An Introduction to Symplectic Geometry and Topology

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1) Linear Algebra of Symplectic Structures

Let V be a finite dim'l vector space over $\mathbb{R}$ and Ω an alternating 2-form on V. Then Ω is called symplectic if Ω is nondegenerate: For any $u \in V \setminus \{0\}$ there is some v so that $\Omega(u,v) \neq 0$.

Lemma: If Ω is a symplectic form on V then $\dim V$ is even and V has a basis $\mathcal{B} = \{e_1, f_1, \ldots, e_n, f_n\}$ so that $\Omega = e_1^* \wedge f_1^* + \cdots + e_n^* \wedge f_n^*$.

Remarks. 1) Note that

$$\Omega^n = n!\, e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^*$$

is a volume form on V and thus (V, Ω) has a preferred orientation.

2) Ω determines a canonical isomorphism from V to V^*:

$$\varphi_\Omega : V \longrightarrow V^*$$
$$u \longmapsto \Omega(u, \cdot) : V \longrightarrow \mathbb{R}$$
$$v \longmapsto \Omega(u, v).$$

(Similarly, a symmetric non-degenerate 2-form, i.e., an inner product $\langle \cdot , \cdot \rangle$ determines a canonical isomorphism V to V^*.)

2) Symplectic Manifolds:

Let M be a smooth manifold. A nondegenerate closed 2-form $\omega \in \Omega^2(M)$ is called a symplectic structure on M and the pair (M,ω) is called a symplectic manifold.

Examples 1) $\mathbb{R}^{2n} = \mathbb{C}^n$

$$(x_1, y_1, \dots, x_n, y_n) \longleftrightarrow (z_1, \dots, z_n)$$

$$z_j = x_j + i y_j$$

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$$

$$= \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$$

$dz_j = dx_j + i\, dy_j$, $d\bar{z}_j = dx_j - i\, dy_j$

$dz_j \wedge d\bar{z}_j = -2i\, dx_j \wedge dy_j$.

Clearly $d\omega = 0$ and
$\omega^n = n!\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$.

2) S^2, $\omega(u,v) = N(p)\cdot(u \times v)$
$p \in S^2$, $u,v \in T_pS^2$. Indeed, we have
$\omega = dx \wedge dy + dy \wedge dz + dz \wedge dx$.

<u>Remarks</u> 1) A symplectic manifold (M,ω) has a preferred orientation given by ω^n.

2) A symplectic manifold (M,ω) has a canonical isomorphism

$$\begin{aligned} \varphi_\omega : TM &\longrightarrow T^*M \\ X &\longmapsto \omega(X,\cdot) \end{aligned}$$

<u>Definition:</u> Let (M_1,ω_1) and (M_2,ω_2) be symplectic manifolds. A

diffeomorphism $f: M_1 \to M_2$ is called a symplectomorphism if $f^*\omega_2 = \omega_1$.

Remark: If (M, ω) is a closed symplectic manifold then the symplectic volume of M, $\mathrm{vol}_\omega(M) = \int_M \omega^n$ is invariant under symplectomorphism. If $\dim M = 2$ then $\mathrm{vol}_\omega(M)$ is the only symplectic symplectic invariant of (M, ω).

Also note that since $[\omega^n] \neq 0$ in $H^{2n}_{DR}(M)$ we have $[\omega^i] \neq 0$ for all $i = 1, \ldots, n$. They are all symplectic invariant of (M, ω).

On the other hand, a symplectic form has no local invariant:

<u>Theorem</u> (Darboux)
Let (M^{2n}, ω) be a symplectic manifold and $p \in M$ be any point. Then there is a coordinate chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ around p so that ω is given by $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$.

Such a chart is called a Darboux chart.

3) Tautological and Canonical Forms

Let M^n be a smooth manifold and $(U, x_1, \dots, x_n)$ be a coordinate chart on M. For any $\xi \in T^*U$ write ξ as $\xi = \sum_{i=1}^{n} \xi_i \, dx_i$.

Then $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ is a coordinate chart on T^*U.
Now $\alpha_u = \sum_{i=1}^{n} \xi_i \, dx_i$ is called the tautological 1-form on T^*U and $\omega_u = d\alpha_u = \sum_{i=1}^{n} d\xi_i \wedge dx_i$ is called the canonical 2-form on T^*U.

Remark: If $U, V \subseteq \mathbb{R}^n$ are open subsets and $F : U \to V$
$F(x_1, \dots, x_n) = (y_1, \dots, y_n)$ is a diffeo-

morphism then $F^*\alpha_V = \alpha_U$
and thus $F^*\omega_V = \omega_U$.

