X}_3 & \mathbf{X}_2 \\ \mathbf{X}_2 & \mathbf{X}_1 & \mathbf{H}'_{\mathrm{SC}} & \cdots & \mathbf{X}_4 & \mathbf{X}_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{X}_{M-2} & \mathbf{X}_{M-3} & \mathbf{X}_{M-4} & \cdots & \mathbf{H}'_{\mathrm{SC}} & \mathbf{X}_{M-1} \\ \mathbf{X}_{M-1} & \mathbf{X}_{M-2} & \mathbf{X}_{M-3} & \cdots & \mathbf{X}_1 & \mathbf{H}'_{\mathrm{SC}} \end{bmatrix},$$

where

$$\mathbf{H}_{\mathrm{SC}} = \mathbf{H}'_{\mathrm{SC}} + \sum_{\ell=1}^{M-1} \mathbf{X}_{\ell}.$$

What is an MD Protograph?

- **DEFINITION: The matrix $\mathbf{H}_{\mathrm{MD}}^{\mathrm{g}}$ obtained by replacing $\mathbf{H}_{\mathrm{SC}}'$ with $\mathbf{H}_{\mathrm{SC}}'^{\mathrm{g}}$ and $\mathbf{X}_l$'s with $\mathbf{X}_l^{\mathrm{g}}$'s is called the MD protograph.**
  - $\mathbf{X}_l$'s are obtained by replacing each nonzero (zero) entry in $\mathbf{X}_l^{\mathrm{g}}$ with the $z \times z$ circulant $\sigma^{f_{i,j}}$ (the zero matrix $\mathbf{0}_{z\times z}$) that has the appropriate power $f_{i,j}$ from the lifting matrix $\mathbf{L}$.
  - The sum $\mathbf{H}_{\mathrm{SC}}'^{\mathrm{g}} + \sum_{l=1}^{M-1} \mathbf{X}_l^{\mathrm{g}}$ gives the SC protograph.

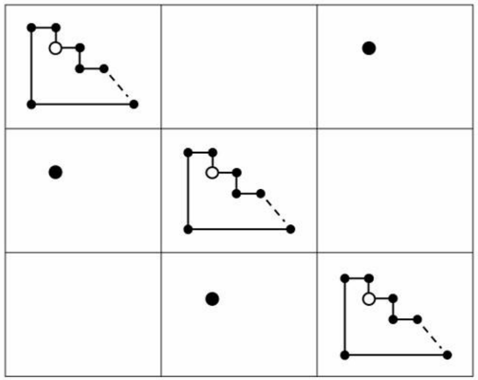
- **Relocations are represented by an MD mapping as follows:**

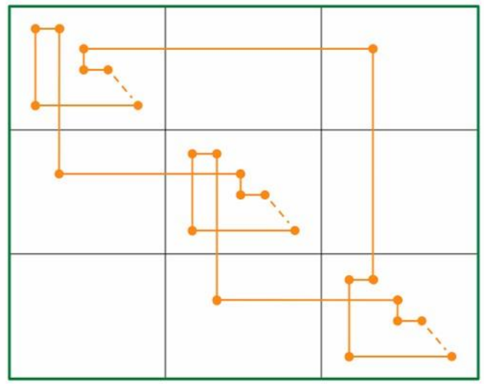
  $F : \{C_{i,j} | 1 \leq i \leq \gamma, 1 \leq j \leq \kappa\} \rightarrow \{0,1, \dots, M-1\},$ where
  - $C_{i,j}$ is the circulant corresponding to 1 at entry (i,j) of the all-one base matrix, and
  - $F(C_{i,j})$ is the index of the auxiliary matrix to which $C_{i,j}$ is located.

- **The percentage of relocated circulants is called the MD density.**

Relocations Eliminate Short Cycles!

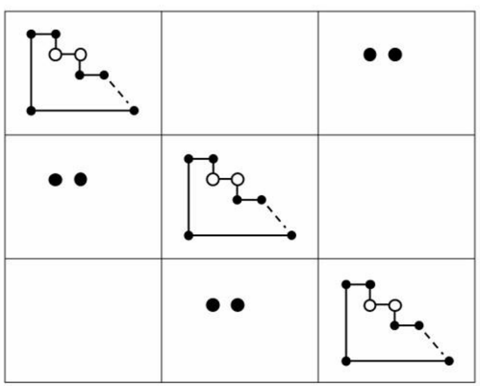
- **EXAMPLE 1** (effective relocation arrangement):

