ME 310

Numerical Methods

Interpolation

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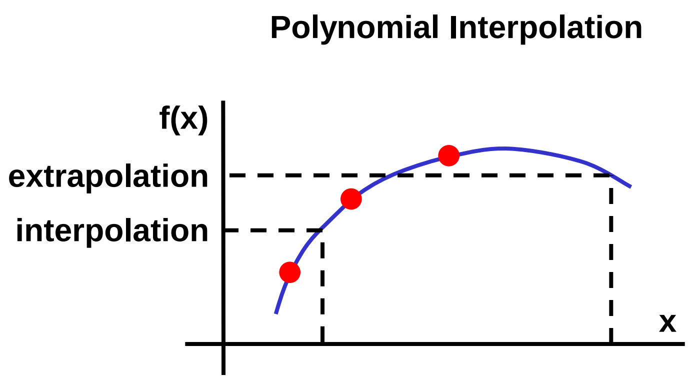
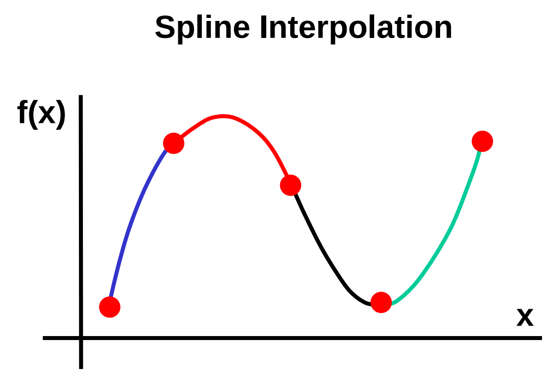
Interpolation

• Estimating intermediate values between precise data points.

• We first fit a function that exactly passes through the given data points and than evaluate intermediate values using this function.





• **Polynomial Interpolation:** A unique nth order polynomial passes through n points.

- Newton's Divided Difference Interpolating Polynomials
- Lagrange Interpolating Polynomials

• **Spline Interpolation:** Pass different curves (mostly 3rd order) through different subsets of the data points.

Polynomial Interpolation

- Given the following n+1 data points

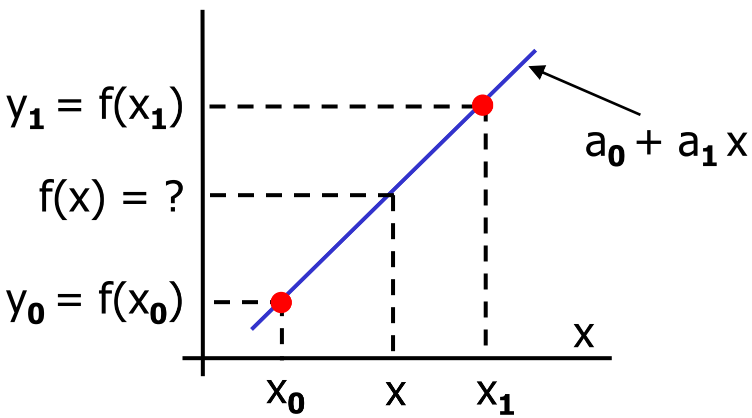
$$(x_1, y_1), \quad (x_2, y_2), \quad (x_3, y_3), \quad \ldots, \quad (x_{n+1}, y_{n+1})$$

there is a unique n^{th} order polynomial that passes through them

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

- The question is to find the coefficients $a_0, a_1, \ldots, a_n$

- **Linear Interpolation:**



- Given: (x_0, y_0) and (x_1, y_1)
- A straight line passes from these two points.
- Using similar triangles

