Chapter 1

Governing Equations of Fluid Flow and Heat Transfer

Following fundamental laws can be used to derive governing differential equations that are solved in a Computational Fluid Dynamics (CFD) study [1]

- conservation of mass
- conservation of linear momentum (Newton's second law)
- conservation of energy (First law of thermodynamics)

In this course we'll consider the motion of single phase fluids, i.e. either liquid or gas, and we'll treat them as continuum. The three primary unknowns that can be obtained by solving these equations are (actually there are five scalar unknowns if we count the three velocity components separately)

- velocity vector $\vec{V}$
- pressure p
- temperature T

But in the governing equations that we solve numerically following four additional variables appear

- density ρ
- enthalpy h (or internal energy e)
- viscosity μ
- thermal conductivity k

Pressure and temperature can be treated as two independent thermodynamic variables that define the equilibrium state of the fluid. Four additional variables listed above are determined in terms of pressure and temperature using tables, charts or additional equations. However, for many problems it is possible to consider ρ, μ and k to be constants and h to be proportional to T with the proportionally constant being the specific heat c_p.

Due to different mathematical characters of governing equations for compressible and incompressible flows, CFD codes are usually written for only one of them. It is not common to find a code that can effectively and accurately work in both compressible and incompressible flow regimes. In the following two sections we'll provide differential forms of the governing equations used to study compressible and incompressible flows.

1.1 Conservation of Mass (Continuity Equation)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V}\right) = 0 \tag{1.1}$$

or equally

$$\frac{D\rho}{Dt} + \rho\left(\nabla \cdot \vec{V}\right) = 0 \tag{1.2}$$

These equations are known to be the conservative and non-conservative forms of mass conservation, respectively. Conservation forms of equations can be obtained by applying the underlying physical principle (mass conservation in this case) to a fluid element fixed in space. Non-conservative forms are obtained by considering fluid elements moving in the flow field. The link between these two equations can be established using the following general equation that relates spatial and material descriptions of fluid flow

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + (\vec{V} \cdot \nabla)A \tag{1.3}$$

The term on the left hand side of this equation is known as the material derivative of property A. First term on the right hand side is the partial time derivative or local derivative. Last term is called the convective derivative of A.

1.2 Conservation of Linear Momentum

Equation for the conservation of linear momentum is also known as the Navier-Stokes equation (In CFD literature the term Navier-Stokes is usually used to include both momentum and continuity equations, and even energy equation sometimes). It is possible to write it in many different forms. One possibility is

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \nabla \cdot \bar{\bar{\tau}} + \rho \vec{f} \tag{1.4}$$

In order to be able to use an Eulerian description, material derivative at the left hand side, which is the acceleration vector, can be replaced with the sum of local and convective accelerations to get

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V}\right] = -\nabla p + \nabla \cdot \bar{\bar{\tau}} + \rho \vec{f} \tag{1.5}$$

where $\vec{f}$ is the body force per unit mass. If the weight of the fluid is the only body force we can replace $\vec{f}$ with the gravitational acceleration vector $\vec{g}$.

$\bar{\bar{\tau}}$ of the above equation is the viscous stress tensor. For Newtonian fluids viscous stresses only depend on the velocity gradient and the dependency is linear. Also it is known that $\bar{\bar{\tau}}$ needs to be symmetric in order to satisfy the conservation of angular momentum. For a Newtonian fluid the relation between $\bar{\bar{\tau}}$ and the velocity components is as follows

$$\tau_{ij} = \mu \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) + \lambda \left(\nabla \cdot \vec{V}\right) \delta_{ij} \tag{1.6}$$

where x_i denote mutually perpendicular coordinate directions. μ is the dynamic viscosity and λ is known as the coefficient of bulk viscosity. It is related to the viscosity through the Stokes' hypothesis

$$\lambda + \frac{2}{3}\mu = 0 \tag{1.7}$$

and using this hypothesis viscous stress tensor becomes

$$\tau_{ij} = \mu \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\left(\nabla \cdot \vec{V}\right)\delta_{ij}\right) \tag{1.8}$$