4) Submanifolds Let (M^{2n}, ω) be a symplectic manifold. A submanifold $i: N^{2k} \hookrightarrow M$ is called symplectic if $i^*\omega$ is a symplectic form on N.
Similarly, a submanifold $i: L^n \hookrightarrow M^{2n}$ is called Lagrangian if $i^*\omega \equiv 0$ on L.

Example: 1) If $i: N \hookrightarrow M$ a submanifold then
$T^*N \hookrightarrow T^*M$ is a symplectic submanifold.

2) $M \simeq M \times \{0\} \hookrightarrow T^*M$ is a Lagrangian submanifold.

3) $\mathbb{R}^{2n}$, $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$.

$L = \{(x_1, 0, x_2, 0, \ldots, x_n, 0) \mid x_i \in \mathbb{R}\}$ is a Lagrangian submanifold.

<u>An observation</u>: $(\mathbb{R}^{2n}, \omega)$ has no closed symplectic submanifold.

<u>Proof</u> If $M^{2k} \subseteq \mathbb{R}^{2n}$ is a closed symplectic submanifold, then

$$\mathrm{vol}_\omega(M) = \int_M \omega^k > 0.$$

However, $\omega = d\alpha$, where $\alpha = \sum x_i \, dy_i$ and then $\omega^k = (d\alpha)^k = d(\alpha\, \omega^{k-1})$ so that $0 < \int_M \omega^k = \int_M (d\alpha)^k = \int_M d(\alpha\omega^{k-1})$

$$= \int_{\partial M = \emptyset} \alpha\omega^{k-1}$$

$$= 0$$

a clear contradiction.

__Exercise:__ (M_i, ω_i) $i = 1, 2$ symplectic manifolds. Then

$\omega = pr_1^* \omega_1 - pr_2^* \omega_2$ is a symplectic form on $M_1 \times M_2$.

Prove that $f : (M_1, \omega_1) \to (M_2, \omega_2)$ is a symplectomorphism if and only if the graph of f,

$\Gamma_f = \{(x, f(x)) \in M_1 \times M_2 \mid x \in M_1\}$

is a Lagrangian submanifold of $(M_1 \times M_2, \omega)$.

5) Some Basic Theorems

Theorem (Moser) Let ω_0 and ω_1 be two symplectic forms on a closed manifold M^{2n}. Suppose that $[\omega_0] = [\omega_1]$ and $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic for all $t \in [0,1]$. Then there is an isotopy $\rho: M \times \mathbb{R} \to M$ s.t. $\rho_t^* \omega_t = \omega_0$ for all $t \in [0,1]$.

Proof uses so called Moser's Trick

First assume that such isotopy $\rho: M \times \mathbb{R} \to M$, $\rho_t^* \omega_t = \omega_0$ exists. Let $v_t \doteq \frac{d\rho_t}{dt} \circ \rho_t^{-1}$, $t \in \mathbb{R}$, the vector field whose flow is ρ_t.

Now, then

$$0=\frac{d}{dt}(\rho_t^* \omega_t)=\rho_t^*\left(\mathcal{L}_{v_t}\omega_t+\frac{d\omega_t}{dt}\right).$$

$$\Longleftrightarrow \mathcal{L}_{v_t}\omega_t+\frac{d\omega_t}{dt}=0 \quad (*)$$

Conversely, let v_t, $t\in\mathbb{R}$ be a vector field and that $(*)$ holds. Since M is compact we can integrate v_t to get its flow $\rho: M\times\mathbb{R}\longrightarrow M$ s.t. $\frac{d}{dt}(\rho_t^*\omega_t)=0$ which implies that $\rho_t^*\omega_t=\rho_0^*\omega_0=\omega_0$.

So we just need to show that such v_t exists.

From $\omega_t=(1-t)\omega_0+t\omega_1$ we get $\frac{d\omega_t}{dt}=\omega_1-\omega_0$ and thus

$$\left[\frac{d\omega_t}{dt}\right] = [\omega_1] - [\omega_0] = 0$$

So $\frac{d\omega_t}{dt} = d\mu$.

Then $L_{v_t}\omega_t = d i_{v_t}\omega_t + i_{v_t}\overbrace{d\omega_t}^{0}$ and

using (*) we get

$$d i_{v_t}\omega_t = L_{v_t}\omega_t = -\frac{d\omega_t}{dt} = -d\mu.$$

$$\Rightarrow d i_{v_t}\omega_t = -d\mu.$$

So we'll be done if we can solve $i_{v_t}\omega_t = -d\mu$, which is possible since ω_t is nondegenerate for all t. $\blacksquare$

__Theorem__ (Weinstein Tubular Neigh. Thm)
Let (M,ω) be a symplectic manifold, X a Lagrangian submanifold, ω_0 canonical symplectic form on T^*X, $i_0 : X \hookrightarrow T^*X$ the Lagrangian embedding as the zero section, and $i : X \hookrightarrow M$ Lagrangian embedding given by the inclusion. Then there are neighborhoods U_0 of X, U of X in M and a diffeomorphism $\varphi : U_0 \to U$ s.t.

$$\begin{tikzcd} U_0 \arrow[rr, "\varphi"] & & U \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes and $\varphi^*\omega = \omega_0$.