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

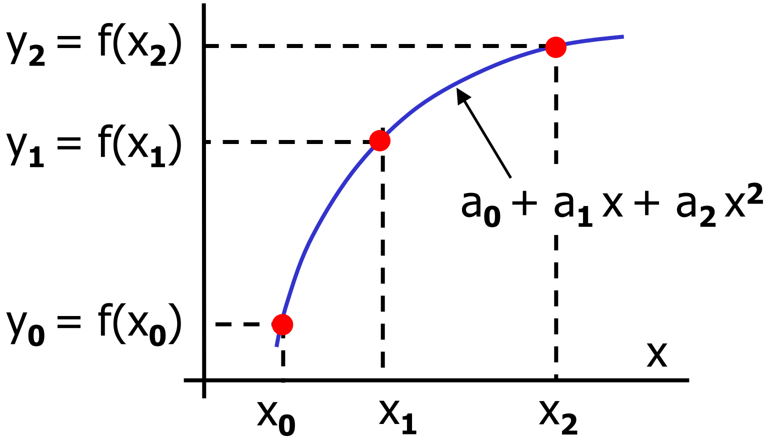
or

$$f_1(x) = b_0 + b_1(x - x_0)$$

Linear interpolation formula

Polynomial Interpolation

- **Quadratic Interpolation:**



- Given: (x_0, y_0) , (x_1, y_1) and (x_2, y_2)
- A parabola passes from these three points.
- Similar to the linear case, the equation of this parabola can be written as

$$\boxed{\mathbf{f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)}}$$

Quadratic interpolation formula

- How to find b_0, b_1 and b_2 in terms of given quantities?
  - at $x=x_0$ $\quad f_2(x) = f(x_0) = b_0 \quad \rightarrow \quad \mathbf{b_0 = f(x_0)}$
  - at $x=x_1$ $\quad f_2(x) = f(x_1) = b_0 + b_1 x_1 \quad \rightarrow \quad \mathbf{b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}$
  - at $x=x_2$ $\quad f_2(x) = f(x_2) = b_0 + b_1(x_2-x_0)+ b_2 (x_2-x_0) (x_2-x_1)$

$$\rightarrow \quad \mathbf{b_2 = \dfrac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}}$$

Newton's Divided Difference Interpolating Polynomials

- We can generalize the linear and quadratic interpolation formulas for an n^{th} order polynomial passing through n+1 points

$$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \dots + b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

where the constants are

$$b_0 = f(x_0) \qquad b_1 = f[x_1, x_0] \qquad b_2 = f[x_2, x_1, x_0] \qquad \dots \qquad b_n = f[x_n, x_{n-1}, \dots, x_1, x_0]$$

where the bracketed functions are finite divided differences evaluated recursively

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j} \qquad 1^{st} \text{ finite divided difference}$$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k} \qquad 2^{nd} \text{ finite divided difference}$$

$$f[x_n, x_{n-1}, \dots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \dots, x_1] - f[x_{n-1}, \dots, x_1, x_0]}{x_n - x_0} \qquad n^{th} \text{ finite divided difference}$$

- There n^{th} order Newton's Divided Difference Interpolating polynomial is

$$f_n(x) = f(x_0) + (x - x_0) f[x_1, x_0] + (x - x_0)(x - x_1) f[x_2, x_1, x_0] + \dots + (x - x_0)(x - x_1)\cdots(x - x_{n-1}) f[x_n, x_{n-1}, \dots, x_1, x_0]$$

Example 29:

The following logarithmic table is given.

x	f(x)=log(x)
4.0	0.60206
4.5	0.6532125
5.5	0.7403627
6.0	0.7781513

(a) Interpolate log(5) using the points x=4 and x=6

(b) Interpolate log(5) using the points x=4.5 and x=5.5

Note that the exact value is log(5) = 0.69897

(a) Linear interpolation. $f(x) = f(x_0) + (x - x_0)\, f[x_1, x_0]$

$x_0 = 4,\ x_1 = 6 \quad \rightarrow \quad f[x_1, x_0] = [f(6) - f(4)] / (6 - 4) = 0.0880046$

$f(5) \approx f(4) + (5 - 4)\, 0.0880046 = 0.690106 \qquad \varepsilon_t = 1.27\ \%$

(b) Again linear interpolation. But this time

$x_0 = 4.5,\ x_1 = 5.5 \quad \rightarrow \quad f[x_1, x_0] = [f(5.5) - f(4.5)] / (5.5 - 4.5) = 0.0871502$

$f(5) \approx f(4.5) + (5 - 4.5)\, 0.0871502 = 0.696788 \qquad \varepsilon_t = 0.3\ \%$

Example 29 (cont'd):

x	f(x)=log(x)
4.0	0.6020600
4.5	0.6532125
5.5	0.7403627
6.0	0.7781513

(c) Interpolate log(5) using the points x=4.5, x=5.5 and x=6

(c) Quadratic interpolation.