where δ_{ij} is the Kronecker-Delta operator which is equal to 1 if $i = j$ and it is zero otherwise. Navier-Stokes equation given in Eqn (1.5) is said to be in non-conservative form. A mathematically equivalent conservative form, given below, can also be derived by using the continuity equation and necessary vector identities

$$\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot (\rho \vec{V} \otimes \vec{V}) = -\nabla p + \nabla \cdot \bar{\bar{\tau}} + \rho \vec{f} \tag{1.9}$$

where $\vec{V} \otimes \vec{V}$ is the tensor product of the velocity vector with itself, as given below

$$\vec{V} \otimes \vec{V} = \begin{bmatrix} V_1V_1 & V_1V_2 & V_1V_3 \\ V_2V_1 & V_2V_2 & V_2V_3 \\ V_3V_1 & V_3V_2 & V_3V_3 \end{bmatrix} \tag{1.10}$$

divergence of which is the following vector

$$\nabla \cdot (\vec{V} \otimes \vec{V}) = \left\{ \frac{\partial}{\partial x_1}, \ \frac{\partial}{\partial x_2}, \ \frac{\partial}{\partial x_3} \right\} \begin{bmatrix} V_1V_1 & V_1V_2 & V_1V_3 \\ V_2V_1 & V_2V_2 & V_2V_3 \\ V_3V_1 & V_3V_2 & V_3V_3 \end{bmatrix} = \begin{Bmatrix} \frac{\partial V_1V_1}{\partial x_1} + \frac{\partial V_2V_1}{\partial x_2} + \frac{\partial V_3V_1}{\partial x_3} \\ \frac{\partial V_1V_2}{\partial x_1} + \frac{\partial V_2V_2}{\partial x_2} + \frac{\partial V_3V_2}{\partial x_3} \\ \frac{\partial V_1V_3}{\partial x_1} + \frac{\partial V_2V_3}{\partial x_2} + \frac{\partial V_3V_3}{\partial x_3} \end{Bmatrix} \tag{1.11}$$

For compressible flow simulations it is quite common to see the use of Euler's equation instead of Navier-Stokes. Euler's equation is obtained by dropping the viscous term of the Navier-Stokes equation, which makes it a first order PDE. It is frequently used to obtain the pressure distribution of high speed (and therefore high Re) aerodynamic flows around/inside flying bodies where viscous affects are squeezed inside very thin boundary layers. However, one needs to be careful in using the Euler's equation since it can not predict flow fields with separation and circulation zones successfully.

1.3 Conservation of Energy

Energy equation can be written in many different ways, such as the one given below

$$\rho \left[\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{V}) \right] = -\frac{Dp}{Dt} + \nabla \cdot (k \, \nabla T) + \phi \tag{1.12}$$

where h is the specific enthalpy which is related to specific internal energy as $h = e + p/\rho$. T is the absolute temperature and ϕ is the dissipation function representing the work done against viscous forces, which is irreversibly converted into internal energy. It is defined as

$$\phi = (\bar{\bar{\tau}} \cdot \nabla)\vec{V} = \tau_{ij} \frac{\partial V_i}{\partial x_j} \tag{1.13}$$

Pressure term on the right hand side of equation (1.12) is usually neglected. To derive this energy equation we considered that the conduction heat transfer is governed by Fourier's law with k being the thermal conductivity of the fluid. Also note that radiative heat transfer and internal heat generation due to a possible chemical or nuclear reaction are neglected.

Equation of state:

For compressible flows the relation between density, pressure and temperature is given by a special equation called equation of state. The most commonly used one is the following ideal gas relation

$$p = \rho R T \tag{1.14}$$

where R is the gas constant, being equal to $287.1\ J/kgK$ for air. For an ideal gas it is also possible to use the following relations to relate enthalpy and internal energy to temperature so that energy equation can be written as temperature being the only unknown.

$$dh = c_p dT\,, \quad de = c_v dT \tag{1.15}$$

In general all three conservation equations (conservation of mass, momentum and energy) are coupled and they need to be solved simultaneously. Overall we have 6 scalar unknowns (density, pressure, 3 velocity components and temperature) which can be obtained by solving 6 scalar equations (conservation of mass, 3 components of conservation of momentum, conservation of energy and equation of state).

