Video 25-26

Note Title 11.04.2020

Theorem:

$$H^k_{DR}(\mathbb{CP}^n) = \begin{cases} \mathbb{R}, & k=0,2,\dots,2n. \\ 0, & \text{otherwise.} \end{cases}$$

Proof: Induction on n.

$n=1$, $\mathbb{CP}^1 = S^2$ done already. $H^k_{DR}(S^2) = \begin{cases} \mathbb{R}, & k=0,2 \\ 0, & \text{otherwise} \end{cases}$

Assume $n>1$. Let $p=[0: \;:0:1] \in \mathbb{CP}^n$.

$U = \{[z_0:\dots:z_n] \in \mathbb{CP}^n \mid z_n \neq 0\}$, $V = \mathbb{CP}^n \setminus \{p\}$.

Since $p \in U$ and hence $U \cup V = \mathbb{CP}^n$.

$$U \longleftrightarrow \mathbb{C}^n = \mathbb{R}^{2n}, \; [z_0:\dots:z_n] \longleftrightarrow \left(\frac{z_0}{z_n}, \dots, \frac{z_{n-1}}{z_n}\right).$$

$$p = [0:\dots:0:1] \longleftrightarrow (0,\dots,0) \in \mathbb{C}^n = \mathbb{R}^{2n}.$$

$$U \cap V \longleftrightarrow \mathbb{C}^n = \mathbb{R}^{2n} \setminus \{(0,\dots,0)\}.$$
$$\cong S^{2n-1} \times \mathbb{R}$$

$\mathbb{R}^{2n} \setminus \{(0,\dots,0)\} \longleftrightarrow S^{2n-1} \times \mathbb{R}$ is a diffeomorphism

$$x \longmapsto \left(\frac{x}{\|x\|}, \ln\|x\|\right)$$

To compute the cohomology of V consider the homotopy of maps

$$P_t : V \times [0,1] \longrightarrow V, \; P_t([z_0: \;:z_n]) = [z_0:\dots:z_{n-1}:tz_n]$$

$$P_1 = \mathrm{id}_V, \; P_0([z_0: \;:z_n]) = [z_0: \;:z_{n-1}:0]$$

$$P_0(V) = H = \{[z_0: \;:z_n] \in \mathbb{CP}^n \mid z_n = 0\} \simeq \mathbb{CP}^{n-1}.$$

On the other hand, the inclusion map

$i : H \longrightarrow V$, $i([z_0 : \; : z_{n-1} : 0]) = [z_0 : \; : z_{n-1} : 0]$.

Now, $P_0 \circ i = Id_H$. Moreover, the homotopy $P_t : V \to V$ going $P_1 = Id_V$ to $P_0 = i \circ P_0$.

Hence, V and H are homotopy equivalent. (P_t is a strong deformation retraction)

$$\begin{tikzcd} V \arrow[r, "P_0"] & H \arrow[l, bend left, "i"] \end{tikzcd}$$

$P_0 \circ i = Id_H$

$i \circ P_0 = P_0$ is homotopic to $P_1 = Id_V$ via the homotopy P_t.

So $i^* : H^k_{DR}(V) \longrightarrow H^k_{DR}(H) = H^k_{DR}(\mathbb{CP}^{n-1})$ is an isomorphism.

Assume that $H^k_{DR}(\mathbb{CP}^{n-1}) = \begin{cases} \mathbb{R}, & k = 0, 2, \dots, 2n-2 \\ 0 & \text{otherwise.} \end{cases}$

$\mathbb{CP}^n = U \cup V$. Use Mayer–Vietoris sequence!

$$\to H^{k-1}_{DR}(\underset{\substack{\cong \\ S^{2n-1}\times\mathbb{R}}}{U \cap V}) \xrightarrow{\delta} H^k_{DR}(\underset{\substack{\| \\ \mathbb{CP}^n}}{U \cup V}) \longrightarrow H^k_{DR}(\underset{\substack{\| \\ \mathbb{C}^n}}{U}) \oplus H^k_{DR}(\underset{\substack{\| \\ \mathbb{CP}^{n-1}}}{V}) \longrightarrow H^k_{DR}(U \cap V) \to$$

$$\overset{(k>0)}{\Longrightarrow} \cdots \to H^{k-1}_{DR}(S^{2n-1}) \xrightarrow{\delta} H^k_{DR}(\mathbb{CP}^n) \to H^k_{DR}(\mathbb{CP}^{n-1}) \to H^k_{DR}(S^{2n-1})$$

If $0 < k < 2n-2$, then $H^{k-1}_{DR}(S^{2n-1}) = 0 = H^k_{DR}(S^{2n-1})$.

$$\Rightarrow 0 \longrightarrow H^k_{DR}(\mathbb{CP}^n) \longrightarrow H^k_{DR}(\mathbb{CP}^{n-1}) \longrightarrow 0$$

$\Rightarrow H^k_{DR}(\mathbb{CP}^n) \simeq H^k_{DR}(\mathbb{CP}^{n-1})$ if $0 < k \leq 2n-2$.

If $k = 2n-1$, then $(n>1)$

$$\cdots \to \underbrace{H^{2n-2}_{DR}(S^{2n-1})}_{\substack{\shortparallel \\ 0}} \xrightarrow{\delta} H^{2n-1}_{DR}(\mathbb{CP}^n) \to \underbrace{H^{2n-1}_{DR}(\mathbb{CP}^{n-1})}_{\substack{2n-2 \\ \shortparallel \\ 0}} \to \cdots$$

$\Rightarrow H^{2n-1}_{DR}(\mathbb{CP}^n) = 0$ ✓

$H^k_{DR}(\mathbb{CP}^n) = 0$ if $k > 2n$.

So, we just need to show that $H^{2n}_{DR}(\mathbb{CP}^n) \simeq \mathbb{R}$.

$k = 2n$, the sequence gives

$0 \to H^{2n-1}_{DR}(S^{2n-1}) \xrightarrow{\delta} H^{2n}_{DR}(\mathbb{CP}^n) \to 0$ because

$H^{2n-1}_{DR}(\mathbb{CP}^{n-1}) = 0$ and $H^{2n}_{DR}(\underset{\substack{\shortparallel \\ \mathbb{C}^n}}{U}) \oplus H^{2n}_{DR}(\underset{\substack{\wr\wr \\ \mathbb{CP}^{n-1}}}{V}) = 0$.

Hence, $H^{2n}_{DR}(\mathbb{CP}^n) \simeq H^{2n-1}_{DR}(S^{2n-1}) \simeq \mathbb{R}$.

Remark: 1) $\displaystyle\int_{\mathbb{CP}^n} \omega^n_{FS} > 0 \Rightarrow [\omega^n_{FS}] \neq 0$ in $H^{2n}_{DR}(\mathbb{CP}^n)$.

However, $[\omega_{FS}] \in H^2_{DR}(\mathbb{CP}^n)$ and

$[\omega_{FS}]^n = [\omega^n_{FS}] \neq 0$, which implies that

$[\omega_{FS}] \neq 0, \dots, [\omega^n_{FS}] \neq 0$.

$\therefore\ H^{2k}_{DR}(\mathbb{CP}^n) \simeq \mathbb{R} = \langle [\omega_{FS}]^k \rangle.$

2) If $i : M^n \longrightarrow \mathbb{CP}^N$ is an embedding of an complex manifold into $\mathbb{CP}^N$ as a complex submanifold, then $(\dim_{\mathbb{C}} M^n = n)$ and

$$\int_{M^n} i^* \omega^n_{FS} > 0 \Rightarrow [\omega^n_{FS}] \neq 0 \text{ in } H^{2n}_{DR}(M).$$

Thus $H^{2k}_{DR}(M) \neq 0$ for all $0 \leq k \leq n$ and $[\omega^k_{FS}] \neq 0$ in $H^{2k}_{DR}(M)$.

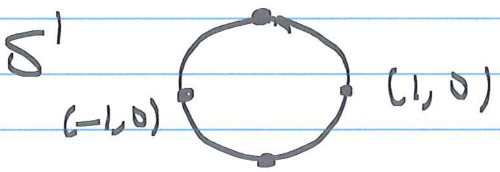
Example $M = S^1 \times S^3$

Claim: $H^k_{DR}(S^1 \times S^3) = \begin{cases} \mathbb{R}, & k = 0, 1, 3, 4 \\ 0, & \text{otherwise.} \end{cases}$

In particular, $H^2_{DR}(S^1 \times S^3) = 0$ and thus by the above remark $S^1 \times S^3$ cannot be embedded into any $\mathbb{CP}^N$ as a complex submanifold.

Proof: $U = S^3 \times (S^1 - \{(-1,0)\})$ and

$V = S^3 \times (S^1 \setminus \{(1,0)\})$.



$S^1 \setminus \{(0,\pm 1)\} \simeq \mathbb{R}$

$U \simeq V \simeq S^3 \times \mathbb{R}$

$$U \cap V = S^3 \times (S^1 \setminus \{(1,0),(-1,0)\})$$

$$\cong S^3 \times S^0 , \quad S^0 = \{(0,1),(0,-1)\}$$

So, $H^k_{DR}(U) \simeq H^k_{DR}(S^3 \times \mathbb{R}) \simeq H^k_{DR}(S^3) = H^k_{DR}(V)$

and $H^k_{DR}(U \cap V) \cong H^k_{DR}(S^3) \oplus H^k_{DR}(S^3)$

Using Meyer–Vietoris we obtain

$$\dashrightarrow \underset{\underset{0}{\parallel}}{H^1_{DR}(U\cap V)} \to H^2_{DR}(S^3 \times S^1) \to \underset{\underset{0}{\parallel}}{H^2_{DR}(U)} \oplus \underset{\underset{0}{\parallel}}{H^2_{DR}(V)} \to \cdots$$

$$\Rightarrow H^2_{DR}(S^3 \times S^1) = 0$$

Also,

$$\cdots \to \underset{\underset{0}{\parallel}}{H^2_{DR}(U\cap V)} \to H^3_{DR}(S^3 \times S^1) \to \underset{\underset{\mathbb{R}}{\parallel}}{H^3_{DR}(U)} \oplus \underset{\underset{\mathbb{R}}{\parallel}}{H^3_{DR}(V)} \to$$

$$\to \underset{\underset{\mathbb{R}\oplus\mathbb{R}}{\parallel}}{H^3_{DR}(U\cap V)} \to H^4(S^3 \times S^1)$$

$$0 \to H^3_{DR}(S^3 \times S^1) \to \mathbb{R}\oplus\mathbb{R} \to \mathbb{R}\oplus\mathbb{R} \to H^4(S^3\times S^1) \to 0$$

$$[\omega] \longmapsto ([\omega],[\omega])$$

$$([\eta],[\nu]) \longmapsto [\eta]-[\nu]$$

$\mathbb{R}\oplus\mathbb{R} \longrightarrow \mathbb{R}\oplus\mathbb{R}$ is the linear map given by $(a,b) \longmapsto (a-b, a-b)$.

Its kernel is one dimensional spanned by $(1,1)$ and image is one dimensional spanned by $(1,1)$.

Hence $H^3_{DR}(S^3 \times S^1) \simeq \mathbb{R}$ and

$H^4_{DR}(S^3 \times S^1) \simeq \mathbb{R}$.

So,
$$H^k_{DR}(S^3 \times S^1) = \begin{cases} \mathbb{R}, & k = 0, 1, 3, 4 \\ 0, & \text{otherwise.} \end{cases}$$

__Exercise:__ Prove that

$$H^k_{DR}(S^n \times S^m) = \begin{cases} \mathbb{R}, & k = 0, n, m, n+m \\ 0, & \text{otherwise} \end{cases}$$

if $n \neq m$.

$$H^k(S^n \times S^n) = \begin{cases} \mathbb{R}, & k = 0, 2n \\ \mathbb{R} \oplus \mathbb{R}, & k = n \\ 0, & \text{otherwise.} \end{cases}$$

Compactly Supported Cohomology:

$U \subseteq M$ open subset of a smooth manifold.

$\Omega^k_c(U) = \{\omega \in \Omega^k(U) \mid \overline{\mathrm{supp}(\omega)} \text{ is compact in } U\}$,

$\mathrm{supp}(\omega) = \{x \in U \mid \omega(x) \neq 0\}$.

