CHARACTERISTIC CLASSES AND ALGEBRAIC HOMOLOGY OF REAL ALGEBRAIC VARIETIES

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1. Introduction

For real algebraic sets $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$ a map $F : X \to Y$ is said to be entire rational if there exist $f_i, g_i \in \mathbb{R}[x_1, \ldots, x_r]$, $i = 1, \ldots, s$, such that each g_i vanishes nowhere on X and $F = (f_1/g_1, \ldots, f_s/g_s)$. We say X and Y are isomorphic to each other if there are entire rational maps $F : X \to Y$ and $G : Y \to X$ such that $F \circ G = id_Y$ and $G \circ F = id_X$. A complexification $X_{\mathbb{C}} \subseteq \mathbb{CP}^N$ of X will mean that X is a nonsingular algebraic subset of some $\mathbb{RP}^N$ and $X_{\mathbb{C}} \subseteq \mathbb{CP}^N$ is the complexification of the pair $X \subseteq \mathbb{RP}^N$. We also require the complexification to be nonsingular (blow up $X_{\mathbb{C}}$ along smooth centers away from X defined over reals if necessary). We refer the reader for the basic definitions and facts about real algebraic geometry to [1, 5]. Let $KH_*(X, R)$ be the kernel of the induced map

$$i_* : H_*(X, R) \to H_*(X_{\mathbb{C}}, R)$$

on homology, where $i : X \to X_{\mathbb{C}}$ is the inclusion map and R is either $\mathbb{Z}$, $\mathbb{Z}_2$ or $\mathbb{Q}$. In [6] it is shown that $KH_*(X, R)$ is independent of the complexification $X \subseteq X_{\mathbb{C}}$. Dually, denote the image of the homomorphism

$$i^* : H^*(X_{\mathbb{C}}, R) \to H^*(X, R)$$

by $ImH^*(X, R)$. In [6] and [7] $KH_*(X, R)$ is studied and computed on some examples. Moreover some applications on the nonexistence of entire rational maps of real algebraic varieties are given. In this note, we will present a relation between $ImH^*(X, R)$ and characteristic classes of strongly algebraic vector bundles over X (see Section 2 for the definition of strongly algebraic vector bundles) and some applications.

All compact manifolds and nonsingular real or complex algebraic sets are R oriented so that Poincaré duality and intersection of homology classes make sense.

2. Results

For a compact nonsingular real algebraic set X, define $H_k^A(X, \mathbb{Z}_2) \subseteq H_k(X, \mathbb{Z}_2)$ to be the subgroup of classes represented by algebraic subsets of X and let $H^k_A(X,\mathbb{Z}_2)$ be the Poincaré dual of $H^A_{n-k}(X,\mathbb{Z}_2)$. These are well known and very useful in the study of real algebraic sets. Also we define $H^k_A(X,\mathbb{Z}_2)^2$ to be the subgroup

$$\{\alpha^2 \mid \alpha \in H^k_A(X,\mathbb{Z}_2)\} \subseteq H^{2k}_A(X,\mathbb{Z}_2)$$

(cup product preserves algebraic cycles [2]).

It is well known that Grassmann varieties together with their canonical bundles have canonical real algebraic structures. Pullbacks of these canonical bundles via entire rational maps, from X into the Grassmannians, are called strongly algebraic vector bundles over X. A continuous vector bundle $E \to X$ is said to have a strongly algebraic structure if it is continuously isomorphic to a strongly algebraic vector bundle, or equivalently, if the continuous map classifying E is homotopic to an entire rational map.

Akbulut and King showed that $H^k_A(X,\mathbb{Z}_2)^2$ and Pontrjagin classes of X are pullbacks of some classes of $X_{\mathbb{C}}$ ([3]). Indeed, the same works for any strongly algebraic vector bundle $E \to X$ over X, not just for the tangent bundle, because the complexification (as a vector bundle) of any strongly algebraic vector bundle over X extends over some complexification $X_{\mathbb{C}}$ of X. The reason is that the real Grassmann variety, $G_{\mathbb{R}}(n,k)$, of real k-planes in $\mathbb{R}^n$ has the complex Grassmann variety, $G_{\mathbb{C}}(n,k)$, of complex k-planes in $\mathbb{C}^n$ as its natural complexifications and therefore any entire rational map from X into $G_{\mathbb{R}}(n,k)$ gives rise to a regular map, maybe after some blowing-ups of the domain, from $X_{\mathbb{C}}$ into $G_{\mathbb{C}}(n,k)$. We can summarize this as follows:

Theorem 2.1. *Let X be a nonsingular compact connected real algebraic variety and set*

$$P = \{e^2(E),\ p_i(E) \mid E \to X \textit{ is a strongly algebraic vector bundle over } X\}$$

and

$$W^2 = \{w_i^2(E) \mid E \to X \textit{ is a strongly algebraic vector bundle over } X\}$$

which are subsets of $H^*(X,\mathbb{Q})$ and $H^*(X,\mathbb{Z}_2)$ respectively, where $e(E)$, $p_i(E)$ and $w_i(E)$ are the Euler, the Pontrjagin and the Stiefel-Whitney classes of E. Then $ImH^*(X,\mathbb{Q})$ and $ImH^*(X,\mathbb{Z}_2)$ contains the subalgebras generated by P and W^2 respectively.

Any closed smooth manifold M has an algebraic model X so that any vector bundle over X has a strongly algebraic structure. This follows from the facts that Grassmann varieties have totally algebraic homology and the K-groups of a compact manifold are finitely generated. Hence, for this X both P and W^2 are maximal. Some geometric consequences of this theorem are as follows:

Corollary 2.2. *Let X be as in above theorem and $H \subseteq X$ an algebraic hypersurface. If $\alpha \in H_2(X,\mathbb{Z}_2)$ with $[H]\cdot[H]\cdot\alpha \neq 0$ then $\alpha \notin KH_2(X,\mathbb{Z}_2)$.*

Corollary 2.3. *If M is a smooth closed manifold having an algebraic model X with $H_2(X,\mathbb{Z}_2) = KH_2(X,\mathbb{Z}_2)$, then $\alpha^2 = 0$ for any $\alpha \in H^1_A(X,\mathbb{Z}_2)$.*

Moreover, if X is such that $H^1(X,\mathbb{Z}_2) = H^1_A(X,\mathbb{Z}_2)$, then $\alpha^2 = 0$ for any $\alpha \in H^1(M,\mathbb{Z}_2)$.

Remark: The converse of the above corollary is not true. Indeed, if F is an oriented closed 2-manifold then clearly $\alpha^2 = 0$ for any $\alpha \in H^1(F,\mathbb{Z}_2)$. However, if F has even genus then for any algebraic model X of F we have $H_2(X,\mathbb{Z}_2) = \mathbb{Z}_2 \neq 0 = KH_2(X,\mathbb{Z}_2)$. The last equality follows from the Bockstein homology sequence

$$\cdots \to H_2(F_{\mathbb{C}},\mathbb{Z}) \xrightarrow{\times 2} H_2(F_{\mathbb{C}},\mathbb{Z}) \longrightarrow H_2(F_{\mathbb{C}},\mathbb{Z}_2) \xrightarrow{\partial} H_1(F_{\mathbb{C}},\mathbb{Z}) \to \cdots$$

and the fact that the Euler class of F is not divisible by 4.

The rational cohomology ring of the quaternionic projective n-space $\mathbb{Q}P^n$ is generated by the Pontrjagin classes of its tangent bundle which is clearly strongly algebraic and thus combining this with the above considerations we get:

Theorem 2.4. *For any real algebraic model X of*

- **i)** *the quaternionic projective n-space $\mathbb{Q}P^n$ we have $KH_k(X,\mathbb{Z}) = 0$, for all k;*
- **ii)** *the complex projective n-space $\mathbb{C}P^n$ we have $KH_{2n}(X,\mathbb{Z}) = 0$ and $KH_{4k}(X,\mathbb{Z}) = 0$, for all k;*
- **iii)** *the real projective $2n$-space $\mathbb{R}P^{2n}$ we have $KH_{2k}(X,\mathbb{Z}_2) = 0$, for all k.*

Parts (ii) and (iii) of the above theorem are proved in [6].

References

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