1.4 Incompressible Flows

For incompressible flows density has a known constant value, i.e. it is no longer an unknown. Also for an incompressible fluid it is not possible to talk about an equation of state.

Conservation of Mass:

For constant density, Eqn (1.2) simplifies to

$$\nabla \cdot \vec{V} = 0 \tag{1.16}$$

which means that the velocity field of an incompressible flow should be divergence free, which is known as the divergence free constraint in CFD literature. Note that there is no time derivative in the continuity equation even for unsteady flows, which is one of the reasons that make numerical solution of incompressible flows difficult.

Conservation of Linear Momentum:

For incompressible flows second term of the viscous stress tensor given in Eqn (1.8) is zero due to the incompressibility constraint given in Eqn (1.16). Considering this simplification together with viscosity being constant, Eqn (1.5) can be written as follows

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \right] = -\nabla p + \mu \nabla^2 \vec{V} + \rho \vec{f} \tag{1.17}$$

Dividing the equation by density we get the following form of the Navier-Stokes equation

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{V} + \vec{f} \tag{1.18}$$

where ν is the constant kinematic viscosity.

The term $(\vec{V} \cdot \nabla)\vec{V}$ on the right hand side is known as the convective term. It is the term which makes the Navier-Stokes equation nonlinear. $\nu\nabla^2\vec{V}$ is known as the viscous term or the diffusion term. For diffusion dominated flows the convective term can be dropped and the simplified equation is called the Stokes equation, which is linear. Stokes equations can be used to model very low speed flows known as creeping flows or flows with very small length scales (micro or nano flows) where Reynolds number is small. Convection dominated flows, which are typically characterized by high Reynolds numbers, are much more difficult to solve numerically compared to diffusion dominated flows. For most solid mechanics problems convection (flow of material) does not exist, which is the main reason of the differences seen in mathematical modeling (Eulerian vs. Lagrangian formulations) and in numerical solution techniques (e.g. need for upwinding) used in the disciplines of fluid and solid mechanics.

Conservation of Energy:

Conservation of energy given in Eqn (1.12) can be simplified by considering the fact that density is constant for incompressible flows. Also using the definition of enthalpy given previously and $dh = c_p dT$ relation, Eqn (1.12) takes the following form

$$\rho c_p \left[\frac{\partial T}{\partial t} + (\vec{V} \cdot \nabla)T\right] = k\nabla^2 T + \phi \tag{1.19}$$

where c_p is the specific heat at constant pressure. Note that $c_p \approx c_v$ for incompressible flows.

Here it is important to note that for incompressible flows equation of state does not exist. In practice this means that the energy equation is decoupled from the other two equations. Therefore we can first solve continuity and Navier-Stokes equations to find the unknown velocity and pressure distribution without knowing the temperature (We assume that fluid properties are taken to be constant, i.e. not functions of temperature. If fluid properties change with temperature all equations becomes coupled as in the case of compressible flows). After finding the velocity field, energy equation can be solved by itself to find the temperature distribution. However for buoyancy driven flows (natural convection) where the density changes due to temperature variations are considered in the body force term of the momentum equation (Boussinesq approximation), all three conservation equations again become coupled.

Heat transfer and therefore the energy equation is not always a primary concern in an incompressible flow. For isothermal (constant temperature) incompressible flows energy equation (and therefore temperature) can be dropped and only the mass and linear momentum equations are solved to obtain the velocity and pressure fields.

Numerical solution of incompressible flows is usually considered to be more difficult compared to compressible flows. The main numerical difficulty of solving incompressible flows lies in the role of pressure. For incompressible flows pressure is no longer a thermodynamic quantity and it can not be related to density or temperature through an equation of state. It just establishes itself instantaneously in a flow field so that the velocity field always remains divergence free. In the continuity equation there is no pressure term and in the momentum equation there are only the derivatives of pressure, but not the pressure itself. This means that the actual value of pressure in an incompressible flow solution is not important, only the changes of pressure in space are important. Additionally there is no time derivative of pressure, even for incompressible flows.