Some applications

1) Symplectic sum

$(M_1^{2n}, \omega_1), (M_2^{2n}, \omega_2)$ symp. mfd.

$N \hookrightarrow M_1$ and $N \hookrightarrow M_2$ symplectic submanifold of dimension $2n-2$. Assume that $e_{M_1}(N_1) = -e_{M_2}(N_2)$.

Then there is an orientation reversing isomorphism φ from $\nu(N_1)$ to $\nu(N_2)$ taking ω_2 to ω_1.

$\Longrightarrow M_1 \setminus \nu(N_1) \underset{\varphi}{\cup} M_2 \setminus \nu(N_2)$

is a symplectic manifold.

2) Lagrangian Surgery (Lattimer Surgery).

(M_1^4, L_1), (M_2^4, L_2)

$L_i \hookrightarrow M_i^4$ Lagrangian subm.

$\Rightarrow \left(M_1^4 \setminus \nu(L_1)\right) \cup \left(M_2^4 \setminus \nu(L_2)\right)$

is a symplectic manifold.

6) <u>Compatible Triples</u>:

Given a symplectic vector space (V, Ω) a compatible complex structure is an endomorphism J of V such that

i) $J^2 = -Id_V$ and

ii) $G_J(u,v) \doteq \Omega(u, Jv)$ is an inner product on V.

<u>Example</u>: $V = \mathbb{R}^{2n}, \Omega = \sum_{i=1}^{n} e_i^* \wedge f_i^*$

$J: V \to V$, $J(e_i) = f_i$, $J(f_i) = -e_i$

$G_J(u,v) = \Omega(u, Jv)$

$G_J(e_i, e_i) = \Omega(e_i, Je_i) = 1$

$G_J(e_i, f_j) = \Omega(e_i, Jf_j) = 0$

$G_J = \sum_{i=1}^{n} (e_i^* \otimes e_i^* + f_i^* \otimes f_i^*)$

This example indeed shows that any symplectic structure has a compatible complex structure. Indeed this can be done on a symplectic manifold.

Given (M, ω) choose any Riemannian metric G on M.