$x_0 = 4.5,\ x_1 = 5.5,\ x_2 = 6 \rightarrow \quad f[x_1, x_0] = 0.0871502$ (already calculated)

$f[x_2, x_1] = [f(6) - f(5.5)] / (6 - 5.5) = 0.0755772$

$f[x_2, x_1, x_0] = \{f[x_2, x_1] - f[x_1, x_0]\} / (6 - 4.5) = -0.0077153$

$f(5) \approx 0.696788 + (5 - 4.5)(5 - 5.5)(-0.0077153) = 0.698717 \qquad \varepsilon_t = 0.04\ \%$

- Note that 0.696788 was calculate in part (b).
- Errors decrease when the points used are closer to the interpolated point.
- Errors decrease as the degree of the interpolating polynomial increases.

Finite Divided Difference (FDD) Table

Finite divided differences used in the Newton's Interpolating Polynomials can be presented in a table form. This makes the calculations much simpler.

x	f( )	f [,]	f [, ,]	f [, , ,]
x_0	$f(x_0)$	$f[x_1, x_0]$	$f[x_2, x_1, x_0]$	$f[x_3, x_2, x_1, x_0]$
x_1	$f(x_1)$	$f[x_2, x_1]$	$f[x_3, x_2, x_1]$	
x_2	$f(x_2)$	$f[x_3, x_2]$		
x_3	$f(x_3)$			

Exercise 27: The first two columns of the following table is given. Calculate the missing finite divided differences.

x	f( )	f [,]	f [, ,]	f [, , ,]
4	0.6020600	?	?	?
4.5	0.6532125	?	?	
5.5	0.7403627	?		
6	0.7781513			

- The numbers decrease as we go right in the table. This means that the contribution of higher order terms are less than the lower order terms.
- This is expected. The opposite behavior is an indication of an inappropriate interpolation (see exam questions of Fall 2006).

Example 30:

x	f( )	f[,]	f[, ,]	f[, , ,]
4	0.6020600	0.1023050	-0.0101032	0.001194
4.5	0.6532125	0.0871502	-0.0077153	
5.5	0.7403627	0.0755772		
6	0.7781513			

Use this previously calculated table to interpolate for log(5).

(a) Using points x=4 and x=4.5.

$\log(5) \approx 0.60206 + (5 - 4)\, 0.102305 = 0.704365$ $\quad \varepsilon_t = 0.8\ \%$ (this is extrapolation)

(b) Using points x=4.5 and x=5.5.

$\log(5) \approx 0.6532125 + (5 - 4.5)\, 0.0871502 = 0.696788$ $\quad \varepsilon_t = 0.3\ \%$

(c) Using points x=4 and x=6.

The entries of the above table can not be used for this interpolation.

(d) Using points x=4.5 , x=5.5 and x=6.

$\log(5) \approx 0.6532125 + (5-4.5)\, 0.0871502 + (5-4.5)(5-5.5)(-0.0077153) = 0.698717$ $\quad \varepsilon_t = 0.04\ \%$

(e) Using all four points.

$$\log(5) \approx 0.60206 + (5 - 4)\, 0.102305 + (5 - 4)(5 - 4.5)(-0.0101032)$$
$$+ (5 - 4)(5 - 4.5)(5 - 5.5)(0.001194) = 0.6990149 \qquad \varepsilon_t = 0.006\ \%$$

Exercise 28:

x	f( )
-2	-0.909297
-1	-0.841471
0	0.000000
1	0.841471
3	0.141120
4	-0.756802
6	-0.279415

Create the FDD table for the given data set. Use it to interpolate for f(2).

- For a linear interpolation use the points x=1 and x=3.
- For a quadratic interpolation either use the points x=0, x=1 and x=3 or the points x=1, x=3 and x=4.
- For a third cubic interpolation use the points x=0, x=1, x=3 and x=4.

Important: Always try to put the interpolated point at the center of the points used for the interpolation.