1.5 Flow Equations in Cartesian and Cylindrical Coordinate Systems

Conservation of mass, momentum and energy given in equations (1.1), (1.5) and (1.12) (or alternatively given in (1.16), (1.18) and (1.19) for incompressible flows) are valid for any coordinate system. In order to write them for a specific coordinate system first we need to define the velocity vector components in these systems, such as the following ones

$$\text{Cartesian :} \qquad \vec{V} = u\,\vec{\imath} + v\,\vec{\jmath} + w\,\vec{k}$$
$$\text{Cylindrical :} \qquad \vec{V} = V_r\,\vec{\imath_r} + V_\theta\,\vec{\imath_\theta} + V_z\,\vec{\imath_z} \tag{1.20}$$

Furthermore we need to use the following mathematical identities

$$\text{Cartesian :} \qquad \nabla A = \frac{\partial A}{\partial x}\,\vec{\imath} + \frac{\partial A}{\partial y}\,\vec{\jmath} + \frac{\partial A}{\partial z}\,\vec{k}$$
$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$$
$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$
$$\vec{V} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{1.21}$$
$$\text{Cylindrical :} \qquad \nabla A = \frac{\partial A}{\partial r}\,\vec{\imath_r} + \frac{1}{r}\frac{\partial A}{\partial \theta}\,\vec{\imath_\theta} + \frac{\partial A}{\partial z}\,\vec{\imath_z}$$
$$\nabla^2 A = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} + \frac{\partial^2 A}{\partial z^2}$$
$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$$
$$\vec{V} \cdot \nabla = V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta} + V_z\frac{\partial}{\partial z}$$

Using these identities governing equations for incompressible flows in Cartesian coordinate system can be obtained as follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1.22}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + f_x \tag{1.23 a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + f_y \tag{1.23 b}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + f_z \tag{1.23 c}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \phi \tag{1.24}$$

where the kinematic viscosity and the thermal conductivity are taken to be constants. u, v and w are the velocity components in the x, y and z directions, respectively. Dissipation function of the energy equation is given by

$$\phi = \mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right\} \tag{1.25}$$

which, as seen, always has a positive value. This term is rarely important (e.g. for high speed flows in long, narrow capillaries, where viscous heating is not negligible).

Incompressible flow equations in cylindrical coordinate system are

$$\frac{1}{r}\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0 \tag{1.26}$$

$$\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} + \frac{\partial^2 V_z}{\partial z^2} - \frac{V_r}{r^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}\right] + f_r \tag{1.27 a}$$

$$\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} - \frac{V_r V_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{\partial^2 V_\theta}{\partial z^2} - \frac{V_\theta}{r^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta}\right] + f_\theta \tag{1.27 b}$$

$$\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2} + \frac{\partial^2 V_\theta}{\partial z^2}\right] + f_z \tag{1.27 c}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + V_r\frac{\partial T}{\partial r} + \frac{V_\theta}{r}\frac{\partial T}{\partial \theta} + V_z\frac{\partial T}{\partial z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right] + \phi \tag{1.28}$$

1.6 Nondimensionalization of Governing Equations

It is possible, and sometimes preferable, to write governing equations in nondimensional form. To do this we need to select a characteristic quantities that describe the flow problem, such as a characteristic length L, characteristic velocity U_∞, characteristic pressure p_∞ and characteristic temperature T_∞. Using these characteristic quantities the following nondimensional parameters can be defined

$$t^* = \frac{t}{L/U_\infty} \;,\quad x^* = \frac{x}{L} \;,\quad y^* = \frac{y}{L} \;,\quad z^* = \frac{z}{L} \;,\quad u^* = \frac{u}{U_\infty} \;,\quad v^* = \frac{v}{U_\infty} \;,\quad w^* = \frac{w}{U_\infty} \;,\quad \ldots$$

$$\ldots \;,\quad p^* = \frac{p - p_\infty}{\rho {U_\infty}^2} \;,\quad T^* = \frac{T - T_\infty}{\Delta T} \tag{1.29}$$

where ΔT is a known reference temperature difference in the flow field such as the one between a constant wall temperature (if it exists) and T_∞. Note that these nondimensionalizations are not unique and can also be done in other ways. Using these definitions in the governing equations nondimensional forms of them can be obtained. For example for an incompressible flow without body forces, equations (1.22)-(1.24) can be converted into the following ones