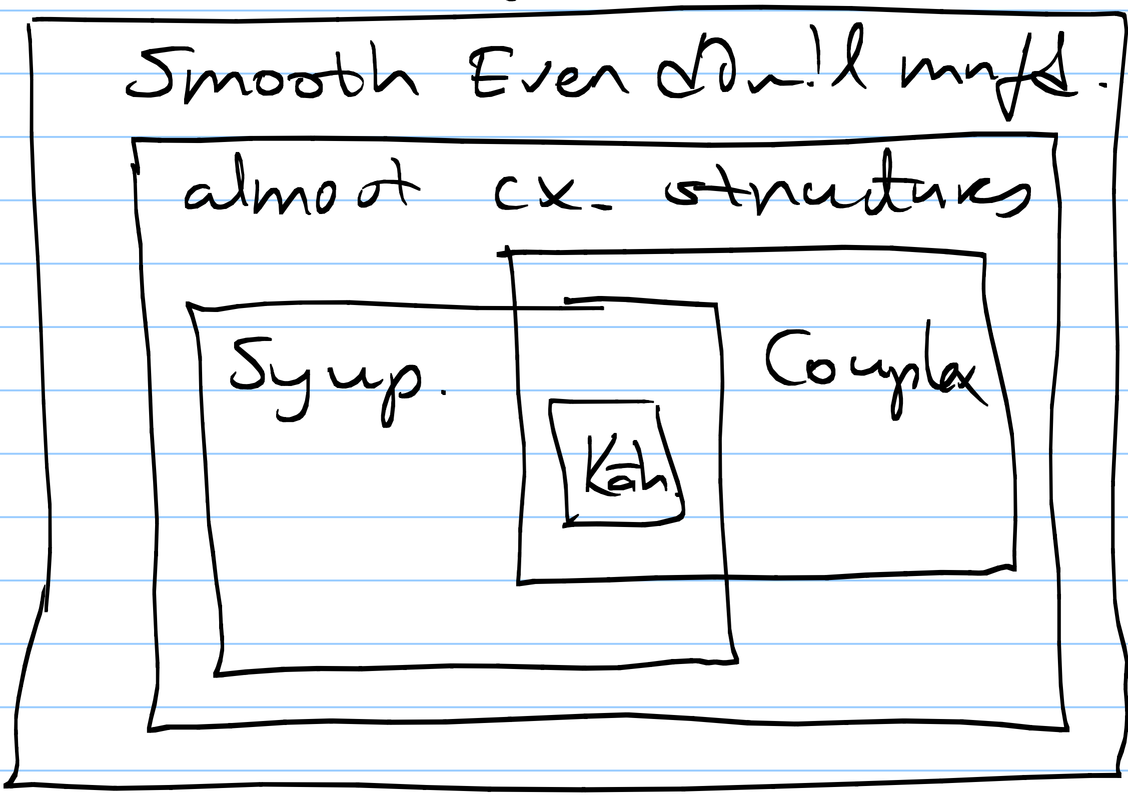
Then the equation

$\omega(u, v) = G(Au, v)$ determines an isomorphism $A: T_x M \to T_x M$.

Then $J = (\sqrt{AA^*})^{-1} A$ is a compatible almost complex structure on (M, ω).

Moreover, this can be done parametrically and this implies that the set of all such structures is path connected.

Definition: Let (M,ω) be a symplectic structure and J an integrable almost cx str. on M. Then (M,ω,J) is called a Kähler manifold.



$S^1\times S^3 = \mathbb{C}^2\setminus\{0\}/(z_1,z_2)\sim(2z_1,2z_2)$ is complex but not symplectic.

Example: Fubini–Studi form on $\mathbb{CP}^n$ and on its subvarieties.

$\mathbb{CP}^1$: $[z_0 : z_1]$ $\frac{z_1}{z_0} = x + iy$

$$\omega_{FS} = \frac{dx \wedge dy}{(x^2+y^2+1)^2} = \frac{1}{4} 4\pi = \pi$$

$\mathbb{CP}^n$ $\omega_{FS} = \frac{i}{2} \partial\bar\partial \log(|z|^2+1)$ on $\mathbb{C}^{n+1}$

descents to a symplectic form on $\mathbb{CP}^n$.

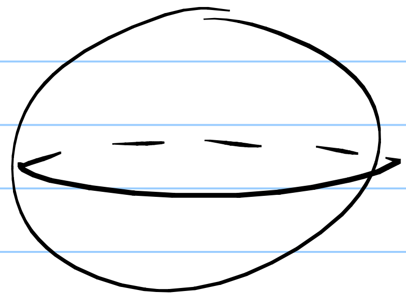
7) Symplectic Reduction:

A vector field X on (M, ω) is called symplectic if its flow preserves ω: $f_t^* \omega = \omega$ for all t. This is equivalent to saying that $\mathcal{L}_X \omega = 0$.

Since $\mathcal{L}_X \omega = i_X d\omega + d i_X \omega$ and $d\omega = 0$ the condition $\mathcal{L}_X \omega = 0$ is equivalent to $d i_X \omega = 0$.

If $i_X \omega$ is not only closed but exact, $i_X \omega = d\mu$ for some smooth function $\mu : M \to \mathbb{R}$ then we say that the action is Hamiltonian.

Ex $\omega = dx \wedge dy + dy \wedge dz + dz \wedge dx$



$\omega = d\theta \wedge dz$

$X = \frac{\partial}{\partial \theta}$

$i_X \omega = dz$, $z: S^2 \to \mathbb{R}$

So X is Hamiltonian.

Ex $\omega = d\theta_1 \wedge d\theta_2$ on $T^2 = S^1 \times S^1$

θ_1, θ_2

$X = \frac{\partial}{\partial \theta_1}$ is symplectic but not Hamiltonian.

Symplectic Reduction: Suppose that S^1 acts on (M, ω) in a Hamiltonian fashion, where X is the corresponding Hamiltonian

vector field and
$\iota_X \omega = d\mu$ for some smooth
$\mu: M \to \mathbb{R}$. Assume that
$0 \in \mathbb{R}$ is a regular value for
μ. Then $\mu^{-1}(0)/S^1$ is a
symplectic manifold called a
symplectic reduction of M by
the Hamiltonian action via S^1.

<u>Example:</u> S^1 acts on $\mathbb{C}^{n+1}$ via

$$\theta \cdot (z_0, z_1, \ldots, z_n) = (e^{i\theta} z_0, \ldots, e^{i\theta} z_n)$$

$$\mu: \mathbb{C}^{n+1} \longrightarrow \mathbb{R}, \quad z \longmapsto -\frac{|z|^2}{2} + \frac{1}{2}$$

$$\mu^{-1}(0)/S^1 = S^{2n+1}/S^1 = \mathbb{CP}^n.$$

Toric Geometry Hamiltonian

T^2 action of $\mathbb{CP}^2$.

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$$

$$\mu[z_0 : z_1 : z_2] = -\frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + \dots + |z_1|^2}, \frac{|z_1|^2}{\dots}\right)$$

$\mu(\mathbb{CP}^2) =$