Turk J Math
(2007) , 1 – 15.


# Barely Transitive Groups

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**Abstract**

This is a survey article on barely transitive groups. It also involves some recent results in the case of a non-locally finite barely transitive group.

**Key Words:** Locally finite group, barely transitive group, infinite permutation group.

A group is said to satisfy the normalizer condition if every proper subgroup is distinct from its normalizer. It is well known that a finite group satisfies the normalizer condition if and only if it is nilpotent. Equivalently hypercentral. Kuroš and Černikov asked in [19, Question XXI, page 327]; whether a group satisfying the normalizer condition need be hypercentral or not. The negative answer was first given by Heineken and Mohamed in [17]. They constructed groups $G$ satisfying:

(i) $G$ is a countable locally finite $p$-group for a prime $p$.

(ii) $G/G'$ is isomorphic to $C_{p^\infty}$ and $G'$ is elementary abelian.

(iii) Every proper subgroup of $G$ is nilpotent and subnormal, so $G$ satisfies the normalizer condition.

(iv) $Z(G) = \{1\}$

(v) The set of normal subgroups of $G$ contained in $G'$ is linearly ordered by set inclusion [15, page 334 ].

(vi) For every proper subgroup $K$ of $G$, $KG'$ is a proper subgroup of $G$ [17, Lemma 1 (a)].

Similar constructions have been given in [27], [13], [14], and [28]. In 1973 Hartley gave examples of groups satisfying (i)-(v) as subgroups of $C_p \wr C_{p^\infty}$, see [14]. In Hartley's

2000 *AMS Mathematics Subject Classification:* 20F50, 20B07, 20F24.

construction, it is relatively easy to detect the properties of the subgroups. In 1974 Hartley constructed, for each natural number $n$, a group $G_n$ such that $G_n/G_n' \cong C_{p^\infty}$. They satisfy the properties (i)-(v) except that (ii) $G_n'$ is an abelian group of exponent $p^n$; in particular the exponent of the commutator subgroup of locally finite groups constructed by Hartley could be arbitrarily large. Furthermore the groups $G$ and $G_n$ constructed by Hartley satisfy additional properties, namely they have barely transitive representations.

A group $G$ has a **barely transitive representation** if $G$ acts on an infinite set $\Omega$ transitively and faithfully and every orbit of every proper subgroup is finite. A group is called a **barely transitive group** if it has a barely transitive representation. One can show easily that $G$ is barely transitive if and only if $G$ has a subgroup $H$ such that $|G : H|$ is infinite, $\bigcap_{x \in G} H^x = \{1\}$ and for every proper subgroup $K$ of $G$, $|K : K \cap H| < \infty$ where $H$ corresponds to a stabilizer of a point.

It is easy to see that an abelian barely transitive group is isomorphic to $C_{p^\infty}$ [21, Lemma 2.3]. Examples of non-abelian locally finite barely transitive groups with $G \neq G'$ are given by Hartley and their abstract properties are studied by B. Love in [15].

**Theorem 1.1** *[15, B. Love] If $G$ is a non-abelian locally finite barely transitive group with $G' \lneqq G$, then*

*(i) $G$ is a $p$-group for some prime $p$.*

*(2) $G/G' \cong C_{p^\infty}$.*

*(3) Every proper normal subgroup is nilpotent of finite exponent.*

*(4) $KG' \lneqq G$ for all $K \lneqq G$*

*(5) $Z_2(G) = Z(G)$ and $Z(G)$ is finite.*

Therefore the structure of a locally finite barely transitive group with $G \neq G'$ is fairly well understood. Basic properties of locally finite barely transitive groups are studied in [21] and the following properties are proved.

**Lemma 1.2** *[21] Let $G$ be a locally finite barely transitive group. Then*

*(1) $G$ has no proper subgroup of finite index*

*(2) $G$ is a countable group.*

*(3) Every proper subgroup of $G$ is residually finite.*

*(4) If $G$ is a locally soluble, locally finite barely transitive group, then $G$ is a $p$-group and every proper normal subgroup is nilpotent of finite exponent.*

The following lemma gives some insight about the structure of a locally finite barely transitive group.