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z} = 0 \tag{1.30}$$

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*} = \frac{\partial p^*}{\partial x^*} + \frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}}\right) \tag{1.31a}$$

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + w^* \frac{\partial v^*}{\partial z^*} = \frac{\partial p^*}{\partial y^*} + \frac{1}{Re}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\partial^2 v^*}{\partial z^{*2}}\right) \qquad (1.31b)$$

$$\frac{\partial w^*}{\partial t^*} + u^* \frac{\partial w^*}{\partial x^*} + v^* \frac{\partial w}{\partial y^*} + w^* \frac{\partial w^*}{\partial z^*} = \frac{\partial p^*}{\partial z^*} + \frac{1}{Re}\left(\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}}\right) \qquad (1.31c)$$

$$\frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} + w^* \frac{\partial T^*}{\partial z^*} = \frac{1}{RePr}\left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} + \frac{\partial^2 T^*}{\partial z^{*2}}\right) + \frac{Ec}{Re}\phi^* \qquad (1.32)$$

As seen the equations are very similar to their dimensional counterparts with additional nondimensional numbers. Reynolds number ($Re = U_\infty L/\nu$) is seen in the momentum and energy equations. In the energy equation we also have Eckert and Prandtl numbers.

Reynolds number is a measure of the balance between convective and diffusive terms of the Navier-Stokes equation. High and low Re flows are said to be convection and diffusion dominated, respectively. The nondimensional time given in equation (1.29) is suitable for convection dominated (high Re) flows. For diffusion dominated problems using a diffusion time scale as $t^* = \frac{t}{L^2/\nu}$ is more suitable. In this case the form of the nondimensional momentum equations will be different.

Eckert number ($Ec = U_\infty^2/(c_p \Delta T)$ is the ratio of flow's kinetic energy to a representative enthalpy difference. As it gets larger the importance of viscous dissipation is amplified. Prandtl number ($Pr = \nu/\alpha = \nu/(k/\rho c_p)$) is the ratio of momentum and thermal diffusivities. As it gets larger the importance of diffusion term on the right hand side of equation (1.32) diminishes and the convective heat transfer modeled by the terms on the left hand side becomes more dominant. Multiplication of Reynolds and Prandtl numbers is called the Peclet number (Pe).

There are many other important nondimensional numbers in fluid mechanics and heat transfer. Some of them appear in the equations as the ones seen above (such as the Grashof number seen in natural convection flows), and some others appear inside the boundary conditions (such as the Nusselt number).

1.7 Turbulence Modeling

One of today's most important challenge for the numerical solution of fluid flow problems is the modeling and simulation of turbulence. Although Navier-Stokes equations are believed to be capable of describing turbulent flows in full detail, with today's computational resources it is simply impossible to have simulations that will yield all the details of a turbulent flow in a realistically complicated 3D domain with realistically high Reynolds numbers. The difficulty arises from the fact that turbulent flows are violently unsteady by nature and the length scales that need to be resolved involve both large and extremely small eddies. Therefore a computational study is restricted to use extremely small space and time discretizations, which exceeds today's computational power.

Solving N-S equations with very fine computational grids and very small time steps to get the whole detail of a turbulent flow is called Direct Numerical Simulation (DNS). Today DNS is a very active research area. However, for practical real world problems it is too restrictive and instead people try to model the effect of viscous dissipation mechanism of turbulent flows by supporting the governing conservation laws with extra equations and unknowns, which is known as turbulence modeling. Turbulence modeling is a very active CFD discipline with dedicated researchers, conferences and books.

Today there are tens of different turbulence models in use and they show a wide range of complexity. Although some of them are clearly superior to the others based on the physics they involve, unfortunately none of them can predict all types of turbulent flows more accurately and efficiently than the others. They all contain empiric constants, which are tuned numbers so that numerical results fit to known experimental and/or analytical ones better. The use of turbulence models together with the governing equations bring terms such as turbulent viscosity or turbulent thermal diffusivity along their laminar counterparts. These additional terms simply model the increased momentum and heat transfer exchange typically seen turbulent flows.