$\Omega^k_c(U) \subseteq \Omega^k(U)$ $\mathbb{R}$-linear subspace

$d : \Omega^k_c(U) \longrightarrow \Omega^{k+1}_c(U)$, where $d^2 = 0$.

$$H^k_c(U) = \frac{\ker(d : \Omega^k_c(U) \to \Omega^{k+1}_c(U))}{\mathrm{Im}(d : \Omega^{k-1}_c(U) \to \Omega^k_c(U))}.$$

$\omega \in \Omega^k_c(N)$, $f : M \to N$, $f^*(\omega) \in \Omega^k_c(M)$?

Definition: A function $f : X \to Y$ between topological spaces is called proper if $f^{-1}(C)$ is compact whenever $C \subseteq Y$ is compact.

If $f : M \to N$ is a proper smooth map of manifolds then we get a homomorphism

$f^* : \Omega^k_c(N) \longrightarrow \Omega^k_c(M)$, which induces

a homomorphism

$f^* : H^k_c(N) \longrightarrow H^k_c(M)$.

Remark: If $f_t : M \times [0,1] \to N$ is a smooth proper homotopy from f_0 to f_1

then $f_0^* = f_1^*$ as maps $H^k_c(N) \to H^k_c(M)$.

<u>Example</u>: $M = \mathbb{R}$

$H^0_c(\mathbb{R})$ $\quad f : \mathbb{R} \to \mathbb{R}$ 0-form (smooth function)

$f \in \Omega^0_c(\mathbb{R})$, f is closed $\Rightarrow df = f'(x)\,dx = 0$

and hence f must be constant. So $f(x) = c$

for all $x \in \mathbb{R}$ (for some c). f is compactly supported implies that there is some $m \in \mathbb{R}$ with $f(x) = 0$ for all $|x| \geq m$.

$c = f(x) = 0$ if $x \geq m$. $\Rightarrow f(x) = 0 \;\; \forall x \in \mathbb{R}$.

$H^0_c(\mathbb{R}) = (0)$.

What about $H^1_c(\mathbb{R})$?

Consider the map $I : H^1_c(\mathbb{R}) \to \mathbb{R}$, given by

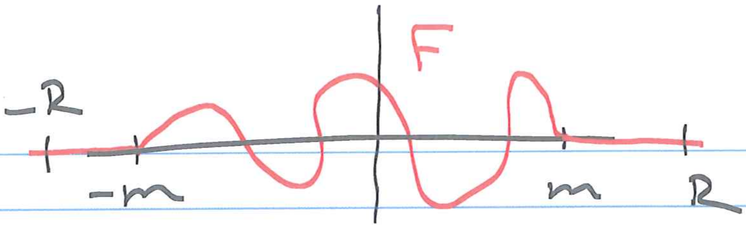
$$I([\omega]) = \int_{\mathbb{R}} \omega .$$

Since ω is compactly supported $\int_{\mathbb{R}} \omega$ is well defined.

<u>must show</u>: If $[\omega] = 0$ then $\int_{\mathbb{R}} \omega = 0$.

<u>proof</u> $[\omega] = 0 \Rightarrow \omega = dF$, for some

$F \in \Omega^0_c(\mathbb{R})$.



$$\int_{\mathbb{R}} \omega = \lim_{R\to\infty}\int_{-R}^{R}\omega = \lim_{R\to\infty}\int_{-R}^{R} F'(x)\,dx$$

$$= \lim_{R\to\infty}\left(\underset{\overset{\shortparallel}{0}}{F(R)} - \underset{\overset{\shortparallel}{0}}{F(-R)}\right) \quad (\text{if } R \geq m)$$

$$= 0.$$

Hence, $I : H^1_c(\mathbb{R}) \longrightarrow \mathbb{R}, [\omega] \longmapsto \int_{\mathbb{R}} \omega$, is well-defined.

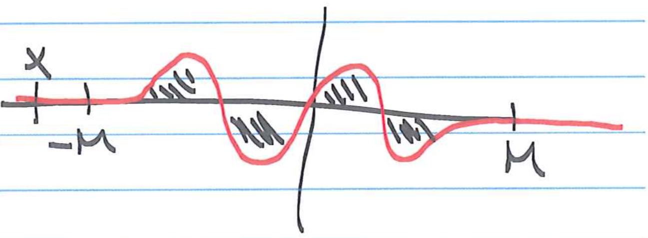
Claim: I is an isomorphism.

Proof: I is one to one: Let $[\omega] \in H^1_c(\mathbb{R})$ so that $I([\omega]) = \int_{\mathbb{R}} \omega = 0$.

must show: $[\omega] = 0$.

$\omega \in \Omega^1_c(\mathbb{R}) \Rightarrow \omega(x) = f(x)dx$, f is compactly supported and

$$0 = \int_{\mathbb{R}} \omega = \int_{-\infty}^{+\infty} f(x)\,dx.$$



Let $F(x) = \int_{-\infty}^{x} f(t)\,dt$. Then $dF = f(x)dx = \omega$.

$F(x)$ is compactly supported because

1) If $x \leq -m$, then $F(x) = \int_{-\infty}^{x} f(t)\,dt = 0$ because f is zero on $(-\infty, x]$.

2) If $x \geq m$, then

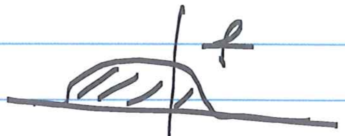
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} f(t)\,dt + \overbrace{\int_{x}^{+\infty} f(t)\,dt}^{=0}$$

$$= \int_{-\infty}^{+\infty} f(t)\,dt = 0.$$

$\Rightarrow \operatorname{supp}(F) \subseteq [-m, m]. \Rightarrow F \in \Omega^0_c(\mathbb{R})$.

$\omega = dF \Rightarrow [\omega] = 0$ in $H^1_c(\mathbb{R})$.
Hence, I is injective.

I is onto: let $\omega = f(x)\,dx$



Then $[\omega] \in H^1_c(\mathbb{R})$ and $I([\omega]) = \int_{\mathbb{R}} f > 0$.

$\Rightarrow I : H^1_c(\mathbb{R}) \to \mathbb{R}$ is onto.

This finishes the proof. ■

Theorem: (Poincaré Lemma for compactly supported cohomology)
If M is smooth manifold, then

$$H^{k+1}_c(M \times \mathbb{R}) \cong H^k_c(M).$$

Proof is similar to that of Poincaré lemm and left as an exercise. ∎

Corollary: $H^n_c(\mathbb{R}^n)\cong\mathbb{R}$ and $H^k_c(\mathbb{R})=0$ if $k\neq n$.

Proof: $H^n_c(\mathbb{R}^n) = H^n_c(\mathbb{R}^{n-1}\times\mathbb{R}) \cong H^{n-1}_c(\mathbb{R}^{n-1}) \cong \dots$

$$\cong H^1_c(\mathbb{R}) \cong \mathbb{R}.$$

If $k>n$, $H^k_c(\mathbb{R}^n)=0$, since $\Omega^k_c(\mathbb{R}^n)\subseteq\Omega^k(\mathbb{R}^n)$

If $k<n$, then

$$H^k_c(\mathbb{R}^n)\cong H^{k-1}_c(\mathbb{R}^{n-1})\cong\dots\cong H^0_c(\mathbb{R}^{n-k}),$$

where $H^0_c(\mathbb{R}^{n-k})=(0)$ and the proof is the same as that of $H^0_c(\mathbb{R})$. ∎

Remark: If M is already compact then

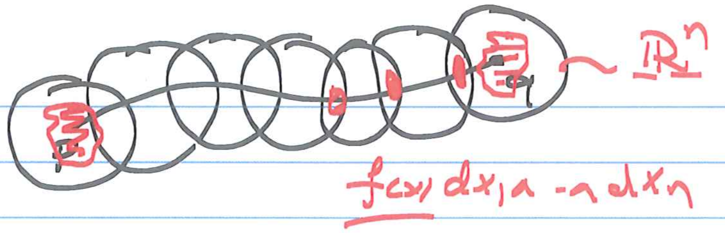
$$H^k_c(M) = H^k_{DR}(M).$$

Corollary For any oriented n-dimensional smooth manifold the homomorphism

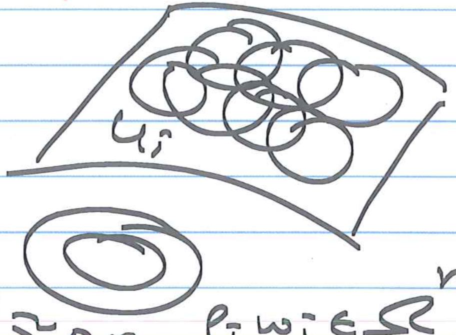
$$I: H^n_c(M)\longrightarrow\mathbb{R},\ [\omega]\longmapsto\int_M\omega,$$ is an

isomorphism (provided that M is connected).

Proof: $\omega\in\Omega^n_c(M)$



$\omega \in \Omega^k_c(M)$



$\omega = \sum \rho_i \omega_i$

$[\omega] = [\omega_0]$

$U_i \simeq_{\mathbb{R}^n}$ $\rho_i \omega_i \in \Omega^n_c(U_i)$

where ω_0 is supported in a coordinate chart $U \simeq \mathbb{R}^n$.

Let $\omega_1 \in \Omega^n_c(U)$ with $\int_M \omega_1 = 1$.

So $\omega_0, \omega_1 \in \Omega^n_c(U) \simeq \Omega^n_c(\mathbb{R}^n)$ and

$\int_{\mathbb{R}^n} (\omega_0 - \lambda \omega_1) = 0$, where $\lambda = \int_U \omega_0 = \int_M \omega$.

$\Rightarrow [\omega_0 - \lambda \omega_1] = 0$ in $\underline{H^n_c(\mathbb{R}^n) \simeq \mathbb{R}}$

$[\omega] = [\omega_0] = \lambda [\omega_1]$.

$\Rightarrow H^n_c(M) \simeq \mathbb{R}$ via I. ∎

<u>Exercise</u>: $I: H^n_c(\mathbb{R}^n) \longrightarrow \mathbb{R}$, $[\omega] \longmapsto \int_{\mathbb{R}} \omega$,

is an isomorphism.

Video 27-28

Note Title 15.04.2020

Compactly Supported Relative Cohomology

$U \subseteq M$ open subset. Then we have an exact sequence of $\mathbb{R}$-vector spaces

$$\begin{tikzcd} & {} \arrow[d, "d"] & {} \arrow[d, "d"] & {} \arrow[d, "d"] & \\ 0 \arrow[r] & \Omega_c^k(U) \arrow[r] \arrow[d, "d"] & \Omega_c^k(M) \arrow[r] \arrow[d, "d"] & \Omega_c^k(M)/\Omega_c^k(U) \arrow[r] \arrow[d, "d"] & 0 \\ & {} & {} & {} & \end{tikzcd}$$

for each $k=0,1,2,\ldots$ This induces a long exact sequence of vector spaces:

$$\cdots \xrightarrow{\delta} H_c^k(U) \to H_c^k(M) \to H_c^k(M,U) \xrightarrow{\delta} H_c^{k+1}(U) \to \cdots,$$ where

$$H_c^k(M,U) \doteq \frac{\ker\left(d : \Omega_c^k(M)/\Omega_c^k(U) \to \Omega_c^{k+1}(M)/\Omega_c^{k+1}(U)\right)}{\operatorname{Im}\left(d : \Omega_c^{k-1}(M)/\Omega_c^{k-1}(U) \to \Omega_c^k(M)/\Omega_c^k(U)\right)}.$$

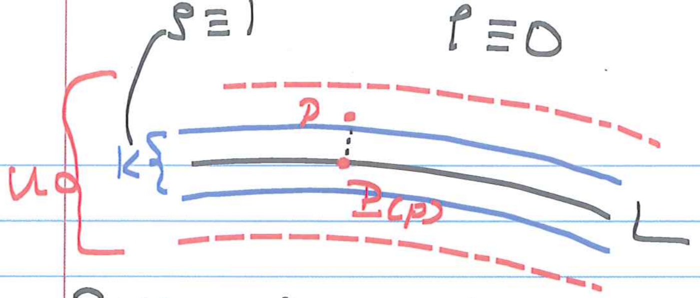
An Important Special Case: Let $L \subseteq M$ be a closed submanifold and set $U = M \setminus L$, an open subset. Then we get a long exact sequence of vector spaces:

$$\cdots \xrightarrow{\delta} H_c^k(M\setminus L) \to H_c^k(M) \to H_c^k(M, M\setminus L) \xrightarrow{\delta} H_c^{k+1}(M\setminus L) \to \cdots$$

Theorem: $H_c^k(M, M\setminus L) \xrightarrow{\cong} H_c^k(L)$ via the pull back homomorphism induced by the inclusion map

$$i : (L, \emptyset) \to (M, M\setminus L).$$

Proof: Let $L \subseteq U \subseteq M$ be a tubular neighborhood of L, and $\pi : U \to L$ be a projection map

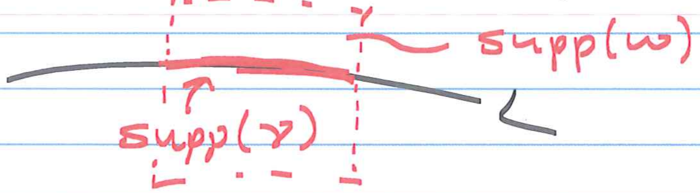


Let $\rho: M \to [0,1]$ be a smooth map with $\rho|_K \equiv 1$ on a closed neighborhood K of L with $K \subseteq U$ and $\rho\equiv 0$ $M\setminus U$.