**Lemma 1.3** *[21, Lemma 2.10] Let $G$ be a locally finite barely transitive permutation group. Then $G$ has an infinite chain of subgroups $H < H_1 < H_2 \ldots$ such that $G = \bigcup_{i=1}^{\infty} H_i$ and for every proper subgroup $K$ of $G$, there exists $n \in N$ such that $K \leq H_n$ . Moreover any two proper subgroups of $G$ generate a proper subgroup.*

In particular it follows from Lemma 1.3 that a locally finite barely transitive group does not have maximal subgroups. Therefore a locally finite barely transitive group cannot be a primitive permutation group. As every 2-transitive permutation group is primitive, a locally finite barely transitive permutation group cannot be 2-transitive. Therefore the degree of transitivity of barely transitive groups can not be increased.

For the quotient groups of barely transitive groups we have:

**Lemma 1.4** *Let $G$ be a barely transitive permutation group on a set $\Omega$, $N$ be a proper normal subgroup of $G$ and $H$ be a point stabilizer. If $\bigcap_{gN}(HN/N)^{gN} = \bar{1}$, then $G/N$ is a barely transitive permutation group on $\overline{\Omega}$ where $\overline{\Omega}$ is the set of all orbits of $N$.*

It follows from Lemma 1.4 that if N is a maximal normal subgroup of $G$, then $G/N$ is a barely transitive permutation group on the set of orbits of $N$.

The name **Heineken and Mohamed type** groups is also used for non-nilpotent locally finite $p$-groups in which every proper subgroup is subnormal and nilpotent. In this general context one can ask the following:

**Question 1.5** *Do all groups of Heineken and Mohamed type have faithful barely transitive representations?*

We prove the following.

**Theorem 1.6** *(Belyaev, Kuzucuoğlu [8]) The groups constructed by Heineken and Mohamed have faithful barely transitive representations.*

The question in the above generality remains open.

The barely transitive groups constructed by Hartley are different from their commutator subgroups. So the following question is of interest.

**Question 1.7** *(Hartley 1974) Does there exist a perfect locally finite barely transitive group?*

A special case of this:

**Question 1.8** *Does there exist a simple locally finite barely transitive group?*

We answer the Question 1.8 negatively.

**Theorem 1.9** *(Hartley, Kuzucuoğlu [16, 1997]) There exists no simple locally finite barely transitive group.*

The proof of the above theorem uses centralizers of elements in countable locally finite simple groups and the classification of finite simple groups. In order to give some ingredients of this proof we spell out some of the salient facts on locally finite simple groups. In order to get some information about the structure of locally finite simple groups one wishes to use the classification of finite simple groups. At this point the notion of Kegel sequences is a very useful tool for infinite locally finite simple groups, as it is well known that a countable locally finite simple group is not necessarily a union of finite simple groups, see [33] and [18]. As barely transitive groups are countable we consider here the countable locally finite simple groups. Let $G$ be a countable locally finite simple group. The sequence $\mathcal{K} = (G_i, M_i)_{i \in N}$ is called a **Kegel sequence** of $G$ if $G = \cup G_i$, where $G_i$ is a strictly ascending sequence of finite subgroups of $G$ and $M_i$ is a maximal normal subgroup of $G_i$ satisfying $G_i \cap M_{i+1} = 1$ for all $i$. By [29, Theorem 4.5] every countable locally finite simple group has a Kegel sequence.

