B.C Lemma 2: Let $A_1, A_2 \dots$ be independent events with $P(A_k) = p_k$; such that $\sum_{k=1}^{\infty} p_k$ diverges. Then, the A_i's happen infinitely often w/ Pr. 1.

Proof: $C \triangleq \bigcap_{k \geq 1} \bigcup_{n=k}^{\infty} A_n$ ("A_i's happen i.o.)

$$C^c = \bigcup_{k \geq 1} \underbrace{\bigcap_{n=k}^{\infty} A_n^c}_{\text{Let this be } B_k}$$

$P(B_k) = \prod_{n=k}^{\infty} P(A_n^c)$ (by independence)

$$= \prod_{n=k}^{\infty} (1 - p_n) = (1-p_k)(1-p_{k+1}) \dots (1-p_M) \dots$$

$$\leq \exp\left(-\sum_{n=k}^{M} p_n\right) \xrightarrow[M \to \infty]{} 0$$

Also, $B_1 \subseteq B_2 \subseteq \dots$, so

$P(C^c) = P\left(\bigcup_{k \geq 1} B_i\right) \leq \lim_{n \to \infty} P(B_n) = 0$ hence $P(C) = 1$.