1.8 Model Differential Equations

Governing equations of fluid mechanics and heat transfer problem are usually second or PDEs. There are more than one unknown and coupled PDEs need to be solved simultaneously. Also Navier-Stokes equations are nonlinear. In short, these equations are not the most appropriate ones for learning the basics of a numerical technique, such as the Finite Element Method. Instead we prefer to start with simplified model ordinary or partial differential equations. These model equations usually involve a single space dimension and maybe time. It is possible to construct problems governed by them with known analytical solutions so that numerical codes can be validated easily. Although these model equations are much simpler than the actual governing equations of fluid mechanics, they still give us a chance to get an experience about the numerical difficulties that we'll face with when we start working with more realistic ones.

Commonly used model differential equations are

- 1D wave equation : $\frac{\partial^2 \phi}{\partial t^2} = c^2 \, \frac{\partial^2 \phi}{\partial x^2}$
- 1D unsteady advection diffusion equation : $\frac{\partial \phi}{\partial t} + U\left(\frac{\partial \phi}{\partial x}\right) = \alpha \, \frac{\partial^2 \phi}{\partial x^2}$
- 1D nonlinear Burgers equation : $\frac{\partial \phi}{\partial t} + \phi\left(\frac{\partial \phi}{\partial x}\right) = \nu \, \frac{\partial^2 \phi}{\partial x^2}$
- 2D Laplace's equation : $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$
- 2D Poisson equation : $\nabla^2 \phi = f(x, y)$
- 2D unsteady heat conduction (diffusion) equation : $\frac{\partial \phi}{\partial t} = \alpha \, \nabla^2 \phi$
- 2D unsteady advection-diffusion equation : $\frac{\partial \phi}{\partial t} + \vec{V} \cdot \nabla \phi = \alpha \, \nabla^2 \phi + S$

1.9 Mathematical and Physical Classification of PDEs

It is possible to classify PDEs in three categories

- Elliptic
- Parabolic
- Hyperbolic

This classification is related to the "characteristics" of PDEs. Characteristics are paths (curved surfaces in $xyzt$ hyperspace in general) in the solution domain along which information propagates. If a PDE possesses real characteristics, then information propagates along these characteristics. If no real characteristics exist, then there are no preferred paths of information propagation. The presence or absence of real characteristics has a significant impact on the solution of a PDE, both analytically and numerically.

Mathematical procedure of identifying the type of a PDE depends on the mathematical details such as the order of the PDE and it can be studied from Hoffmann [2] and Hoffmann and Chiang [3] (related chapters of these two references are available as PDF files at the Files tab of the course web site). Here let's concentrate on the difference of the physics of the problems that are governed by different types of PDEs.

Parabolic and hyperbolic PDEs have real characteristics. Problems governed by these two types of PDEs are called propagation problems, which are actually initial value problems in which the solution starts from a known initial condition and propagates in time. In the meantime the solution is guided by the boundary conditions. Solution domains of parabolic and hyperbolic PDEs are said to be open in the sense that in theory the solution may continue infinitely long in the time domain.

2D unsteady heat conduction equation given in the previous section is a parabolic PDE. For simplicity consider its 1D version, which can be used to study the temperature distribution of a bar which is left to cool down from a known initial temperature distribution. We are interested in the temperature distribution at various stages of this cooling. To solve this problem, boundary conditions need to be specified at both ends of the bar, e.g. two ends are kept at fixed temperatures. Starting from the known initial temperature distribution, new temperature values at different time levels can be calculated numerically. In practice the solution does not continue infinitely long in time, but it ends at a proper final time, e.g. when the process reaches steady-state, if such a state exists, where the temperature of the bar no longer changes.

As said before, parabolic PDEs have real characteristics. From a physical standpoint this divides the solution domain into a zone of dependence and a zone of influence as seen in Figure 1.1. In the "cooling of a bar" problem defined above consider the mid-point P of the bar. Initially this point has a certain temperature and this temperature value affects the temperature of all points on the bar at all future times. That is if we start the solution with a different temperature at point P, temperature all along the bar at all time levels will change. Therefore at the beginning of a solution all the problem domain, both in space and in time, is zone of influence for point P. Now consider the same point P at $t = 5$ seconds after cooling starts. Point P has a certain temperature value at $t = 5$, but this value can affect only the solution ahead in time, i.e. it can only affect the solution between $5 < t < \infty$. Now the part of the problem domain corresponding to $0 < t < 5$ is the zone of dependence and the part $5 < t < \infty$ is the zone of influence. The solution at point P at $t = 5$ depends on the solution in the zone of dependence and will affect the solution in the zone of influence.