Exercise 29: Complete the following table given for the log function. Do you observe anything strange? Comment.

x	f( )	f [,]	f [, ,]	f [, , ,]	f [, , , ,]	f [, , , , ,]
0.5						
1						
3						
5						
8						
10						

Errors of Newton's DD Interpolating Polynomials

$$f_n(x) = f(x_0) + (x - x_0) f[x_1, x_0] + (x - x_0)(x - x_1) f[x_2, x_1, x_0] + \dots$$
$$+ (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f[x_n, x_{n-1}, \dots, x_1, x_0]$$

- The structure of Newton's Interpolating Polynomials is similar to the Taylor series.
- Remainder (truncation error) for the Taylor series was $R_n = \frac{f^{n+1}(\xi)}{(n+1)!}(x_{i+1} - x_i)^{n+1}$
- Similarly the remainder for the n^{th} order interpolating polynomial is

$$R_n = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\dots(x - x_n)$$

where ξ is somewhere in the interval containing the interpolated point x and other data points.

- But usually only the set of data points is given and the function f is not known.
- An alternative formulation uses a finite divided difference to approximate the $(n+1)^{th}$ derivative.

$$R_n \approx f[x, x_n, x_{n-1}, \dots, x_0](x - x_0)(x - x_1)\dots(x - x_n)$$

- But this includes f(x) which is not known.
- Error can be predicted if an additional data point (x_{n+1}) is availbale

$$R_n \approx f[x_{n+1}, x_n, x_{n-1}, \dots, x_0](x - x_0)(x - x_1)\dots(x - x_n)$$

which is nothing but $f_{n+1}(x) - f_n(x)$

Newton's Interpolating Polynomials for Equally Spaced Data

• If the data points are equally spaced and in ascending order, that is,

$$(x_0, y_0)\ ,\quad (x_0 + h, y_1)\ ,\quad (x_0 + 2h, y_1)\ ,\ \ldots\ldots\ ,\quad (x_0 + nh, y_n)$$

finite divided difference simplify.

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f(x_0)}{h}$$

$$f[x_2, x_1, x_0] = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} = \frac{f(x_2) - 2f(x_1) + f(x_0)}{2h^2} = \frac{\Delta f^2(x_0)}{2h^2}$$

$$\text{or in general}\quad f[x_n, x_{n-1}, \ldots, x_0] = \frac{\Delta f^n(x_0)}{n!\, h^n}$$

where $\Delta f^n(x_0)$ is the n^{th} forward difference.

• With this notation Newton's DD Interpolating polynomials simplify to

$$f_n(x) = f(x_0) + \Delta f(x_0)\, \alpha + \Delta^2 f(x_0)\, \alpha(\alpha - 1) / 2! + \ldots + \Delta^n f(x_0)\, \alpha(\alpha - 1) \cdots (\alpha - n + 1) / n! + R_n$$

where $\alpha = (x - x_0) / h$ and $R_n = f^{(n+1)}(\xi)\, h^{n+1}\, \alpha(\alpha - 1) \cdots (\alpha - n) / (n+1)!$

• This is called the forward Newton-Gregory formula.

Lagrange Interpolating Polynomials

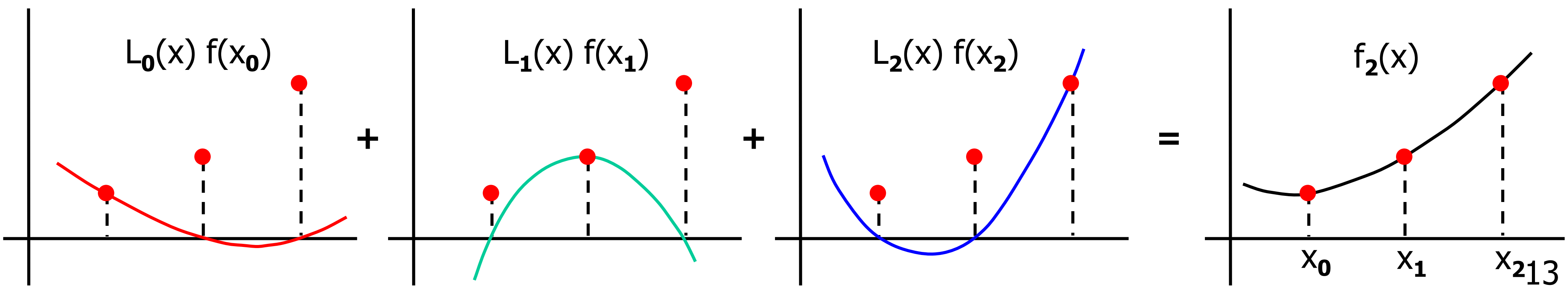
- It is a reformulation of Newton's Interpolating Polynomials.