For the sketch of the proof of Theorem 1.9, assume that there exists a simple locally finite barely transitive group $G$. Since a locally finite barely transitive group is countable by Theorem 1.2 (2) and by Lemma 1.3, $G$ cannot be generated by two proper subgroups. Then by [6, Corollary 1.9], $G$ can be embedded in a finitary linear group $FGL(V)$ on a vector space $V$ over a field of characteristic $p$. Now by [7, Theorem B] for an infinite simple periodic group $G$ of finitary transformations on a vector space over a field of characteristic $p$ the following are valid:

1) If $p = 0$, then for each finite subgroup $K$ of $G$, there exists a finite quasi-simple subgroup $H$ that contains $K$ and satisfying $K \cap Z(H) = \{1\}$.

2) If $p > 0$, then for each finite subgroup $K$ of $G$, there exists a finite subgroup $M$ that contains $K$ and satisfying $M = M'$, $M/O_p(M)$ is a quasi-simple group and

$K \cap S(M) = \{1\}$ where $S(M)$ is the maximal soluble normal subgroup of $M$.

In the first case, $G$ has a Kegel sequence $(G_i, Z(G_i))$, $i = 1, 2, 3, \cdots$ . By [21, Theorem 4.3] such a group $G$ cannot be a barely transitive group. For the second case, let $G = \cup_{i=1}^{\infty} G_i$, where the $G_i/O_p(G_i)$ are finite quasi-simple groups. We show that there exists an element $x$ in $G$ such that $C_G(x)$ involves an infinite non-linear, locally finite simple group; then by [21, Lemma 4.2] we obtain a final contradiction.

In connection with the above result we prove:

**Theorem 1.10** *(Belyaev-Kuzucuoğlu [8, Theorem 1]) A locally finite barely transitive group containing an element of order $p$ is a $p$-group where $p$ is prime.*

## 2. The non-locally finite torsion case

Recall that an infinite group is called **quasi-finite** if all its proper subgroups are finite. It is clear that a group is barely transitive in its regular permutation representation if and only if all its proper subgroups are finite, i.e. it is quasi-finite. Therefore the quasi-finite groups are examples of barely transitive groups. In this context the study of barely transitive groups can be thought of as a natural generalization of the study of questions that arise from the Schmidt problem. For well known examples of quasi-finite groups, we may consider the groups constructed by Olshanskii.

**Theorem 2.1** *(Ol'shanskii) [30, Theorem 28.2] There are infinite simple $p$-groups $G_{(\infty)}$ in which every proper subgroup is finite cyclic of order $p$ where $p$ is a sufficiently large prime.*

Therefore the groups in Theorem 2.1 are examples of periodic, countable, simple barely transitive $p$-groups. Although it is known that a locally finite barely transitive group is a $p$-group for a prime $p$, the groups constructed by Ol'shanskii in [31] show that there are infinite two generator groups all of whose proper subgroups are cyclic of prime order where the set of primes occurring as orders is infinite. Since in a barely transitive group the stabilizer of a point does not contain any nontrivial normal subgroup, the center of a barely transitive group is always finite. In all the known examples of non-abelian locally finite barely transitive groups the center is trivial. The following example shows that there are non-abelian, barely transitive quasi-finite groups with non-trivial center.

**Theorem 2.2** *[30, Olshanskii 31.8] For any sufficiently large prime $p$ and any integer*

*$k \geq 0$ there is a group $K$ such that $K/Z(K) \cong G_{(\infty)}$ where $G_{(\infty)}$ is the group in Theorem 2.1 and $Z(K)$ is a finite cyclic group of order $p^k$ such that every proper subgroup of $K$ is cyclic and either contains $Z(K)$ or is contained in $Z(K)$.*

Moreover by [30, Corollary 35.3] there exists a barely transitive group which contains an isomorphic copy of every finite group of odd order. These examples show that the structure of periodic non-locally finite barely transitive groups is quite complicated. So without any restriction on the group or stabilizer of a point, a characterization of such groups seems difficult.