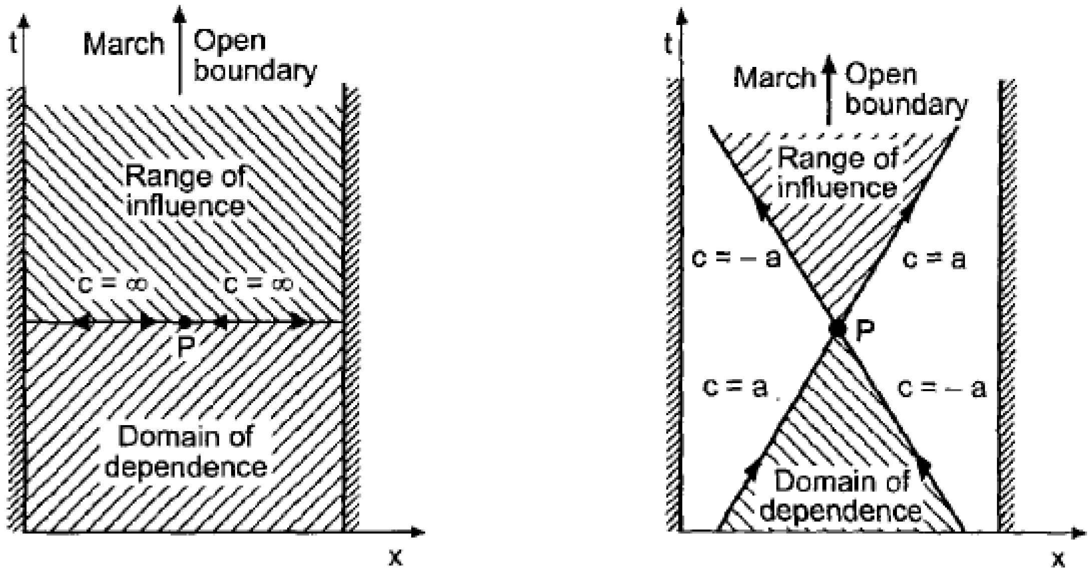

Figure 1.1 Domain of dependence and domain and influence for a parabolic PDE (left) and a hyperbolic PDE (right) [2]

Hyperbolic PDEs also have real characteristics and they also govern propagation problems. Consider the 1D wave equation given in the previous section, which is a hyperbolic PDE. It can be used to study the travel of a pressure disturbance (acoustic pressure) with a known initial shape in a 1D domain with a constant speed of sound c. The value of c determines how fast the information can propagate in the solution domain. Since c has a finite value, the solution at a certain point P of the solution domain can only affect certain parts of the future solutions. Similarly it can be affected by only certain parts of the previous solutions.

The lines drawn in Figure 1.1 (right) are called characteristic lines. These are the lines along which information, i.e. the pressure disturbance travels. The slopes of these lines are determined by the constant speed c of the pressure wave. As the wave speed increases the slopes of these lines will change and in the limiting case of infinite wave speed the characteristic lines will be parallel to the x axis, which is the case for the previously discussed parabolic PDEs. So for a parabolic PDE information travels with infinite speed and can reach and influence all problem domain immediately.

Finally elliptic problems govern equilibrium problems, which are used to obtain steady state solutions in closed domains. 2D Laplace's equation given in the previous section that may be used to calculate the temperature distribution over a square plate heated at the center with a known heat source. Here the problem domain is in the xy plane and time is not an independent variable. Depending on the amount of heat source and the boundary conditions specified at the four edges of the plate a certain steady-state temperature distribution can be calculated.

Elliptic PDEs have no real characteristics and both the domain of dependence and the domain of influence is the whole problem domain for all points. Solution at every point of the problem domain is influenced by the solution at all other points, and the solution at each point influences the solution at all other points.