$P: U \to L$ projection map

Also let $V = M\setminus K$. Then $U \subseteq V$.

Let $[\nu] \in H^k_c(L)$, then $\omega = \rho\cdot P^*(\nu) \in \Omega^k_c(M)$



Then $\omega|_L = \nu$ since $\rho\equiv 1$ on L and $P|_L = \mathrm{Id}_L$.

Writing $\omega = \rho\omega + (1-\rho)\omega$ and on $\mathring{K}$ we get
$$d\omega = d(\rho\omega) + d(\underbrace{(1-\rho)\omega}_{\text{"}0\text{ on }K}) = d(\underset{\text{"}1\text{ on }K}{\rho}\omega) = d(P^*\nu) = P^*(\overset{\text{"}0}{d\nu}) = 0.$$

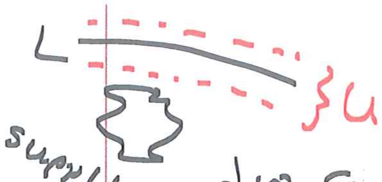
$\Rightarrow d\omega \in \Omega^{k+1}_c(V) \subseteq \Omega^{k+1}_c(M\setminus L)$.

So ω is closed in $\Omega^k_c(M)/\Omega^k_c(M\setminus L)$, and it defines a class $[\omega]$ in $H^k_c(M, M\setminus L)$.

Moreover, if $\iota: L \hookrightarrow M$ is the inclusion map
$$\iota^*\omega = \omega|_L = \nu \quad \text{so that } \iota^*[\omega] = [\nu].$$

Hence, $\iota^*: H^k_c(M, M\setminus L) \to H^k_c(L)$ is onto.

To show that it is 1-1, let $\iota^*([\omega]) = 0$ for some $[\omega] \in H^k_c(M, M\setminus L)$. By the definition of this cohomology group we must have



$d\omega \in \Omega_c^{k+1}(M \setminus L)$. Since L is closed and $d\omega$ is compactly supported by shrinking the tubular neighborhood U if necessary we may assume that $d\omega = 0$ on U.

However, $i^* : H^k_{DR}(U) \to H^k_{DL}(L)$ is an isomorphism because $P : U \to L$ is a homotopy equivalence having $i : L \hookrightarrow U$ as its inverse.

Since by assumption $i^*([\omega]) = 0$ we have $d\nu = i^*(\omega)$ for some $\nu \in \Omega_c^{k-1}(L)$. So we have

$$P^* i^*(\omega) = P^*(d\nu) = d P^*(\nu).$$

Also $P^* i^*$ is identity on $H^k_{DR}(U)$ and thus we see that

$d\mu = \omega - P^* i^*(\omega) = \omega - d P^*(\nu)$, for some $\mu \in \Omega^{k-1}(U)$. In fact, on the closed neighborhood K we have

$\omega = d\mu + d P^*(\nu) = d(\rho\mu) + d(\rho P^* \nu)$, because $\rho \equiv 1$ on K. The form $\rho\mu$ may not compactly supported but $d(\rho\mu)$ is since both ω and $\rho P^* \nu$ and hence $d(\rho P^* \nu)$ are compactly supported.

Moreover, on K $\rho \equiv 1$ and thus

$$d(\rho\mu) = \omega - d(\rho P^* \nu) = \omega - d(P^* \nu) = \omega - P^* d\nu = \omega - P^* i^* \omega$$

However, on L $i \circ P = id_L$ and thus $\omega - P^* i^* \omega = 0$ on L. In other words, $d(\rho\mu) \in \Omega_c(M \setminus L)$.

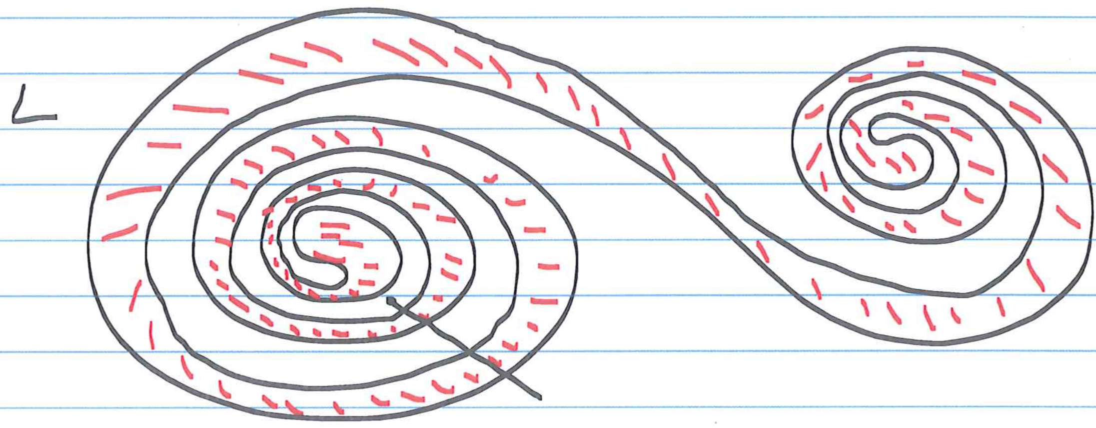
Finally, $\omega = d(\rho\mu) + d(\rho P^* \nu)$ on K and is zero

on L. Hence, ω represents the zero class in the quotient space $\Omega^k_c(M) \Big/ \Omega^k_c(M\setminus L) + d\left(\Omega^{k-1}_c(M)\right)$,
since $d(\rho\nu) \in \Omega^k_c(M\setminus L)$ and $P^*\nu \in \Omega^{k-1}_c(\rho P^*\nu)$.

This implies that $[\omega]=0$ in $H^k_c(M, M\setminus L)$, and the proof finishes. ■

An Application: Generalized Jordan Closed Curve Theorem.

Theorem: If L is a smoothly embedded simple closed curve in $\mathbb{R}^2$ then $\mathbb{R}^2\setminus L$ has two components, one bounded and on unbounded. Both have L as their boundary.



General Case Theorem: Let $M^n \subseteq \mathbb{R}^{n+1}$ be a closed connected embedded submanifold. Then $\mathbb{R}^{n+1}\setminus M^n$ has exactly two components. One of the components is bounded while the other is unbounded and both have M as their topological boundary.

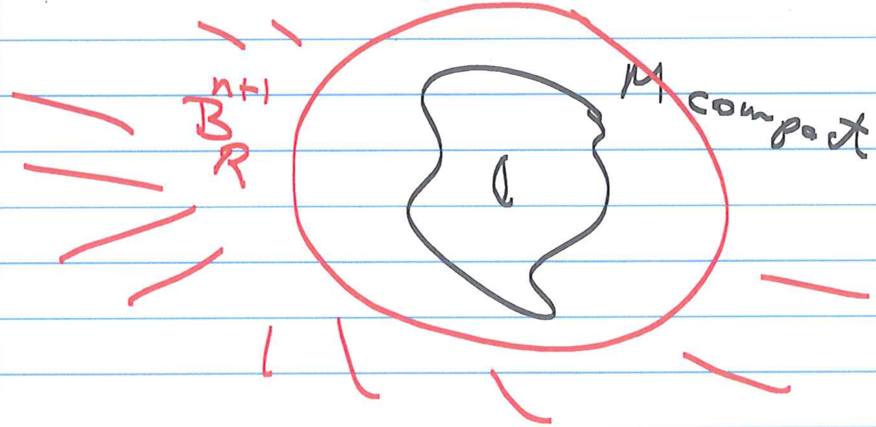
Proof: $U = \mathbb{R}^{n+1} \setminus M^n$. Consider the compactly supported cohomology long exact sequence of the pair $(\mathbb{R}^{n+1}, U)$:

$$\cdots \to \underset{\overset{\parallel}{0}}{H^n_c(\mathbb{R}^{n+1})} \to \underset{\overset{\text{\rotatebox{90}{$\cong$}}}{H^n_c(M) \simeq \mathbb{R}}}{H^n_c(\mathbb{R}^{n+1}, U)} \to H^{n+1}_c(U) \to \underset{\overset{\text{\rotatebox{90}{$\cong$}}}{\mathbb{R}}}{H^{n+1}_c(\mathbb{R}^{n+1})}$$

$$\to \underset{\overset{\text{\rotatebox{90}{$\cong$}}}{H^{n+1}_c(M) = 0.}}{H^{n+1}_c(\mathbb{R}^{n+1}, U)} \to \cdots$$

$$0 \to \mathbb{R} \to H^{n+1}_c(U) \to \mathbb{R} \to 0$$

$\Rightarrow H^{n+1}_c(U) \simeq \mathbb{R} \oplus \mathbb{R}$, $U \subseteq \mathbb{R}^{n+1}$ an open subset of $\mathbb{R}^{n+1}$. Hence, U has exactly two connected components.



Choose a large ball B^{n+1}_R containing M inside. Since outside of S^n_R is connected it must lie in one of the connected components of $\mathbb{R}^{n+1} \setminus M^n$, because

$$M^n \subset B_R^{n+1} \Rightarrow \mathbb{R}^{n+1} \setminus B_R^{n+1} \subseteq \mathbb{R}^{n+1} \setminus M^n.$$

Let V_1 be the connected component of $\mathbb{R}^{n+1} \setminus M^n$ containing $\mathbb{R}^{n+1} \setminus B_R^{n+1}$, which is unbounded. Hence, V_1 is unbounded.

It follows that the other component say V_0 of $\mathbb{R}^{n+1} \setminus M^n$ lies inside B_R^{n+1}, and thus it is bounded.

$$\mathbb{R}^{n+1} \setminus M^n = V_0 \,\dot{\cup}\, V_1$$

Claim $\partial V_i = M^n$, $i = 1, 2$.

Proof: Exercise. ∎

Another application is the Alexander Duality

Theorem: $K \subseteq \mathbb{R}^n \simeq S^n \setminus \{p\} \subseteq S^n$, $n \geq 2$, a compact embedded submanifold.