In this context it might be interesting to know which conditions on $G$ force $G$ to be locally finite. We have the following:

**Proposition 2.3** *A barely transitive group $G$ satisfying $G \neq G'$ is a locally finite group.*

**Proof:** If $N$ is a proper normal subgroup of $G$, then by bare transitivity each orbit of $N$ is finite and $G$ acts on the set of orbits of $N$ transitively. The kernel of this action is locally finite since it is represented as a subdirect product of a finite group.

Let $H$ be a stabilizer of a point. Clearly $HG'$ is a proper subgroup of $G$. Then each proper subgroup of the abelian group $G/HG'$ is finite. Hence $G/(HG')$ is isomorphic to $C_{p^\infty}$. Now by the first part $G'H$ becomes a normal locally finite subgroup of $G$. Hence $G$ is locally finite.

Zelmanov proved in [34] and [35] that, if a finite group $G$ is generated by $n$ elements and is of finite exponent $m$, then the order of $G$ is bounded by a function of $m$ and $n$. One of the well known applications of this is the following. A residually finite group of finite exponent $m$ is a locally finite group. Indeed assume that there exists a finitely generated, infinite residually finite group $F = \langle x_1, \cdots, x_n \rangle$ of finite exponent. Since $F$ is an infinite group, given any natural number $k$, there exists $N_k \trianglelefteq F$ such that $|F/N_k| > k$. Since $F$ has finite exponent, clearly $F/N_k$ has finite exponent for any $k$. Then for any normal subgroup $N$ of $F$, by Zelmanov's theorem the order $|F/N|$ must be bounded. Hence we obtain a contradiction. So $F$ must be finite. By using this result and the fact that each proper normal subgroup of a barely transitive group is residually finite and of finite exponent one can prove that each proper normal subgroup of a barely transitive group is locally finite as in [11].

Then the proof of the next proposition is elementary but it gives a necessary and sufficient condition for a barely transitive group to be a locally finite barely transitive group.

**Proposition 2.4** *[23, Proposition 1] A barely transitive group $G$ is a union of an increasing sequence of proper normal subgroups of $G$ if and only if $G$ is locally finite.*

## 3. The torsion-free case

Another question of Hartley about barely transitive groups is the following.

**Question 3.1** *(Hartley [15, page 335]) Does there exist a torsion-free barely transitive group?*

The answer to this question in this generality is still unknown. As every proper normal subgroup of a barely transitive group is locally finite, if a torsion-free barely transitive group exists, then it is simple. This is also observed by Arıkan in [2].

Therefore the collection of results below is an attempt to prove that a torsion-free barely transitive group does not exist and answer Hartley's question negatively.

The following theorem shows that if a torsion-free barely transitive group exists, then the centralizer of a non-trivial element cannot contain a stabilizer of a point [23].

**Theorem 3.2** *Let $G$ be a simple barely transitive group, and $H$ be the stabilizer of a point. If for some non-identity element $x \in G$ the group $C_G(x)$ is infinite, then $C_G(x)$ cannot contain $H$.*

On the other hand, in [23] we prove the following.

**Proposition 3.3** *Let $G$ be a torsion-free barely transitive group and $H$ be a stabilizer of a point. Then for all non-identity element $x \in G$, the index $|H : C_H(x)|$ is infinite.*

Then we have an easy application of this.

**Corollary 3.4** *Let $G$ be a torsion-free barely transitive group. Then the FC-center of $H$ is trivial.*

In [22] we ask how restrictions on a stabilizer of a point affect the structure of a barely transitive group. In the locally finite case, if the stabilizer of a point is abelian, then the commutator subgroup is a proper subgroup.