1.10 Advection-Diffusion (A-D) Equation

A-D equation, given below, is the simplest model equation that can be used to test the performance of different numerical schemes for problems involving advection (convection) and diffusion phenomena.

$$\frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T = k \nabla^2 T + S \qquad (1.33)$$

A-D equation can be seen as a linearized and simplified scalar form of the Navier-Stokes equation with a single variable. It is also very similar to the energy equation solved separately by itself for incompressible flows. The scalar unknown T is advected (convected) with a known velocity field $\vec{V}$, which can be taken to be divergence free to simulate the constraint due to the continuity equation of incompressible flows. At the same time T is diffused with a known constant and isotropic diffusivity of k. S represents the known source term. Physically the problem corresponds to the calculation of the temperature field of a heat transfer problem or concentration field of a species transport problem with the use of a known velocity field.

In this course we'll use A-D equation extensively to study the difficulties faced with highly convective cases and alternative solutions methods. Note that for a special case of no velocity ($\vec{V} = 0$, pure diffusion), we obtain the transient heat equation which is parabolic. If $\vec{V} = 0$ and $\frac{\partial T}{\partial t} = 0$, we get the steady Poisson equation which is elliptic. For the pure advection case ($k = 0$) the equation becomes hyperbolic.

To discuss dimensionless form of the A-D diffusion equation we can consider the following 1D form of it and neglect the source term for simplicity

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial x^2} \qquad (1.34)$$

Using a characteristic length L, a characteristic velocity U_0 and a characteristic time L/U_0, following dimensionless variables can be defined

$$T^* = \frac{T}{\Delta T} \ , \qquad x^* = \frac{x}{L} \ , \qquad u^* = \frac{u}{U_0} \ , \qquad t^* = \frac{t}{L/U_0} \qquad (1.35)$$

where ΔT is the characteristic driving temperature difference for a heat transfer problem. Using these dimensionless variables the following nondimensional form of the A-D equation can be derived

$$\frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} = \frac{1}{Pe} \frac{\partial^2 T^*}{\partial x^{*2}} \qquad (1.36)$$

where Peclet number ($Pe = U_0 L/k$) is similar to the Reynolds number of the Navier-Stokes equation. It represents a ratio between the "strength" of advection and diffusion processes. Convection dominated flows are characterized by high Peclet values. Note that for diffusion dominated problems using a characteristic time of L^2/k is more appropriate.

1.11 Exercises

E-1.1. In this course fluids are treated as continuum. Define continuum. What is its relation with the nondimensional Knudsen number? Give engineering examples for which it is no longer is valid. Which equations need to be solved when continuum does not hold? What are "slip" and "temperature jump" boundary conditions? What are the popular numerical techniques specifically used to simulate non-continuum flows?

E-1.2. What is the axisymmetric flow assumption? How is it different than the 2D planar flow assumption? Provide examples of engineering problems involving axisymmetric fluid flow and/or heat transfer.

E-1.3. Fluid properties such as kinematic viscosity and thermal conductivity are commonly assumed to be constant. However, in general they are known to be functions of pressure and temperature. How do these two properties change with temperature (at standard atmospheric pressure) for the most common fluids air and water? What about their change with pressure at 20 ^oC? Provide your results as figures.

E-1.4. What is the physical simplification behind the ideal gas assumption? Under which conditions does it hold nicely and under which conditions do fluids show non-ideal behavior?

E-1.5. In fluid mechanics it is possible to define two different pressures; thermodynamic and mechanical. How are they defined for incompressible and compressible flows? What is the role of Stoke's hypothesis in their relation?

E-1.6. What is the Boussinesq approximation mentioned in Section 1.4?

References

[1] F. M. White, Viscous Fluid Flow, 2nd ed., McGraw-Hill, 1991 (QA929 .W48)

[2] J. D. Hoffman, Numerical Methods for Engineers and Scientists, Marcel Dekker, 2001 (QA297 .H588)

[3] K. A. Hoffmann, S. T. Chiang, Computational Fluid Dynamics, Engineering Education System, 2000 (QA911 .H54)