Then

$$H^0_c(\mathbb{R}^n \setminus K) = 0$$

$$H^i_c(\mathbb{R}^n \setminus K) \cong H^{i-1}_{DR}(K) \quad i = 1, \dots, n-1.$$

$$H^n_c(\mathbb{R}^n \setminus K) \cong H^{n-1}_{DR}(K) \oplus \mathbb{R}$$

Proof Exercise! $\left(H^k_{DR}(K) = H^k_c(K) \right)$

Meyer–Vietoris Sequence for Compactly Supported Cohomology:

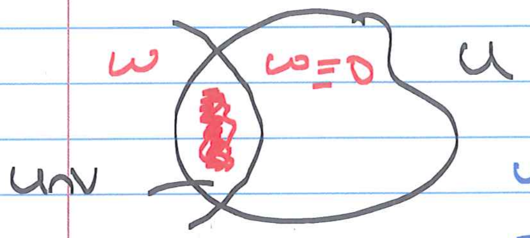
$M = U \cup V$, M smooth manifold written as the union of two open subsets. Then we have an exact sequence

$$\omega \longmapsto (\omega_1, \omega_2) \longmapsto \bar{\omega}_1 - \bar{\omega}_2 = 0$$

$$0 \to \Omega^k_c(U \cap V) \to \Omega^k_c(U) \oplus \Omega^k_c(V) \to \Omega^k_c(U \cup V) \to 0$$

$$i_U \oplus i_V \qquad (\nu, \eta) \xmapsto{\ j_U - j_V\ } \bar{\nu} - \bar{\eta}$$

$i_u : \Omega^k_c(U \cap V) \longrightarrow \Omega^k_c(U)$



$j_u : \Omega^k_c(U) \longrightarrow \Omega^k_c(U \cup V)$

$\omega_1|_{U \cap V} = \omega = \omega_2|_{U \cap V}$

$\bar{\nu}|_U = \nu, \quad \bar{\eta}|_V = \eta$

Exercise: Fill the details that this sequence is short exact.

So we get

Theorem: If a smooth manifold M is written as $M = U \cup V$ for some open subsets U and V, then we have a long exact sequence of compactly supported cohomology groups

$$\cdots \to H^k_c(U \cap V) \to H^k_c(U) \oplus H^k_c(V) \to H^k_c(M) \to H^{k+1}_c(U \cap V) \to \cdots$$

Degrees of Proper Functions:

Let $f: M \to N$ be a smooth proper function between two manifolds of the same dimension n. Assume that M and N are both connected and oriented. Now consider the homomorphism induced on compactly supported cohomology vector spaces

$$f^*: H_c^n(N) \to H_c^n(M),$$ where both vector spaces are isomorphic to $\mathbb{R}$. Then f^* is given by a scalar multiplication: There is some $\lambda \in \mathbb{R}$ so that

$$f^*([\omega_N]) = \lambda [\omega_M],$$ where ω_N and ω_M are compactly supported n-forms on N and M, respectively, with

$$\int_N \omega_N = 1, \quad \int_M \omega_M = 1.$$

The real number λ is called the degree of the proper map $f: M \to N$.

Let $p \in N$ be a regular value of $f: M \to N$. Since $\{p\}$ is compact and f is proper the inverse image

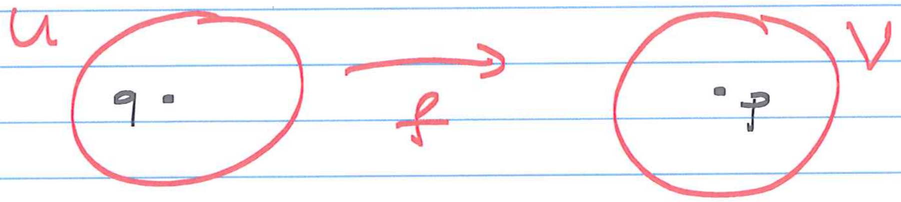
$$f^{-1}(\{p\}) = f^{-1}(p) \subseteq M$$ is a compact subset.

If $q \in f^{-1}(p)$ is any point then

$df_q : T_qM \to T_pN$, $p = f(q)$ is surjective and thus it is an isomorphism because
$\dim T_qM = n = \dim T_pN$.

Hence, by the inverse function theorem f is a local diffeomorphism at q: Hence, there are open subsets U, V with

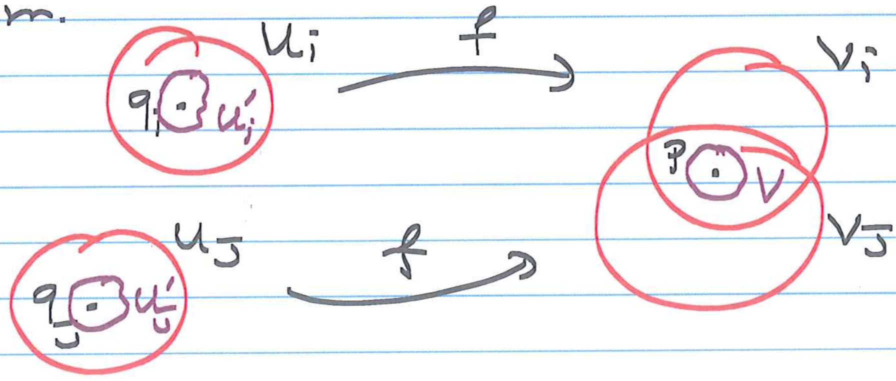
$q \in U$, $p \in V$, $f : U \to V$ diffeomorphism.



Since $f : U \to V$ is a diffeomorphism the only point in U that is mapped to p via f is q. Hence, the set $f^{-1}(p)$ consists of isolated points. Since $f^{-1}(p)$ is both compact and isolated we see that $f^{-1}(p)$ is a finite set, say

$$f^{-1}(p) = \{q_1, q_2, \dots, q_N\}.$$

For each $i = 1, 2, \dots, N$, choose U_i and V_i so that $q_i \in U_i$, $p \in V_i$, $f : U_i \to V_i$, diffeomorphism.



Choosing smaller U_i's we may assume that $U_i \cap U_j = \phi$ if $i \neq j$. Let V be an open ball in $V_1 \cap V_2 \cap \cdots \cap V_N$ containing the point p. Let $U_i' = f^{-1}(V)$ considered as the restriction map

$$f : U_i \longrightarrow V_i.$$

Let $\omega \in \Omega^n_c(V) \subseteq \Omega^n_c(N)$, be so that

$\int_V \omega = \int_N \omega = 1$. Then $H^n_c(N) \cong \mathbb{R} = \langle [\omega] \rangle$.

$$f : U_i' \longrightarrow V, \qquad \int_{U_i'} f^* \omega = \pm \int_V \omega, \text{ where}$$

the sign $\pm$ is determined by the derivative of f at q_i:

$$df_{q_i} : T_{q_i} M \longrightarrow T_p N$$

$$\underset{\text{oriented basis}}{\beta = \{v_1, \dots, v_n\}} \longmapsto \{w_1, \dots, w_n\}, \quad w_j = df_{q_i}(v_j)\}$$

Sign is "+" if df_{q_i} is orientation preserving.

Sign is "−" if df_{q_i} is orientation reversing.

Notation: This sign is called the local degree of f at q_i and denoted by

$$\deg(f)_{q_i}.$$

We know that deg(f) satisfies

$$\int_M f^*\omega = \deg(f)\int_N \omega = \deg(f)$$

$$\int_M f^*\omega = \sum_{i=1}^{N} \int_{U_i} f^*\omega = \sum_{i=1}^{N} \deg(f)_{q_i} \overbrace{\int_V \omega}^{\overset{1}{\shortparallel}}$$

$$= \sum_{i=1}^{N} \deg(f)_{q_i}$$

$$\therefore \deg(f) = \sum_{i=1}^{N} \deg(f)_{q_i}$$

Theorem: If $f: M \to N$ is a smooth proper function of oriented connected manifolds then $\deg(f)$ is an integer and it is equal to the sum local degrees at q_i's, when

$$\{q_1, \dots, q_N\} = f^{-1}(p)$$

for any regular value $p \in N$ of f.

Note Title Video 29-30 18.04.2020

Properties of the Degree of a function.

$f: M^n \to N^n$ proper map of oriented connected manifolds, then

$$\begin{tikzcd} f_* : H^n_c(N) \arrow[r] & H^n_c(M) \\ \mathbb{R} \arrow[u, phantom, "\cong" rotate] \arrow[r, "\times \deg(f)"] & \mathbb{R} \arrow[u, phantom, "\cong"] \end{tikzcd}$$

$\deg(f) = \sum_{i=1}^{N} \deg(f)(q_i)$, $\{q_1, \ldots, q_N\} = f^{-1}(p)$

for any regular value $p \in M$ of f.

Theorem: Let $f: M_1 \to N_1$, $g: M_2 \to N_2$ be proper maps of oriented connected manifolds. Then we have:

1) If f and g are homotopic via a proper homotopy $F: M_1 \times I \to N_1$, then $f^* = g^*$ and thus $\deg f = \deg g$ (if $M_1 = M_2$ and $N_1 = N_2$)

In particular, if $M_1 = M_2 = N_1 = N_2 = M$ and $g = Id_M$

$\deg f = \deg g = 1$.

2) Suppose $N_1 = M_2$, then the degree of the composition

$$M_1 \xrightarrow[f]{} N_1 \xrightarrow[g]{} N_2$$

satisfies $\deg(g \circ f) = \deg(f) \deg(g)$

3) If f is <u>not</u> onto then $\deg(f) = 0$.

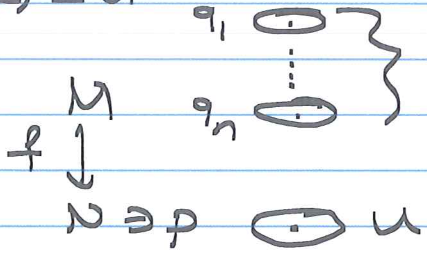
4) If $f: M_1 \to N_1$ is a n-sheeted covering space then $\deg(f) = \pm n$.

<u>Proof</u>: 2)

$$\begin{tikzcd} H^n_c(N_2) \arrow[r, "g^*"] \arrow[d, phantom, "\cong"] & H^n_c(N_1) \arrow[r, "f^*"] \arrow[d, phantom, "\cong"] & H^n_c(M_1) \arrow[d, phantom, "\cong"] \\ \mathbb{R} \arrow[r, "\times \deg(g)"] \arrow[rr, bend right, "\times(\deg(f)\cdot\deg(g))"'] & \mathbb{R} \arrow[r, "\times \deg(f)"] & \mathbb{R} \end{tikzcd}$$

3) Suppose $f: M_1 \to N_1$ is <u>not</u> onto. Let $p \in N_1 \setminus f(M_1)$, then p is a regular value and $f^{-1}(p) = \emptyset$. Hence, there is <u>no</u> local degree, so that $\deg(f) = 0$.

4)



$f^{-1}(U) = V_1 \cup V_2 \cup \cdots \cup V_n$

$f: V_1 \to U$ diffeo.

<u>Exercise</u>: Since both M_1 and N_2 are oriented

$\deg(f)(q_i) = \deg(f)(q_j)$ for all i, j.

∎

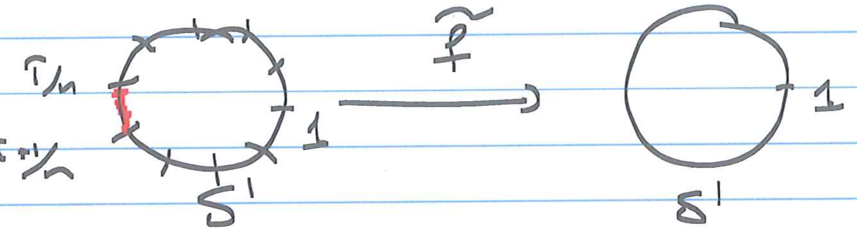
<u>Examples</u>: 1) $f: \mathbb{R} \to \mathbb{R}$, $f(t) = \lambda t$, $\lambda \in \mathbb{R} \setminus \{0\}$.

f is a diffeomorphism of $\mathbb{R}$ and f preserves orientation if and only if $\lambda > 0$.

$\deg(f) = \operatorname{sgn}(\lambda)$. If $\lambda = 0$, then $\deg(f) = 0$.

Now assume $\lambda = n \in \mathbb{Z}$, then f induces a map $\tilde{f}: \mathbb{R}/\mathbb{Z} \simeq S^1 \to S^1 \simeq \mathbb{R}/\mathbb{Z}$ because

$f(\mathbb{Z}) = n\mathbb{Z} \subseteq \mathbb{Z}$. $n \neq 0$



$\tilde{f}^{-1}(1) = \{ 1/n, 2/n, \dots, \frac{n-1}{n}, 1 \}$

$\deg(\tilde{f})_{(i/n)} = sgn(n) = \begin{cases} 1 & \text{if } n > 0 \\ -1 & \text{if } n < 0 \end{cases}$ and hence

$\Rightarrow \deg(\tilde{f}) = n$.