**Proposition 3.5** *Let $G$ be a locally finite barely transitive group with an abelian point stabilizer. Then $G$ is metabelian. Furthermore*

*(i) $G$ is a $p$-group, $p$ prime.*

*(ii) $G/G'$ is isomorphic to $C_{p^\infty}$.*

*(iii) $G'$ is an abelian group of finite exponent.*

In the general case we have the following characterization:

**Theorem 3.6** *[12](Betin -Kuzucuoğlu) A barely transitive group $G$ with an abelian point stabilizer $H$ satisfies one of the following:*

*(1) $G$ is a metabelian $p$-group satisfying $G/G' \cong C_{p^\infty}$.*

*(2) $G$ is a finitely generated quasi-finite group. In particular $H$ is a finite group.*

*(3) $G$ is finitely generated, $H$ is infinite and $G$ has a maximal normal subgroup $M$ containing $H$ such that $M$ is locally finite, $M^{''} = 1$ and $G/M$ is a simple quasi-finite group. In particular in all the above cases $G$ is periodic.*

The groups constructed by Hartley form an example for (1), the groups constructed by Ols'hanskii form an example for (2). For the $3^{rd}$ part of the theorem we do not have examples but private communication with Ols'hanskii indicates that it should be possible to construct extensions of Ols'hanskii groups of type $G_{(\infty)}$ in Theorem 2.1.

In the case of infinitely generated groups (i.e. every finitely generated subgroup is a proper subgroup of $G$) we obtain the following relatively stronger result.

**Corollary 3.7** *If $G$ is an infinitely generated barely transitive group and the stabilizer of a point is abelian, then $G$ is locally finite and $G \neq G'$.*

In all the examples of barely transitive groups (known to the author) it is possible to find a barely transitive representation with an abelian point stabilizer.

**Question 3.8** *Is it true that for every barely transitive group one can find a barely transitive representation with abelian point stabilizer?*

In general, if the stabilizer of a point is nilpotent, then we have the following characterization:

**Theorem 3.9** *[12] (Betin -Kuzucuoğlu) If $G$ is a barely transitive group with a nilpotent point stabilizer of class $c$, then $G$ satisfies one of the following:*

*(1) $G$ is a soluble locally finite $p$-group of derived length at most $c(c+1)$.*

*(2) $G$ is a finitely generated quasi-finite group.*

*(3) $G$ is finitely generated, $H$ is infinite and $G$ has a maximal normal subgroup $M$ containing $H$, which is locally finite and soluble of derived length at most $c(c+1)$. Moreover $G/M$ is a simple quasi-finite group.*

*In particular in all the above cases $G$ is periodic.*

This answers Hartley's question 3.1 negatively in the case that stabilizer of a point is nilpotent.

**Corollary 3.10** *Let $G$ be an infinitely generated barely transitive group and $H$ be a stabilizer of a point. If $H$ is locally nilpotent by soluble, then $G$ is a locally finite group.*

Recall that a subgroup $H$ is called a **permutable** subgroup of a group $G$, denoted by $HperG$ if $HK = KH$ for every subgroup $K$ of $G$.

By using a theorem of Mal'cev, Stonehewer proved the following:

**Proposition 3.11** *[32] A simple group cannot have a proper non-trivial permutable subgroup.*

Then the following question is of interest: If in a barely transitive group $G$, the stabilizer of a point is permutable, then what can we say about the structure of $G$? We have the following:

**Theorem 3.12** *(Betin -Kuzucuoğlu) Let $G$ be an infinitely generated barely transitive group with a permutable point stabilizer. Then $G$ is locally finite.*

A permutation is called **finitary** if it moves only finitely many symbols. A group is called a **finitary** group if every element of it is a finitary permutation. It is also very natural to ask whether there exists a locally finite barely transitive group $G$ acting on an infinite set $\Omega$ as a finitary permutation group. i.e. a finitary barely transitive permutation representation. Although the question of the existence of a perfect locally finite barely transitive group is still open, one can prove the following ([16, Lemma 1]): If there exists a finitary locally finite barely transitive group $G$, then $G = G'$. So the existence of a finitary barely transitive group will answer positively Question 1.7. We investigated the properties of barely transitive groups in finitary symmetric group in [20, 16] and proved the following:

**Theorem 3.13** *[16, Theorem 2] If there exists a finitary locally finite barely transitive group $G$, then $G$ is a perfect minimal non-FC, $p$-group where $p$ is a prime.*

A group is called an **FC-group** if every element has finitely many conjugates. A **minimal non-FC-group** is a group in which every proper subgroup is an FC-group but itself is not an FC-group. Then the existence of a finitary barely transitive group will answer positively the following long-standing open question.