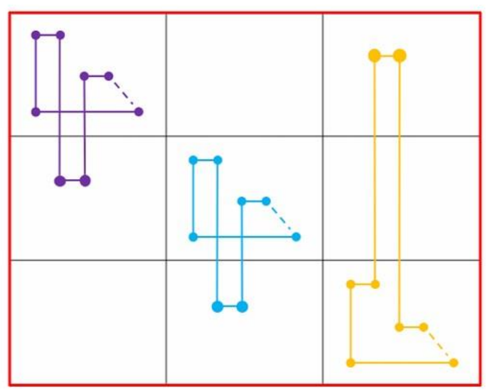




$\mathbf{H}'_{SC}$	$\mathbf{X}_2$	$\mathbf{X}_1$
$\mathbf{X}_1$	$\mathbf{H}'_{SC}$	$\mathbf{X}_2$
$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{H}'_{SC}$

- **EXAMPLE 2** (ineffective relocation arrangement):





Design complexity increases as we have more degrees of freedom.

Solution: Probabilistic design!

Set-up for Probabilistic Framework

- For $m \geq 1$ and $M \geq 2$, consider the matrix

$$\mathbf{P} = \begin{bmatrix} p_{0,0} & p_{0,1} & \cdots & p_{0,M-1} \\ p_{1,0} & p_{1,1} & \cdots & p_{1,M-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m,0} & p_{m,1} & \cdots & p_{m,M-1} \end{bmatrix}_{(m+1)\times M}$$

probability of m^{th} component of the 1^{st} auxiliary matrix

where

$$\sum_{i=0}^{m} \sum_{j=0}^{M-1} p_{i,j} = 1$$

and each $p_{i,j} \in [0,1]$ specifies the probability that a non-zero circulant in the MD protograph is assigned to the i^{th} component of the j^{th} auxiliary matrix.

- $\mathbf{P}$ is referred as the (joint) probability-distribution matrix.

Set-up for Probabilistic Framework

- **The coupling polynomial of an MD-SC code associated with the probability-distribution matrix $\mathbf{P}$ is**

$$f(X,Y,\mathbf{P}) = \sum_{i=0}^{m}\sum_{j=0}^{M-1} p_{i,j}X^{i}Y^{j}$$

  which is abbreviated as $f(X,Y)$.

- **The vector $\mathbf{p}^{\text{con}}$ obtained by concatenating the rows of $\mathbf{P}$ from top to bottom is called the probability-distribution vector.**

- **We express the expected number of cycles-6 and cycles-8 candidates as a function of the probabilities $p_{i,j}$'s after partitioning and relocations.**

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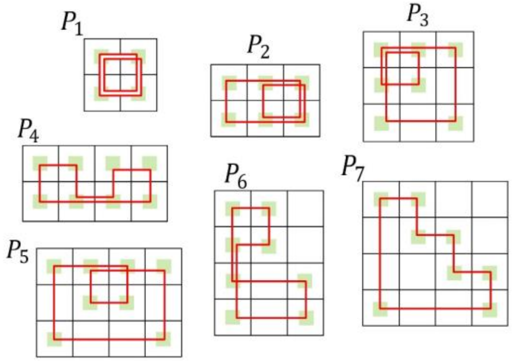


MD-SC Solution Form for Cycles-6

- **<u>THEOREM 1</u>: The expected number $N_6(\mathbf{p}^{\text{con}})$ of cycles-6 in the MD protograph is given by**

$$6\binom{\gamma}{3}\binom{\kappa}{3}\sum_{M|b}[f^3(X,Y)f^3(X^{-1},Y^{-1})]_{0,b}$$

where $[\cdot]_{i,j}$ denotes the coefficient of X^iY^j in a two-variable polynomial.

- **<u>LEMMA 1</u>: For $N_6(\mathbf{p}^{\text{con}})$ to be locally minimized subject to the constraints**

$$p_{i,0}+p_{i,1}+\cdots+p_{i,M-1}=p_i^*$$

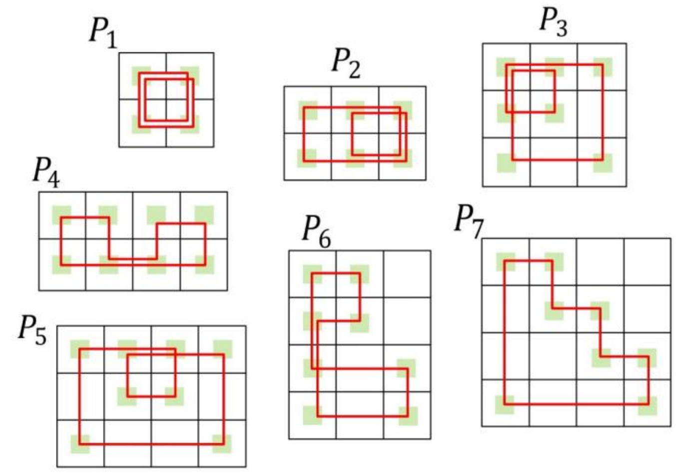
for all $i \in \{0,1,\ldots,m\}$ and $j \in \{0,1,\ldots,M-1\}$, it is necessary that the following equations hold for some $c_i \in R$:

$$\sum_{M|b}[f^3(X,Y)f^2(X^{-1},Y^{-1})]_{i,b+j}=c_i.$$

- **Here, p_i^*'s are obtained by GRADE-AO Algorithm in [R1].**

Cycles-8 Candidates

- **<u>THEOREM 2</u>: The expected number $N_8(\mathbf{p}^{\text{con}})$ of cycles-8 candidates in the MD protograph is given by**



$$N_8(\mathbf{p}^{\text{con}}) = \sum_{M|b} \Big\{ w_1 \left[f^2(X,Y) f^2(X^{-1},Y^{-1})\right]_{0,b} + w_2 \left[f(X^2,Y^2) f(X^{-2},Y^{-2}) f^2(X,Y) f^2(X^{-1},Y^{-1})\right]_{0,b} + w_3 \left[f(X^2,Y^2) f^2(X,Y) f^4(X^{-1},Y^{-1})\right]_{0,b} + w_4 \left[f^4(X,Y) f^4(X^{-1},Y^{-1})\right]_{0,b} \Big\},$$

where $w_1 = \binom{\gamma}{2}\binom{\kappa}{2}$, $w_2 = 3\binom{\gamma}{2}\binom{\kappa}{3} + 3\binom{\gamma}{3}\binom{\kappa}{2}$, $w_3 = 18\binom{\gamma}{3}\binom{\kappa}{3}$, $w_4 = 6\binom{\gamma}{2}\binom{\kappa}{4} + 6\binom{\gamma}{4}\binom{\kappa}{2} + 36\binom{\gamma}{3}\binom{\kappa}{4} + 36\binom{\gamma}{4}\binom{\kappa}{3} + 24\binom{\gamma}{4}\binom{\kappa}{4}$ *if* $\gamma \geq 4$, *and* $w_4 = 6\binom{\gamma}{2}\binom{\kappa}{4} + 36\binom{\gamma}{3}\binom{\kappa}{4}$ *if* $\gamma = 3$, *where* $\kappa \geq 4$.

Expected Number of Cycles

- **We compute the expected number of cycles-k, $k \in \{6,8\}$, under a specific MD probability distribution.**
  - This gives an estimate of what the finite-length algorithmic optimizer (FL-AO) algorithm can produce.
  - These numbers inform us what to expect from incorporating the probability-distribution matrix in designing gradient-descent MD-SC (GD-MD) codes under random partitioning and lifting.

- **THEOREM 3: After random partitioning, relocations, and lifting based on $\mathbf{p}^{\text{con}}$, the expected number of cycles-6 in the Tanner graph of $\mathbf{H}_{\text{MD}}$ is**

$$\approx N_6(\mathbf{p}^{\text{con}}) * \frac{2L-m}{2} * M$$

and the expected number of cycles-8 in the Tanner graph of $\mathbf{H}_{\text{MD}}$ is

$$\approx N_8(\mathbf{p}^{\text{con}}) * (L-m) * M.$$

MD-GRADE Algorithm

- **The algorithm is designed to find**
  - the probability distribution matrix of an MD-SC code with arbitrary number of auxiliary matrices and memory,
  - to estimate the expected number of short cycles along with their upper and lower bounds.

- **A modified version of the gradient-descent algorithm is employed to handle the constraints outlined in Lemma 1, initiated with the locally-optimal edge distribution of the underlying SC code, obtained as in [R1].**

- **MD-GRADE provides a non-trivial solution where**
  - relocation percentages of component matrices are not necessarily the same;
  - typically, more circulants need to be relocated from the middle component matrices of the SC code than from the side ones.

Finite-length Algorithmic Optimizer (FL-AO)

- **The algorithm to construct an MD-SC code starting from an underlying SC code through performing relocations is introduced in [R2] by Esfahanizadeh et al. and modified in [R3] to achieve more reduction in cycle counts.**

- **Our FL-AO initially starts with**
  - a random relocation based on a locally-optimal GD distribution,
  - performs relocations based on the majority rule, and makes random decisions between the best options if there are multiple options.

- **The GD-MD distribution results in a significant reduction in the search space of all possible relocation arrangements that the FL-AO operates on.**
  - This remarkably reduces the framework complexity and latency.