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i) \quad \text{where} \quad L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

- For n=1 (linear): $f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$

- For n=2: $f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$

- To generalize, n^{th} order polynomial is the summation of (n+1) n^{th} order polynomials.
- Each of these n^{th} order polynomials have a value of 1 at one of the data points and have values of 0 at all other data points.
- This is due to the following property of Lagrange functions $L_i(x) = \begin{cases} 1 & \text{at} \quad x = x_i \\ 0 & \text{at} \quad \text{all other data points} \end{cases}$



Example 31:

x	f(x)
1	4.75
2	4.00
3	5.25
5	19.75
6	36.00

Calculate f(4) using Lagrange Interpolating Polynomials

(a) of order 1

(b) of order 2

(c) of order 3

(a) Linear interpolation. Select $x_0 = 3$, $x_1 = 5$

$f_1(x) = L_0(x)\, f(x_0) + L_1(x)\, f(x_1) = (x-5)/(3-5)\ 5.25 + (x-3)/(5-3)\ 19.75$

$f(4) \approx 12.5$

(b) Quadratic interpolation. Select $x_0 = 2$, $x_1 = 3$, $x_1 = 5$

$f_2(x) = L_0(x)\, f(x_0) + L_1(x)\, f(x_1) + L_2(x)\, f(x_2)$

$= (x-3)(x-5)/(2-3)(2-5)\ 4.00 + (x-2)(x-5)/(3-2)(3-5)\ 5.25 + (x-2)(x-3)/(5-2)(5-3)\ 19.75$

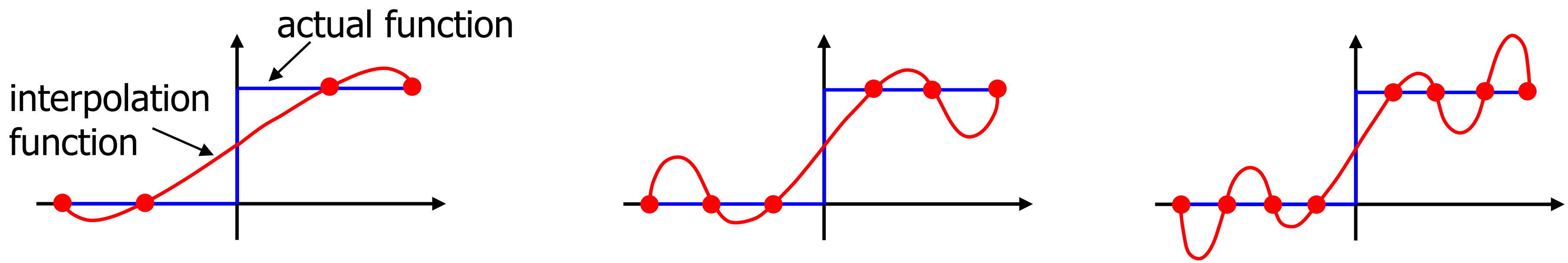
$f(4) \approx 10.5$

Exercise 30: Solve part (b) using the last three points. Also solve part (c).

Spline Interpolation

• We learned how to interpolate between n+1 data points using n^{th} order polynomials.

• For high number of data points (typically n > 6 or 7), high order polynomials are necessary, but sometimes they suffer from oscillatory behavior.

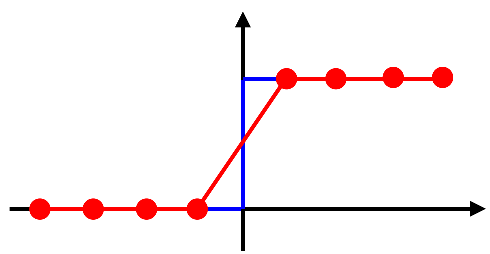


• Instead of using a single high order polynomial that passes through all data points, we can use different lower order polynomials between each data pair.

• These lower order polynomials that pass through only two points are called splines.