**Question 3.14** *Does there exist a locally finite perfect minimal non-FC, $p$-group?*

On the other hand Leinen proved the following reduction Theorem, see [25].

**Theorem 3.15** *(Leinen) Every perfect locally finite minimal non-FC-group has a quotient which acts as a barely transitive $p$-group of finitary permutations on some infinite set.*

By Theorem 3.13 and Theorem 3.15 the question about the existence of perfect locally finite minimal non-FC $p$-groups turns out now to be equivalent to the question about the existence of perfect barely transitive $p$-groups which in addition act finitarily on the underlying set.

The following Lemma [16, Lemma 3] might be of independent interest in finitary permutation groups and it is the main tool for the next theorem.

**Lemma 3.16** *Let $G = \langle\, g_i : \quad i = 1, 2, 3, \dots \rangle$ be a transitive finitary permutation group on a set $\Delta$, where $g_i^p = 1$ for each $i$, and suppose that $\Delta = \cup_{i=1}^{\infty} \Delta_i$ where $\Delta_1 < \Delta_2 \cdots$ and $\Delta_i$ are finite $G$-blocks. Then $G$ has a subgroup isomorphic to $Wr^N C_p$.*

In this lemma we do not assume bare transitivity, therefore all transitive, finitary, totally imprimitive permutation groups that are generated by $p$-elements contain a subgroup isomorphic to $Wr^N C_p$. The next lemma shows that barely transitive finitary permutation groups generated by $p$-elements are indeed totally imprimitive.

**Lemma 3.17** *If there exists a finitary barely transitive group on a set $\Delta$, then $G$ does not have a maximal $G$-block. Moreover $\Delta = \cup_{i=1}^{\infty} \Delta_i$, where $\Delta_i$ are finite $G$-blocks.*

Then by using this information it is easy to prove:

**Theorem 3.18** *If $G$ is a finitary permutation group on $\Delta$ and $G = \langle g_i \mid i = 1, 2, 3 \ldots \rangle$ where $g_i^p = 1$ for each $i$, then $G$ is not a barely transitive group on $\Delta$.*

Recently in an attempt to investigate the existence of a finitary barely transitive group Asar asks the following more general question in [4].

**Question 3.19** *(Asar) Let $G$ be a transitive subgroup of an infinite finitary symmetric group. What are the sufficiency conditions under which $G$ has proper subgroup having an infinite orbit?*

In this direction he proves the following in [4].

**Theorem 3.20** *(Asar) Let $G$ be a totally imprimitive $p$-subgroup of $Fsym(\Omega)$, where $\Omega$ is infinite. Suppose that for every non-normal finite subgroup $F$ of $G$ there exists $y \in G \setminus N_G(F)$ such that $y^p \in C_G(F)$. Then $G$ contains a proper subgroup that has an infinite orbit.*

About the existence of a barely transitive finitary permutation group we have a reduction in [8]. In order to formulate this result, we need to bring in some new concepts. Recall that the set of all elements of a group $X$ having only finitely many conjugates forms a characteristic subgroup, called the FC-center. Clearly the center of a group is always contained in the FC-center. By analogy with the definition of an upper central series, we can form an upper FC-central series. A group $X$ is called an FC-nilpotent group if there exists a finite normal series $1 = X_0 \leq X_1 \leq \ldots \leq X_n = X$, satisfying the property that every factor $X_{i+1}/X_i$ is contained in the FC-center of the group $X/X_i$.