2) $A \in M(n, \mathbb{R})$, $A = (a_{ij})_{n \times n}$ $a_{ij} \in \mathbb{R}$.

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(x_1, \dots, x_n) \longmapsto \left(A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right)^T$.

$\omega = dx_1 \wedge \ \wedge dx_n \in \Omega^n(\mathbb{R}^n)$, orientation form on $\mathbb{R}^n$.

$A^*(\omega) = \det(A)\, \omega$.

$\deg(A) = sgn(\det(A))$.

Similar to the above example A induces a map between the n-tori

$\tilde{A} : \mathbb{R}^n / \mathbb{Z}^n \longrightarrow \mathbb{R}^n / \mathbb{Z}^n$, provided that

$A \in M(n, \mathbb{Z})$ so that $A(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$.

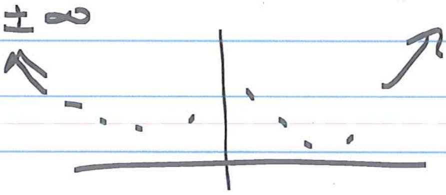
$\tilde{A} : T^n = S^1 \times \dots \times S^1 \longrightarrow S^1 \times \dots \times S^1 = T^n$

$\deg(\tilde{A}) = \det(A)$. (Exercise!)

3) $P : \mathbb{R} \to \mathbb{R}$ degree n polynomial map.

If $\deg P = n$ is an even integer, then

$\lim_{x \to \pm\infty} P(x) = +\infty$ $(P(x) = x^n + \dots + a_1 x + a_0)$

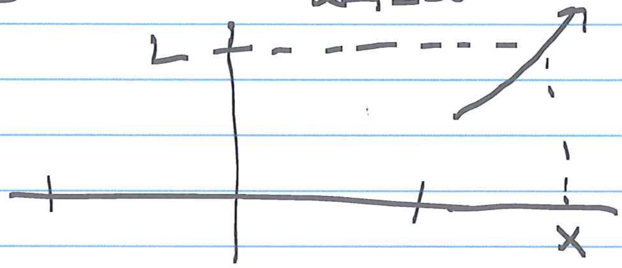


Clearly, P is not onto.

Hence, $\deg P = 0$.

Now let $\deg P = n$ is an odd integer. Then

$\lim_{x \to \infty} P(x) = +\infty$, $\lim_{x \to -\infty} P(x) = -\infty \Rightarrow P$ is onto.



$P^{-1}(L) = \{x\}$

If $L \in \mathbb{R}$ is a large enough regular value for $P(x)$ then $P^{-1}(L)$ is a single point.

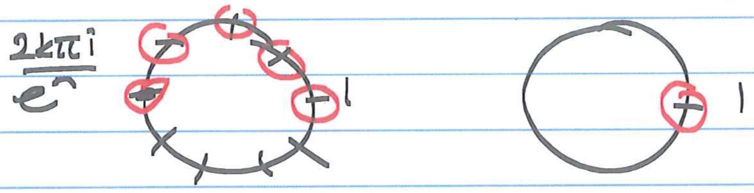
Thus $\deg P = \deg(P)_x = \pm 1$.

$\Rightarrow \deg(P) = 1$, similarly, $\deg(-x^n + \dots) = -1$.

4) Now let $F : \mathbb{C} \to \mathbb{C}$, $F(z) = z^n$, $n \in \mathbb{Z}^+$.

$F'(z) = n z^{n-1} \neq 0$ for all $z \in \mathbb{C} \setminus \{0\}$.
Hence $z = 1$ is a regular value.

$f^{-1}(z) = \{ e^{\frac{2k\pi i}{n}} \mid k = 0, 1, \dots, n-1 \}$.



Any analytic map is orientation preserving and thus $\deg(f)_{z_k} = +1$, $z_k = e^{2k\pi i/n}$.

$$\Rightarrow \deg(f) = \sum_{k=0}^{n-1} \deg(f)_{z_k} = \sum_{k=0}^{n-1} 1 = n.$$

__Claim:__ Let $f(z) = z^n + a_{n-1} z^n + \dots + a_1 z + a_0 \in \mathbb{C}[z]$.

Then $f(z)$ is homotopic to z^n via a proper homotopy. (Exercise).

Claim $\Rightarrow$ $\deg(f) = \deg(z^n) = n$.

Exercise: $f(z) = z$, $g(z) = z^2$. Consider the homotopy $F : \mathbb{C} \times I \to \mathbb{C}$,

$$F(z,t) = (1-t)f(z) + t\,g(z)$$
$$= (1-t)z + t z^2.$$

$F(z,0) = f(z) = z$, $F(z,1) = g(z) = z^2$.

Show that F is <u>not</u> a proper function.

Poincaré Isomorphism:

M connected, oriented n-dimensional manifold and let $k \in \mathbb{Z}$, $0 \leq k \leq n$. Then for any $\omega \in \Omega_c^k(M)$, $\nu \in \Omega^{n-k}(M)$, then the product $\omega \wedge \nu \in \Omega_c^n(M)$ and thus its integral is well defined:

$$\int_M \omega \wedge \nu.$$

This defines a bilinear form

$$H_c^k(M) \times H_{DR}^{n-k}(M) \longrightarrow \mathbb{R}, ([\omega],[\nu]) \longmapsto \int_M \omega \wedge \nu.$$

We may define a linear map

$$D_M : H_{DR}^k(M) \longrightarrow \left(H_c^{n-k}(M)\right)^*, (D_M([\nu]))([\omega]) \doteq \int_M \omega \wedge \nu.$$

Theorem: (Poincaré Isomorphism)

If M is a connected and oriented smooth manifold of dimension n, then for any $0 \leq k \leq n$ the linear map

$D_M : H_{DR}^k(M) \longrightarrow \left(H_c^{n-k}(M)\right)^*$ is an isomorphism

Proof: Lemma (5 Lemma) Suppose we have a ladder of vector space homomorphisms whose rows are exact, where each square is commutative:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

Then, if f_1, f_2, f_4 and f_5 are isomorphisms, then so is f_3.

Proof of the 5-lemma is left as an exercise.

Step 1: Consider the special $M=\mathbb{R}^n$. If $k\neq n$ then $H^k_c(M) = 0 = (H^{n-k}_{DR}(M))^*$.

Now let $k=n$. Choose a form $\omega \in \Omega^n_c(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \omega = 1$. Then $H^n_c(M) = \langle[\omega]\rangle \simeq \mathbb{R}$.

$[1] \in H^0_{DR}(M) \simeq \mathbb{R}$, $\int_{\mathbb{R}^n} \omega \wedge 1 = 1$.

$\Rightarrow$ Bilinear form is non degenerate and thus D_M is an isomorphism.

Step 2: $L^k(M) \doteq (H^k_c(M))^*$.

$U, V \subseteq M$ open subsets. Let's write down the Meyer-Vietoris exact sequences for both cohomologies:

$$\begin{tikzcd}
{} \arrow[r, "\delta"] & H^k_{DR}(U\cup V) \arrow[r] \arrow[d, "D"'] & H^k_{DR}(U)\oplus H^k_{DR}(V) \arrow[r] \arrow[d, "D \oplus D"'] & H^k_{DR}(U\cap V) \arrow[r, "\delta"] \arrow[d, "D"'] & H^{k+1}_{DR}(U\cup V) \arrow[r] \arrow[d, "D"'] & \cdots \\
{} \arrow[r, "\delta^*"'] & L^{n-k}(U\cup V) \arrow[r] & L^{n-k}(U)\oplus L^{n-k}(V) \arrow[r] & L^{n-k}(U\cap V) \arrow[r, "\delta^*"'] & L^{n-k-1}(U\cup V) \arrow[r] & \cdots
\end{tikzcd}$$

The maps labelled D are the Poincaré Duality homomorphism.

<u>Claim</u>: All the squares in the above ladder are commutative.

Proof: Exercise.

Now if the Poincaré map D is an isomorphism for open sets U, V and $U \cap V$ then by the 5-Lemma D is an isomorphism for $U \cup V$.

<u>Step 3</u>: Let $\{U_\alpha\}$ be a collection of disjoint open subset of M, $(U_\alpha \cap U_\beta = \emptyset$ if $\alpha \neq \beta)$.
Let $U = \bigcup_\alpha U_\alpha$.

$$\begin{tikzcd}
\text{Then } H^k_{DR}(U) \arrow[dd, "D_U"] \arrow[r, phantom, "\cong"] & \prod_\alpha H^k_{DR}(U_\alpha) \arrow[dd, bend left, "\prod_\alpha D_{U_\alpha}"] \arrow[ddr, bend left, "D_{U_\alpha}"] & \text{and} \\
H_c^{n-k}(U) \arrow[r, phantom, "\cong"] & \bigoplus_\alpha H_c^{n-k}(U_\alpha). & \\
L^{n-k}(U) = (H_c^{n-k}(U))^* \arrow[r, phantom, "\cong"] & \prod_\alpha (H_c^{n-k}(U))^* \arrow[r, phantom, "="] & \prod_\alpha L^{n-k}(U_\alpha).
\end{tikzcd}$$

If D_{U_α} is an isomorphism for each α, then so is their product $\prod_\alpha D_{U_\alpha}$.

<u>Step 4</u>: Theorem holds for any open subset U of $\mathbb{R}^n$.

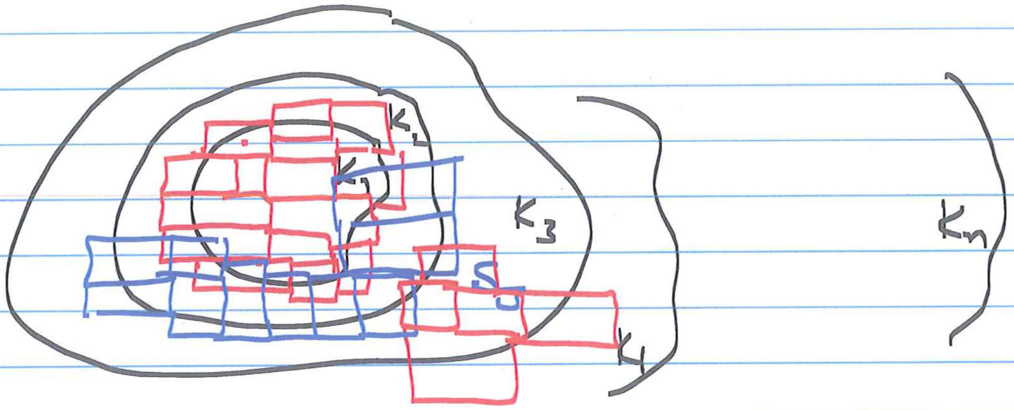
<u>Claim</u>: Any open subset of $\mathbb{R}^n$ can be written as the union of two open subset U and V so that U, V and $U \cap V$ can be written as the

the union of disjoint open subsets each of which is the union of finitely many open rectangles in $\mathbb{R}^n$?

Video 31-32

Proof of the Claim: $O \subseteq \mathbb{R}^n$ any open subset of $\mathbb{R}^n$.

$O = \bigcup_{n=1}^{\infty} K_n$, K_n compact, $K_n \subseteq \text{Int}\, K_{n+1}$, $\forall n$.



Let $R_1, \dots, R_{i_1}$ be open rectangles so that $R_i \subseteq \text{Int}\, K_2$ and $K_1 \subseteq R_1 \cup \dots \cup R_{i_1}$. Now choose open rectangles $S_1, \dots, S_{j_1}$ so that $S_j \subseteq \text{Int}(K_3)$ and
$K_2 \setminus \text{Int}(K_1) \subseteq S_1 \cup \dots \cup S_{j_1}$.