- **FL-AO converges on excellent finite-length MD-SC designs in few iterations.**

Parameters of Codes

- **SC Code 1 is an SC code with parameters** $(\gamma, \kappa, z, L, m) = (4, 17,17,10,1)$ **and girth** 6 **[R3].**
- **SC Code 1.1 is similar to SC Code 1, but with** $L = 30$.
- **GD-MD Code 1.1 with** $M = 3$ **has final MD density** $= 33.82\%$ **and is obtained after** 12 **iterations.**
- **SC Code 3 is an SC code with parameters** $(\gamma, \kappa, z, L, m) = (3, 19,23,10,2)$ **and girth** 8 **[R3].**
- **SC Code 3.1 is similar to SC Code 3, but with** $L = 40$.
- **GD-MD Code 3.1 with** $M = 4$ **has final MD density** $= 31.58\%$ **after** 11 **iterations.**

Code name	Length	Rate
GD-MD Code 1.1	8,670	0.74
SC Code 1.1	8,670	0.76
GD-MD Code 2.1-2.2	210,250	0.81
MD-SC (NR)	210,250	0.81
GD-MD Code 3.1	17,480	0.81
SC Code 3.1	17,480	0.83
GD-MD Code 4.1	107,100	0.81
SC Code 4.1	107,100	0.82
GD-MD Code 4.2	154,700	0.81
SC Code 4.2	154,700	0.82
GD-MD Code 5	43,350	0.75
TC Code 1.1	43,350	0.76

Remarkable Reduction in Cycle Counts!

- The following table compares the population of cycles of interest in our proposed GD-MD codes with their 1D-SC counterparts and/or those in the literature [R3].

Code name	Cycle-6 count
GD-MD Code 1.1	6,375
[R3]	9,078
SC Code 1.1	79,917
GD-MD Code 5	0
TC Code 1.1	47,736

Code name	Cycle-8 count
GD-MD Code 2.2	2,768,485
MD-SC (NR)	16,809,705
GD-MD Code 3.1	239,752
[R3]	249,320
SC Code 3.1	1,397,319
GD-MD Code 4.1	112,959
SC Code 4.1	3,819,480
GD-MD Code 4.2	92,001
SC Code 4.2	5,530,280

Better Cycle Counts with Higher Lengths!

- **Significant reduction in**
  - the number of cycles-6 that ranges between 92% and 100% and
  - the number of cycles-8 that ranges between 83% and 98%

  compared with 1D SC/TC codes of the same lengths.

- **More intriguingly, our GD MD codes offer lower cycle counts compared with their underlying SC/TC codes, which have notably lower lengths.**

- **Despite that the final MD densities are in the range $25\% - 35\%$, the number of iterations is always below 20.**

- **GD-MD codes are constructed in a computationally fast manner. In fact,**
  - [R3, Algorithm 2] (with no relocations initially) terminates at MD density 21% in $\geq$ 21 iterations, whereas
  - our FL-AO terminates after 11 iterations only, to design our GD-MD Code 3.1, yielding fewer cycles-8.

Strength of Theorem 3

- **The final number of cycles we obtain after FL-AO Algorithm is lower than the upper bound on the expected number of cycles obtained in Theorem 3.**

- **For GD-MD Code 3.1, however, Theorem 3 gives a good indicator of the outcome of FL AO Algorithm, where the (rounded) expected number of cycles-8 is $228{,}070$ and the actual final number of cycles-8 is $239{,}752$.**

- **With Theorem 3 in hand, we are able to answer two major questions:**
  - What is the estimated percentage of relocations required to remove all instances of a cycle, assuming it can be removed entirely?
  - What is the optimal approach to handling a cycle given a pre-specified maximum relocation percentage, imposed by decoding latency requirements?

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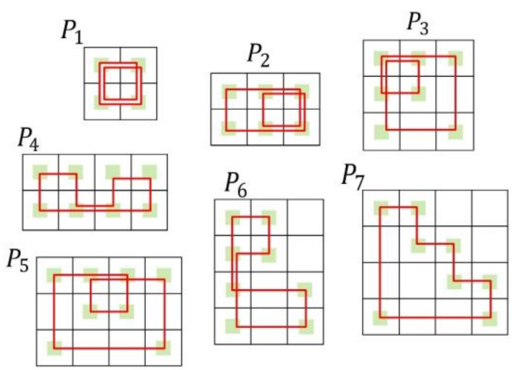

Conclusion and Ongoing Research

- **Our MD-GRADE algorithm provides us with a locally-optimal distribution matrix that guides the FL-AO by reducing its search space.**

- **When fed with a locally-optimal probability-distribution matrix, the FL-AO can converge on excellent finite-length MD-SC designs in a notably fast manner.**

- **Our GD-MD codes achieve notable reduction in the number of short cycles.**

- **Future work includes**
  - simulating the codes’ performance/density/lifetime gains and
  - extending the analysis to objects that are more advanced than short cycles.

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Thank You!