**Theorem 3.21** *(Belyaev-Kuzucuoğlu [8]) A perfect locally finite barely transitive group with FC-nilpotent point stabilizer has a non-trivial barely transitive finitary permutation representation.*

This evokes the question as to whether it is possible to reduce the present situation to finitary permutation groups in the general case. Namely, we ask the following:

**Question 3.22** *Does every perfect locally finite barely transitive group have a non-trivial finitary permutation representation?*

However even with this extra restriction that a perfect barely transitive group might be finitary, still Hartley's question remains unanswered. As is discussed before with some extra restrictions on the structure of a point stabilizer, Hartley's question is answered negatively. On the other hand, the proofs of these results give insight into the general structure of locally finite barely transitive groups. Thus, for instance, in [3] it is shown that a locally nilpotent barely transitive $p$-group differs from its derived group if the point stabilizer is soluble and hypercentral. Continuing research in these directions, we have obtained the following generalizations of the results in [3, Theorem 1] and [16, Theorem 1].

**Theorem 3.23** *(Belyaev-Kuzucuoğlu [8, Theorem 2]) A locally finite barely transitive group with soluble point stabilizer is itself soluble.*

The following lemma is the main tool for the proof of the above theorem. It does not assume the local finiteness of the group and it might be of independent interest.

**Lemma 3.24** *If $G$ contains a soluble subgroup of derived length $n$ and of finite index in $G$, then $G$ contains a soluble, characteristic subgroup of finite index that has derived length at most $n^2$.*

In the above Lemma the derived length of the characteristic subgroup may increase but recently Khukhro and Makarenko [26] proved the following stronger result where the derived length of the characteristic subgroup is the same.

**Theorem 3.25** *If a group $G$ has a subgroup $H$ of finite index $n$ satisfying the identity $\chi(H) = 1$, where $\chi$ is a multilinear commutator of weight $\omega$, then $G$ has also a characteristic subgroup $C$ of finite $(n, \omega)$-bounded index satisfying the same identity $\chi(C) = 1$*

Roughly speaking the stabilizer of a point in a barely transitive group plays a similar role as the centralizer of a point in a minimal non-FC-group. So for the minimal non-FC version of Theorem 3.23 we have the following easy proof.

**Proposition 3.26** *Let $G$ be a locally finite minimal non-FC-group. If there exists $g \in G$ such that $C_G(g)$ is soluble of derived length $m$, then $G$ is soluble of derived length at most $m + 1$.*

**Proof** An imperfect locally finite minimal non-FC-group is a group of Miller-Moreno type, see [9, Theorem 1]. Then by [10, Proposition 4] $G$ is a metabelian group. These groups are completely classified in [10].

So we may assume that $G$ is perfect. By [9, Lemma 8] $G$ is countable and by [24] $G$ is a $p$-group. Then $G = \bigcup_{i=1}^{\infty} N_i$, where each $N_i$ is a proper normal subgroup of $G$. As every two proper subgroup of $G$ generate a proper subgroup [9, Lemma 6], we have that for any proper subgroup $K$ in $G$, the group $\langle K, g \rangle$ is an FC-group, and so $| \langle K, g \rangle : C_{\langle K,g \rangle}(g) | < \infty$.

Let $S = C_G(g)$ be a soluble subgroup of $G$ of derived length $m$. Let $N$ be a proper normal subgroup of $G$. Then $|N : N \cap S| < \infty$ implies that there exists a finite normal subgroup $L$ of $N$ such that $N = (N \cap S)L$. But then $N/L = (N \cap S)L/L \cong (N \cap S)/(N \cap S \cap L)$. Then $N^{(m)}$ is a finite normal subgroup of $G$ contained in $L$. As $G$ cannot have a proper subgroup of finite index, $N^{(m)}$ must be contained in the center of the group $G$. It follows that $N^{(m)}$ is abelian. Hence $N^{(m+1)} = 1$ for any proper normal subgroup $N$ of $G$. This implies $G^{(m+1)} = 1$.

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Received September 10, 2007