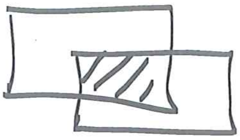
Then choose rectangles $R_{i_1+1}, \dots, R_{i_2}$ so that
$R_i \subseteq \text{Int}(K_4)$, $R_i \cap R_j = \emptyset$ $\forall$ $i \in \{1, \dots, i_1\}$, $j \in \{i_1+1, \dots, i_2\}$
$K_3 \setminus \text{Int}(K_2) \subseteq R_{i_1+1} \cup \dots \cup R_{i_2}$. ▬

We continue this way for each K_n and let

$U = \bigcup R_i$, $V = \bigcup S_j$. Then $O = U \cup V$. U, V and $U \cap V$ are disjoint unions of open sets each of which is a union of open rectangles. ▬

Proof of Step 4. $O = U \cup V$

Any open rectangle say R_1 is diffeomorphic to $\mathbb{R}^n$ and thus by Step 1 D_M is an isomorphism for



R_1. $R_1 \cup R_2 \cup \cdots \cup R_{i_1}$

Result holds for R_1, R_2 and $R_1 \cap R_2$. So by Step 2 the result holds for $R_1 \cup R_2$.

$R_1 \cup R_2$, R_3, $(R_1 \cup R_2) \cap R_3 = (R_1 \cap R_3) \cup (R_2 \cap R_3)$.

$\Rightarrow$ Results holds for $R_1 \cup R_2 \cup R_3$.
Similarly, by induction the results holds for any finite union of open rectangles.

Now by Step 3, the result holds for both U and V. Clearly, it holds for $U \cap V$. Hence, again by Step 2 the result holds for the open subset O of $\mathbb{R}^n$. ■

Step 5: let's write as in the above claim the smooth manifold M as the union

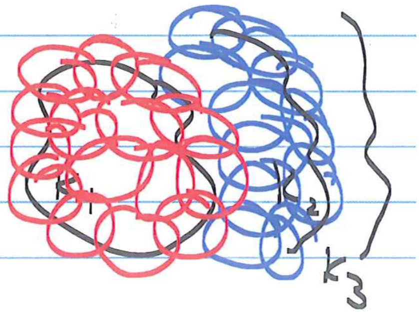
$M = U \cup V$ of open subsets so that U, V and $U \cap V$ are disjoint unions of open subsets which are diffeomorphic to open subsets of $\mathbb{R}^n$.

$U = \dot{\bigcup} U_\alpha$, $V = \dot{\bigcup} V_\alpha$

U_α, V_α, $U_\alpha \cap V_\beta$ is a finite union of open subsets diffeomorphic to open sets in $\mathbb{R}^n$.

$M = \bigcup_{n=1}^{\infty} K_n$, $K_n \subset \mathrm{Int}(K_{n+1})$

So we may finish the proof in a similar fashion. ■



Remark: $D_M : H^k_{DR}(M) \longrightarrow (H^{n-k}_c(M))^*$ is an isomorphism.

Assume M is compact. $H^{n-k}_c(M) = H^{n-k}_{DR}(M)$.

$$H^k_{DR}(M) \simeq (H^{n-k}_{DR}(M))^* \simeq ((H^k_{DR}(M))^*)^*.$$

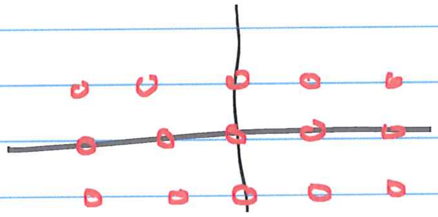
So the vector space $H^k_{DR}(M)$ is isomorphic to its double dual. Hence, it must be finite dimensional.

Theorem: If M is a compact smooth manifold then each $H^k_{DR}(M)$ is finite dimensional.

Definition: Let M^n be a smooth manifold s.t. each $H^k_{DR}(M)$ is finite dimensional. Then the k^{th} Betti number of M is defined as follow:

$$b_k(M) = \dim_{\mathbb{R}} H^k_{DR}(M).$$

Example $M = \mathbb{R}^2 \setminus \mathbb{Z}^2$



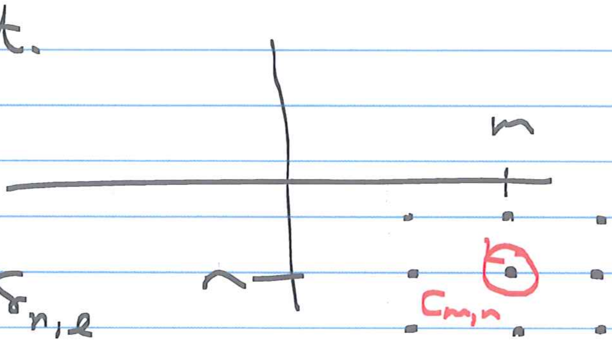
For any $m,n \in \mathbb{Z}$ let

$$\omega_{m,n} = \frac{x\,dy - y\,dx}{2\pi((x-m)^2+(y-n)^2)} \in \Omega^1(M).$$

$$d\omega_{m,n} = 0 \Rightarrow [\omega_{m,n}] \in H^1_{DR}(M).$$

Claim: $\{[\omega_{m,n}]\}_{(m,n)\in\mathbb{Z}^2}$ is a linearly independent set.

Proof



$$\int_{C_{m,n}} \omega_{k,l} = \delta_{m,k}\,\delta_{n,l}$$

This finishes the proof. ■

In particular, the vector space $H^1_{DR}(M)$ is not finite dimensional.

Corollary Let M^n be a connected smooth manifold. Then $H^n_{DR}(M) \simeq \mathbb{R}$ if and only if M is compact without boundary and orientable. Otherwise, $H^n_{DR}(M) = 0$.

Proof Exercise.

Definition: Assume that each de Rham cohomology vector space of M is finite dimensional so that

$b_k(M) \doteq \dim H^k_{DR}(M)$ is defined. Then

$$\chi(M) \doteq \sum_{k=0}^{\dim(M)} (-1)^k b_k(M)$$

is called the Euler number of M.

Remark If M_1 and M_2 are diffeomorphic manifolds. Then $H^k_{DR}(M_1) \simeq H^k_{DR}(M_2)$, and then

$\chi(M_1) = \chi(M_2)$ provided that all cohomologies are finite dimensionals.

Corollary M differentiable manifold, $U, V \subseteq M$ open subsets so that U, V and $U \cap V$ have all finite dimensional cohomologies. Then

$$\chi(U \cup V) = \chi(U) + \chi(V) - \chi(U \cap V).$$

Proof: **Claim.** Assume that we have a finite exact sequence of vector spaces

$\cdots \xrightarrow{\delta_{i-1}} A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \xrightarrow{\delta_i} A_{i+1} \xrightarrow{f_{i+1}} \cdots$, where each vector space is finite dimensional.

Let $\chi(A) = \sum_i (-1)^i \dim A_i$, $\chi(B) = \sum_i (-1)^i \dim B_i$,

$\chi(C) = \sum_i (-1)^i \dim C_i$.

Then $\chi(B) = \chi(A) + \chi(C)$.

Proof of the claim. $\dim A_i = \dim \operatorname{Im} f_i + \dim \ker f_i$.

From exactness, $\operatorname{Im} \delta_i = \ker f_{i+1}$, $\ker \delta_i = \operatorname{Im} g_i$, $\operatorname{Im} f_i = \ker g_i$. Then

$$\chi(C) + \chi(A) = \sum_i (-1)^i \Big[(\dim \operatorname{Im} \delta_i + \dim \ker \delta_i) + (\dim \operatorname{Im} f_i + \dim \ker f_i) \Big]$$

$$= \sum_i \big(\underline{\dim \ker f_{i+1}} + \underline{\dim \operatorname{Im} g_i} \big) (-1)^i$$

$$+ \sum_i (-1)^i (\underline{\dim \ker g_i} + \underline{\dim \ker f_i})$$

$$= \sum_i (-1)^i (\dim \operatorname{Im} g_i + \dim \ker g_i)$$

$$+ \sum (-1)^i (\dim \ker f_i + \dim \ker f_{i+1})$$

$$= \sum_{i=1} (-1)^i \dim B_i + 0.$$

$$= \chi(B).$$

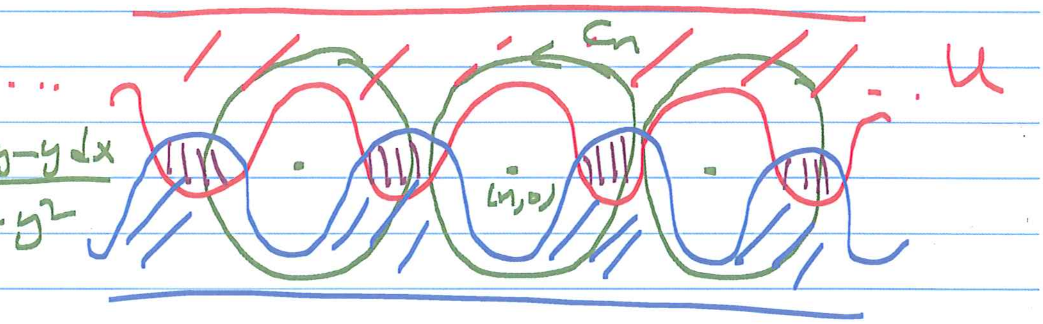
To finish the proof let $A \doteq H^i_{DR}(U \cup V)$, $B \doteq H^i_{DR}(U) \oplus H^i_{DR}(V)$, $C_i \doteq H^i_{DR}(U \cap V)$.

Now the above sequence just becomes the Meyer-Vietoris exact sequence for U, V. This finishes the proof. ∎

Remark:

$$\omega_n = \frac{1}{2\pi} \frac{(x-n)\,dy - y\,dx}{(x-n)^2 + y^2}$$

$$\int_{C_m} \omega_n = \delta_{mn}$$



$H^0_{DR}(U) \simeq \mathbb{R}$, $H^i_{DR}(U) = 0$, if $i > 0$ and

$H^0_{DR}(V) \simeq \mathbb{R}$, $H^i_{DR}(V) = 0$, if $i > 0$.

$$H^0_{DR}(U \cap V) \simeq \prod_{i=-\infty}^{\infty} \mathbb{R}$$

So even if U and V has finite dimensional cohomologies, $U \cap V$ may have finite dimensional cohomology.

Note that $\{[\omega_n] \mid n \in \mathbb{Z}\}$ is a linearly independent set in $H^1_{DR}(U \cup V)$. Hence, $H^1_{DR}(U \cup V)$ is also infinite dimensional.

<u>Corollary</u>: Let M be an odd dimensional compact orientable manifold without boundary. Then

$$\chi(M) = 0.$$

<u>Proof</u>: $\dim M = n$. Since M is compact and orientable, by the Poincaré duality we have

$$H^k_{DR}(M) \simeq \left(H_c^{n-k}(M)\right)^* \simeq \left(H^{n-k}_{DR}(M)\right)^*$$
$$\simeq H^{n-k}_{DR}(M)$$

because each cohomology is finite dimensional.

Hence, $b_k = \dim H^k_{DR} = \dim H^{n-k}_{DR}(M) = b_{n-k}$.

Then $\chi(M) = b_0 - b_1 + b_2 - b_3 + \cdots + b_{n-1} - b_n.$

$=0$

$=0$

$=0$

$=0$

$\underline{n=5}$ $\chi(M) = b_0 - b_1 + b_2 - b_3 + b_4 - b_5 = 0.$

Corollary Let M be a compact orientable manifold without boundary of dimension $4m+2$. Then $\chi(M)$ is an even integer.

Proof: $n=6$, $\chi(M) = b_0 - b_1 + b_2 - b_3 + b_4 - b_5 + b_6$

$$= 2b_0 - 2b_1 + 2b_2 - b_3$$

$$\chi(M) \equiv b_{n/2} \pmod 2$$

So we must show $b_{n/2}$ is an even integer.

$$H^k_{DR}(M) \times H^{n-k}_{DR}(M) \longrightarrow \mathbb{R}, \quad (\omega, \eta) \longmapsto \int_M \omega \wedge \eta$$

is a nondegenerate pairing: For any $\omega \neq 0$ there is some η so that $\int_M \omega \wedge \eta \neq 0$.

Let $k = n/2 = 2m+1$.

$$H^{2m+1}_{DR}(M) \times H^{2m+1}_{DR}(M) \longrightarrow \mathbb{R}$$
$$(\omega, \eta) \longmapsto \int_M \omega \wedge \eta$$ is nondegenerate.