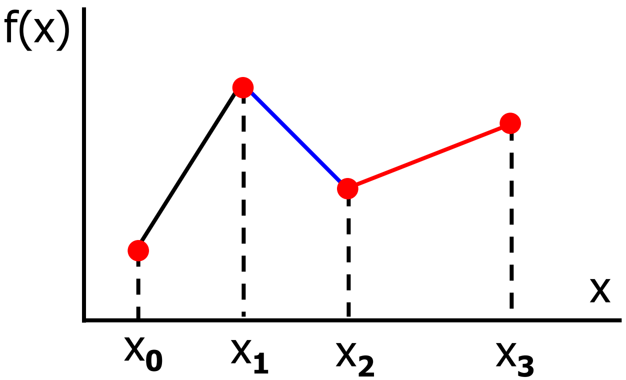
• Third order (cubic) splines are the most preferred ones.

first order splines :



Linear Splines:

- Given a set of ordered data points, each two point can be connected using a straight line.



$f(x) = f(x_0) + m_0(x - x_0)$ for $x_0 \le x \le x_1$

$f(x) = f(x_1) + m_1(x - x_1)$ for $x_1 \le x \le x_2$

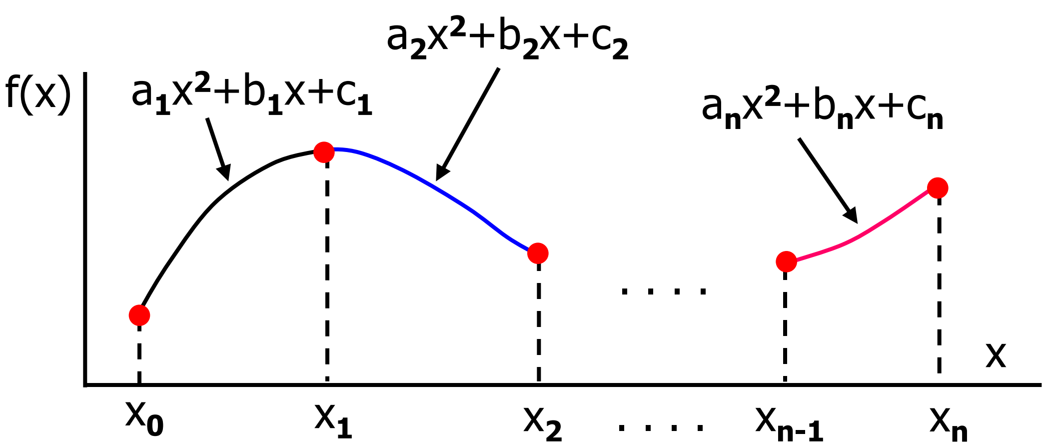
$f(x) = f(x_2) + m_2(x - x_2)$ for $x_2 \le x \le x_3$

where the slopes are $m_i = [f(x_{i+1}) - f(x_i)] / (x_{i+1} - x_i)$

- Functions are not continuous at the interior points.

Quadratic Splines:

- Every pair of data points are connected using quadratic functions.



- For n+1 data points, there are n splines and 3n unknown constants.
- We need 3n equations to solve for them.

Quadratic Splines (cont'd):

- These 3n equations are
  - The first and last functions must pass through the end points (2 equations).

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$
$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$

  - The function values must be equal at interior points (2n-2 equations).

$$a_{i-1} x_{i-1}^2 + b_{i-1} x_{i-1} + c_{i-1} = f(x_{i-1})$$
$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

for i = 2 to n

  - First derivatives must be equal at the interior points (n-1 equations).

$$2 a_{i-1} x_{i-1} + b_{i-1} = 2 a_i x_{i-1} + b_i$$

for i = 1 to n

  - This makes a total of 3n-1 equations. One more equation is necessary and we need to make an arbitrary choice. Among many possibilities we will use the following,

  - Take the second derivative at the first point to be zero (1 equation).

$$a_1 = 0$$

i.e. first two points are connected with a straight line.

- Solve this set of 3n linear algebraic equations with any of the methods that we learned.