Note that this pairing is skew symmetric:

$$\int_M \omega \wedge \eta = -\int_M \eta \wedge \omega.$$

Fact: If a finite dimensional vector space has an skew symmetric nondegenerate pairing then

Its dimension must be even.

This is another way saying that only even dimensional vector spaces admits symplectic forms.

Hence, $b_{2m+1} = \dim H^{2m+1}_{DR}(M)$ is even.

<u>Examples</u>: 1) $H^k_{DR}(S^n) = \begin{cases} \mathbb{R} & k=0,n \\ 0 & \text{otherwise.} \end{cases}$

$$\chi(S^{2n}) = 2 \text{ and } \chi(S^{2n-1}) = 0.$$

2) $H^k_{DR}(\mathbb{CP}^n) = \begin{cases} \mathbb{R} & k=0,2,\dots,2n \\ 0 & \text{otherwise} \end{cases}$

$$\chi(\mathbb{CP}^n) = n+1.$$

In particular, $\chi(\mathbb{CP}^2) = 3$ <u>not</u> an even integer.

Videos 33-34

Intersection Theory

Note Title 23.04.2020

Definition: Let $f : K \to M$ and $g : L \to M$ be differentiable functions and $(p, q) \in K \times L$ with $f(p) = g(q)$. If we have

$$(Df)_p(T_pK) + (Dg)_q(T_qL) = T_{f(p)=g(q)}M$$

then we say that f and g intersects transversally at (p, q).

If $(Df)_p(T_pK) + (Dg)_q(T_qL) = T_{f(p)=g(q)}M$ for all pairs $(p, q) \in K \times L$ with $f(p) = g(q)$, then we say that f and g intersects transversally and we write $f \pitchfork g$.

Remark: 1) Suppose that $\dim K + \dim L < \dim M$. So $f \pitchfork g$ if and only if $f(K) \cap g(L) = \emptyset$.

2) $L = \{q_1, \dots, q_\ell\} \subseteq M$, $g : L \hookrightarrow M$ inclusion map. Let $f : K \to M$ be any smooth map. Then $f \pitchfork g$ at say (p, q_i) then $f(p) = g(q_i) = q_i$ and

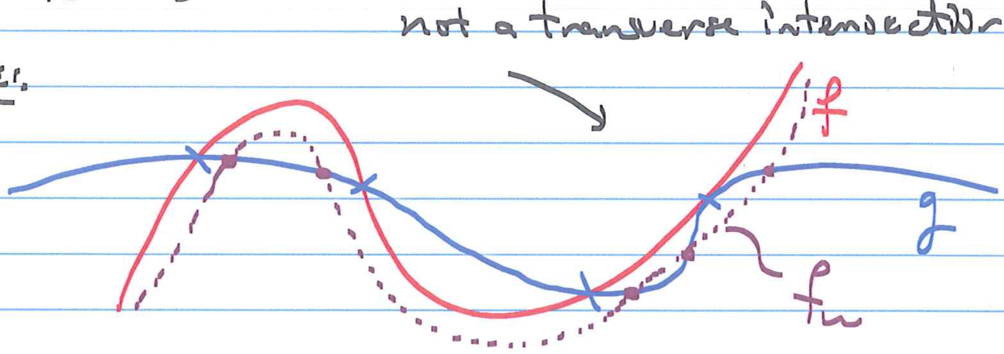
$$(Df)_p(T_pK) + (Dg)_{q_i}(\underbrace{T_{q_i}\{q_i\}}_{0}) = (Df)_p(T_pK) = T_{q_i}M.$$

Hence, $f \pitchfork g$ if and only if $L = \{q_1, \dots, q_\ell\}$ consists of regular values of f.

Theorem: Let $f: K \to \mathbb{R}^n$ and $g: L \to \mathbb{R}^n$ be differentiable functions. Then there is a measure zero set C in $\mathbb{R}^n$ so that for any $w \in \mathbb{R}^n \setminus C$, the map $f_w(x) := f(x) + w$, $x \in K$, is transverse to g.

If we further assume that K is compact and $g: L \to \mathbb{R}^n$ is a proper function, then the set C is closed. Moreover, in this case, if $f \pitchfork g$ then there is some $\delta > 0$ so that, for any $w \in B(0, \delta)$ we have $f_w \pitchfork g$.

Remark:



Proof: Consider the function $g - f: K \times L \to \mathbb{R}^n$, $(g-f)(x,y) = g(y) - f(x)$.

Let $C \subseteq \mathbb{R}^n$ be the set critical values of $g-f$, which has measure zero by Sard's Theorem.

If $w \in C$, then there is some $(x,y) \in K \times L$ such that $(g-f)(x,y) = w$ ($\Leftrightarrow g(y) - f(x) = w$)

$D(g-f)_{(x,y)}: T_{(x,y)} K \times L \longrightarrow T_w \mathbb{R}^n$ is not onto.

$\Leftrightarrow (Dg)_y(T_y L) + (Df)_x(T_x K) \neq T_w \mathbb{R}^n$.

$\Leftrightarrow (Dg)_y (T_y L) + (Df_w)_{(x)} (T_x K) \neq T_w \mathbb{R}^n$.

$\Leftrightarrow (g(y) = f_w(x))$ f_w and g do not intersect transversally at (x,y).

Conversely, if $w \in C$ then f_w and g intersects transversally at (x,y) with $(g(y) - f(x) = w)$

$g(y) = f_w(x)$.

For the second part, we assume that K is compact and $g : L \to \mathbb{R}^n$ is proper.

<u>Claim</u>: $g - f : K \times L \longrightarrow \mathbb{R}^n$ is proper.

<u>Proof</u>: Let $A \subseteq \mathbb{R}^n$ be a compact subset.

<u>must show</u>: $(g-f)^{-1}(A)$ is compact in $K \times L$.

Since f, g and thus $g-f$ is continuous and A is closed, $(g-f)^{-1}(A)$ is a closed subset of $K \times L$.

A point $(x,y) \in K \times L$ is in $(g-f)^{-1}(A)$ if and only if $g(y) - f(x) \in A$. Or equivalently,

$g(y) \in f(x) + A$.

So, for a point $y \in L$, we have $(x,y) \in (g-f)^{-1}(A)$ for some $x \in K$ if and only if $g(y) \in f(K) + A$.

Hence, $(x,y) \in K \times g^{-1}(f(K) + A) \subseteq K \times L$, provided that $(x,y) \in (g-f)^{-1}(A)$

$\Rightarrow (g-f)^{-1}(A) \subseteq K \times g^{-1}(f(K)+A)$.

$f(K)+A$ compact, g proper $\Rightarrow g^{-1}(f(K)+A)$ is compact.

Since K is also compact, we see that

$(g-f)^{-1}(A)$ is a closed set in the compact set $K \times g^{-1}(f(K)+A)$. Hence, $(g-f)^{-1}(A)$ is also compact.

This finishes the proof of the claim. ∎

Let $E \subseteq K\times L$ be the set of critical points of $g-f : K\times L \longrightarrow \mathbb{R}^n$.

So E consists of points $(x,y) \in K\times L$ so that

$$\begin{tikzcd} D(g-f)_{(x,y)} : T_{(x,y)}K\times L \arrow[r] \arrow[d, phantom, "\wr"] & T_{f(x)}\mathbb{R}^n \text{ is not onto.} \\ \mathbb{R}^k\times\mathbb{R}^l & \end{tikzcd}$$

This is equivalent to saying that in any matrix representation of the derivative map has no $n\times n$ minor with nonzero determinant. This is a closed condition and thus E is a closed subset of $K\times L$.

Finally, $C = (g-f)(E)$ is closed in $\mathbb{R}^n$, because, E is closed and $g-f$ is proper.

For the final statement, assume that $f \pitchfork g$. Then $f_0 \pitchfork g$ and thus $0 \notin C$. Since $C \subseteq \mathbb{R}^n$ is closed there is $\delta > 0$ so that $B(0,\delta) \cap C = \emptyset$. Hence, for any $w \in B(0,\delta)$ ($|w| < \delta$) we have $f_w \pitchfork g$. ■

Remark: We can indeed make a stronger statement for the second part of the theorem. Let $w : K \to \mathbb{R}^n$ be a smooth function satisfying the following conditions:

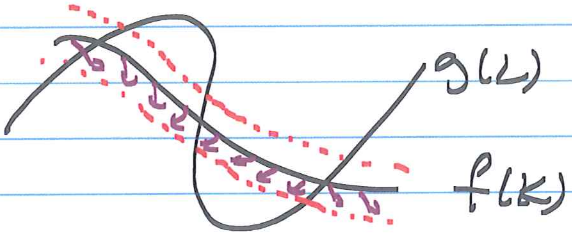
1) $w(K) \cap C = \emptyset$, C the set of critical values of $g-f$.

2) $N(f(K))$, a tubular neighborhood of the compact set $f(K)$ so that for any

$(p,q) \in K \times (L \cap g^{-1}(N(f(K))))$ we have

$\|Dw\|_p < \|D(g-f)\|_{(p,q)}$.

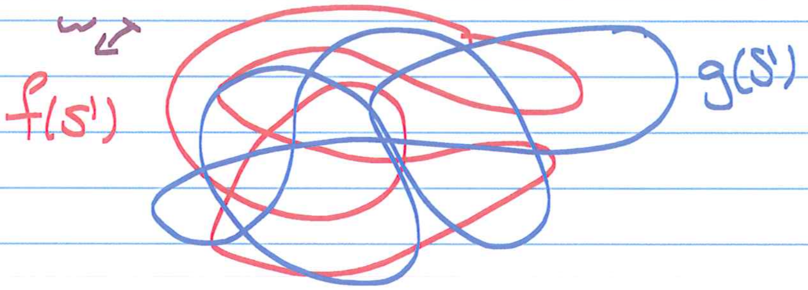
Then $f_w : K \to \mathbb{R}^n$, $f_w(p) = f(p) + w(p)$ is transverse to g.



Proof: Exercise!

Example: Let $f : S^1 \to \mathbb{R}^3$, $g : S^1 \to \mathbb{R}^3$ be smooth functions. Since S^1 is compact the function g is proper. For this case the set C is closed. So for any $\epsilon > 0$ there is some $w \in \mathbb{R}^3$ with $\|w\| < \epsilon$ so that $f_w \pitchfork g$.

$\dim S^1 + \dim S^1 = 1+1 = 2 < 3 = \dim \mathbb{R}^3$. Hence, $f_w \pitchfork g$ is equivalent to saying that $f_w(S^1) \cap g(S^1) = \emptyset$.

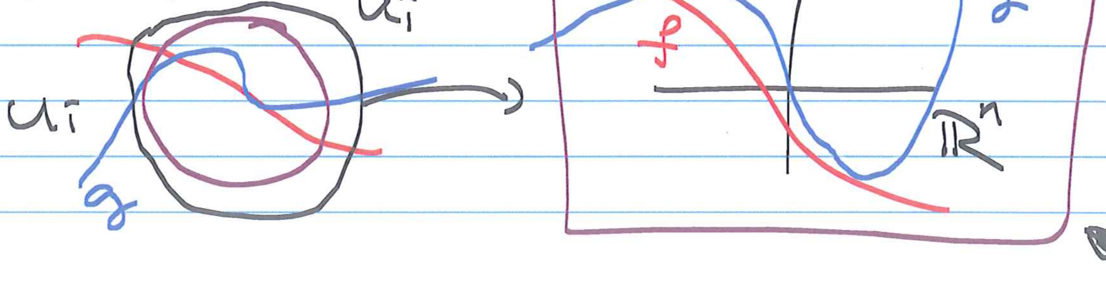


Corollary: Let $f: K \to M$ and $g: L \to M$ be differentiable functions. If $g: L \to M$ is a proper function then we can find a function $\tilde{f}$ homotopic to f and arbitrarily close to f so that

$$\tilde{f} \pitchfork g.$$

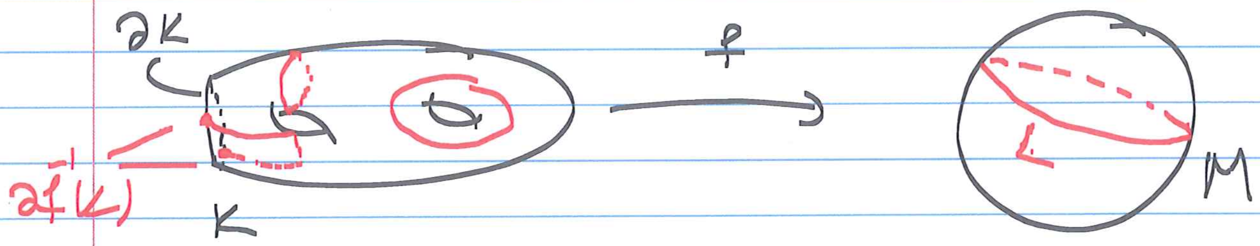
Moreover, if $f \pitchfork g$ to start with then any function $\tilde{f}$ which is homotopic to f and closed enough to f is still transverse to g.