Cubic Splines:

- For n+1 points, there will be n intervals and for each interval there will be a 3rd order polynomial

$$a_i x_i^3 + b_i x_i^2 + c_i x + d_i \qquad \text{for } i = 1 \text{ to } n$$

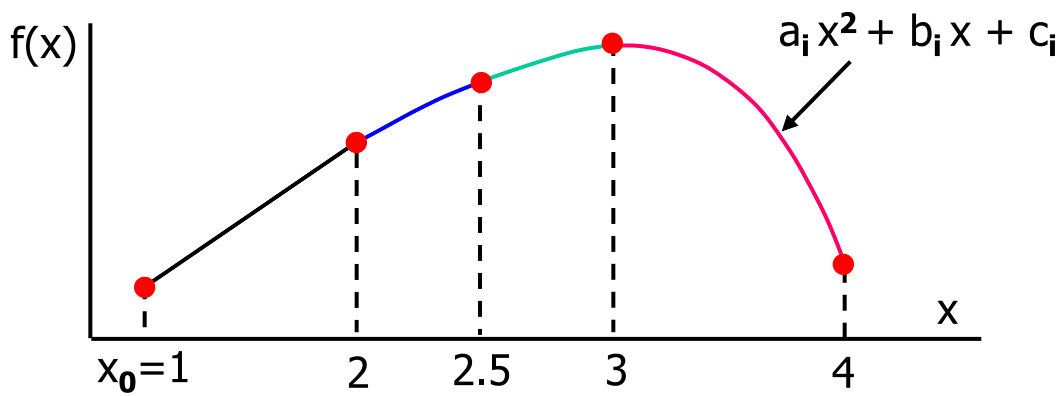
- Totally there are 4n unknowns. They can be solved using the following equations
  - The first and last functions must pass through the end points (2 equations).
  - The function values must be equal at interior points (2n-2 equations).
  - First derivatives must be equal at the interior points (n-1 equations).
  - Second derivatives must be equal at the interior points (n-1 equations).
  - This makes a total of 4n-2 equations. Two extra equations are (other choices are possible)
  - Second derivatives at the end points are zero (2 equations).

- Setting up and solving 4n equations is costly. There is another way of constructing cubic splines that results in only n-1 equations in n-1 unknowns. See pages 502-503 of the book.

Example 32:

x	f(x)
1	1
2	5
2.5	7
3	8
4	2

Develop quadratic splines for these data points and predict f(3.4) and f(2.2)



- There are 5 points and n=4 splines. Totally there are 3n=12 unknowns. Equations are
- End points: $a_1 1^2 + b_1 1 + c_1 = 1$, $a_4 4^2 + b_4 4 + c_4 = 2$
- Interior points: $a_1 2^2 + b_1 2 + c_1 = 5$, $a_2 2^2 + b_2 2 + c_2 = 5$

  $a_2 2.5^2 + b_2 2.5 + c_2 = 7$, $a_3 2.5^2 + b_3 2.5 + c_3 = 7$

  $a_3 3^2 + b_3 3 + c_3 = 8$, $a_4 3^2 + b_4 3 + c_4 = 8$
- Derivatives at the interior points: $2a_1 2 + b_1 = 2a_2 2 + b_2$

  $2a_2 2.5 + b_2 = 2a_3 2.5 + b_3$

  $2a_3 3 + b_3 = 2a_4 3 + b_4$
- Arbitrary choice for the missing equation: $a_1 = 0$

Example 32 (cont'd):

- a_1=0 is already known. Solve for the remaining 11 unknowns.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 4 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6.25 & 2.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.25 & 2.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 3 & 1 \\ 1 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & -5 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 & 0 & -6 & -1 & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 5 \\ 5 \\ 7 \\ 7 \\ 8 \\ 8 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -3 \\ 0 \\ 4 \\ -3 \\ -4 \\ 24 \\ -28 \\ -6 \\ 36 \\ 46 \end{Bmatrix}$$

- Equations for the splines are

  1st spline: $f(x) = 4x - 3$ (Straight line.)

  2nd spline: $f(x) = 4x - 3$ (Same as the 1st. Coincidence)

  3rd spline: $f(x) = -4x^2 + 24x - 28$

  4th spline: $f(x) = -6x^2 + 36x - 46$

- To predict f(3.4) use the 4th spline. $f(3.4) = -6\,(3.4)^2 + 36\,(3.4) - 46 = 7.04$

  To predict f(2.2) use the 2nd spline. $f(2.2) = 4\,(2.2) - 3 = 5.8$