Idea of the proof Cover M with coordinate charts $\varphi_i : U_i \to V_i = \mathbb{R}^n$



Theorem: Let $f: K \to M$ be a differentiable function and $L \subseteq M$ a closed submanifold ($g = i: L \hookrightarrow M$). If $f \pitchfork L$ ($f \pitchfork g$) then $f^{-1}(L) \subseteq K$ is a submanifold of dimension $\dim L + \dim K - \dim M$.

If K is a manifold with boundary ∂K and $f|_{\partial K} \pitchfork L$, then $f^{-1}(L)$ is a manifold with boundary in K with $\partial f^{-1}(L) = \partial K \cap f^{-1}(L)$.



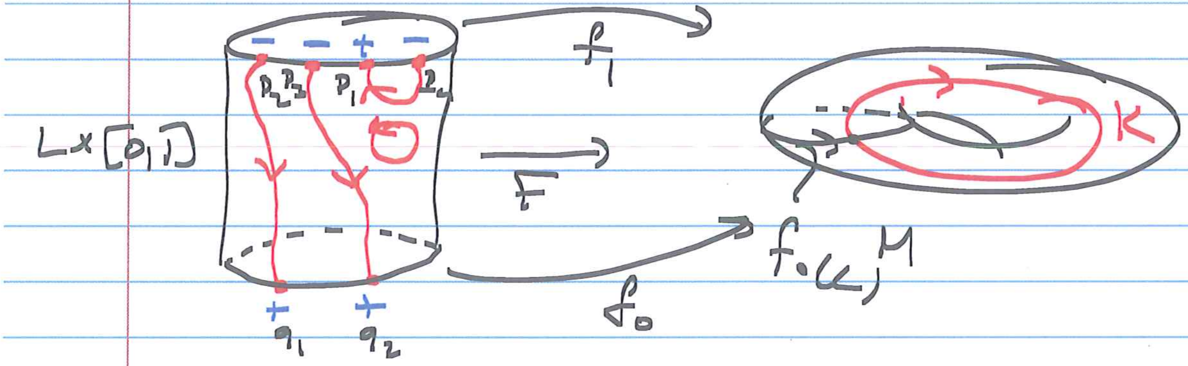
Intersections of Submanifolds and Poincaré Duality:

Proposition: Let $f: L \to M$ and $g: K \to M$ be two proper embedding functions. Also assume that $f_0, f_1: L \to M$ be functions both homotopic to f and transverse to g. Then the $k+l-m$ dimensional submanifolds

$f_0(L) \cap g(K)$ and $f_1(L) \cap g(K)$ form the boundary of a $k+l-m+1$ dimensional submanifold $W \subseteq M$. If K, L and M are oriented then so is W and thus its boundary ∂W.

Proof: $L \times [0,1] \xrightarrow{F} M$, $F(\cdot,0)=f_0$, $F(\cdot,1)=f_1$.

$f_0 \pitchfork g$, $f_1 \pitchfork g$. By the previous Theorem we can change F to a function transverse to g and without changing it on the boundaries



Then $F^{-1}(K) \subseteq L\times[0,1]$ submanifold with

$$\begin{aligned}\partial F^{-1}(K) &= F^{-1}(K) \cap \partial(L\times[0,1]) \\ &= F^{-1}(K) \cap (L\times\{0\} \cup L\times\{1\}) \\ &= f_0^{-1}(L) \cup f_1^{-1}(L)\end{aligned}$$

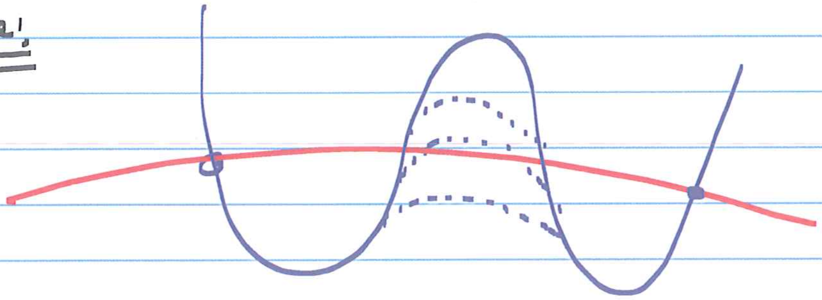
Corollary $L, K \subseteq M$, $\dim L + \dim K = \dim M$, closed submanifolds. Let f_0 and f_1 as above. Assume either L or K is compact. Then $f_0(L)\cap K$ and $f_1(L)\cap K$ are both finite sets. Further assume that L, K and M are all oriented. Then $f_1(L)\cap K$ and $f_0(L)\cap K$ are oriented finite sets. Count the points in each $f_1(L)\cap K$ and $f_0(L)\cap K$ with the orientation signs. Then the two counts are the same once we change the orientation on one of them.

$f_1(L)\cap K = \{p_1, -p_2, -p_3, -p_4\} \to 1-1-1-1=-2$

$f_2(L) \cap K = \{a_1, a_2\} \longrightarrow 1+1 = 2.$

If we ignore the orientations then these numbers are equal modulo 2.

Example:



Videos 35-36

Note Title 23.04.2020

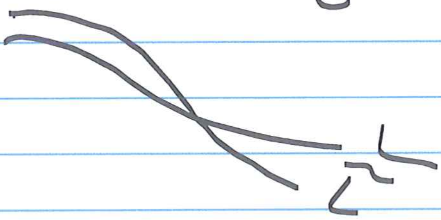
Remark 1) If the manifold M or K or L are non-orientable then Intersection of K and L are defined only in modulo 2.

2) If $K = L$ then the intersection

$Int(K,L) = Int(L,L)$ is called the self

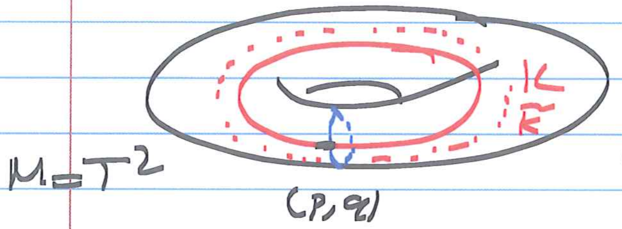
intersection of L with self.

$f : L \longrightarrow M \longrightarrow f_{t_0} : L \longrightarrow M$
$g : L \longrightarrow M \longrightarrow g : L \longrightarrow M$



3) $K^k, L^l \subseteq M^n$, $l+k=n$ $Int(\underline{K,L}) = (-1)^{kl} Int(\underline{L,K})$

Examples: 1) $M = T^2 = S^1 \times S^1$, $L = S^1 \times \{p\}$, $K = \{q\} \times S^1$



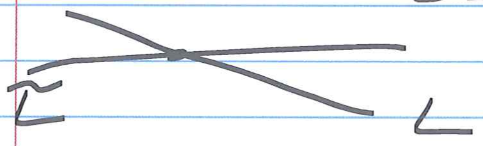
$\mathrm{Int}(K, L) = \pm 1$.

$\mathrm{Int}(K, K) = 0$.

$\dim K = \dim L = 1$, $\mathrm{Int}(K,K) = (-1)^{1 \cdot 1}\,\mathrm{Int}(K,K)$
$= -\,\mathrm{Int}(K,K)$

$\Rightarrow \mathrm{Int}(K,K) = 0$.

2) $M = \mathbb{CP}^2$, $L = \mathbb{CP}^1 : z_0 = 0$
$[z_0 : z_1 : z_2]$
$\tilde{L} = \mathbb{CP}^1 : z_1 = 0$



$t \longmapsto [(1-t)z_0 : t z_1 : z_2]$

$L \cap \tilde{L} = \{[z_0 : z_1 : z_2] \mid z_0 = z_1 = 0\} = \{[0:0:1]\}$

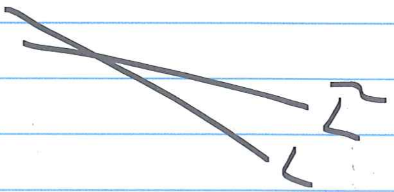
$\mathrm{Int}(L, L) = 1$ if $M, L, \tilde{L}$ are all oriented with the complex orientation

2') $M = \overline{\mathbb{CP}^2}$, $L = \mathbb{CP}^1$ with complex orien.

$\mathrm{Int}(L, L) = -1$.

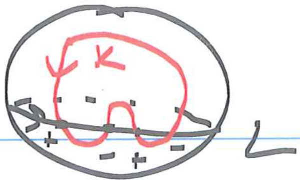
3) $M = \underset{n}{\underline{\mathbb{R}}}\mathbb{P}^2$, $L = \mathbb{RP}^1 : z_0 = 0$

non-orientable

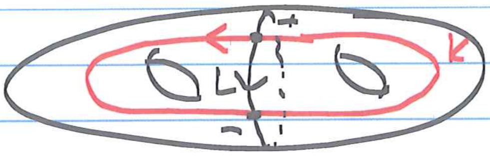


$\mathrm{Int}(L, L) = 1$.

4) $M = S^2$



$\text{Int}(K, L) = 0.$



$\text{Int}(K, L) = 0.$

Definition: $L^\ell \subseteq M^m$ oriented closed submanifold of an oriented manifold M. Define a homomorphism

$$\phi : H^\ell_c(M) \longrightarrow \mathbb{R}, \ [\omega] \longmapsto \int_L \omega.$$

Note that if $[\omega] = [\tilde{\omega}]$ for some $\tilde{\omega}$, then $\omega - \tilde{\omega} = d\eta$, for some $\eta \in \Omega^{\ell-1}_c(M)$.

Then $\int_L \omega - \int_L \tilde{\omega} = \int_L (\omega - \tilde{\omega}) = \int_L d\eta = \int_{\partial L = \phi} \eta = 0.$

$\Rightarrow \int_L \omega = \int_L \tilde{\omega}.$

So $\phi \in (H^\ell_c(M))^* = \text{Hom}(H^\ell_c(M), \mathbb{R})$.

However, by the Poincare Duality we have

$D_M : H^{m-\ell}_{DR}(M) \xrightarrow{\cong} (H^\ell_c(M))^*, \ (D_M([\nu]))([\omega]) = \int_M \omega \wedge \nu.$

Hence, there is a class $[\nu_L] \in H^{m-\ell}_{DR}(M)$ so that $D_M([\nu_L]) = \phi$. In particular,

$\int_L \omega = \phi([\omega]) = D_M([\nu_L])([\omega]) = \int_M \omega \wedge \nu.$

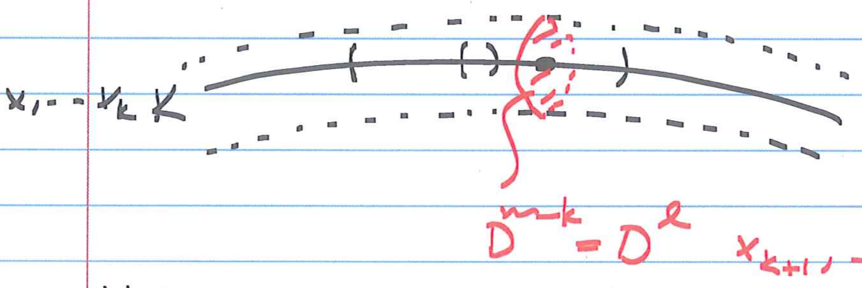
The class $[\nu_L]$ is called the Poincaré Dual of the submanifold L.

Theorem: Let M, L and $[\nu_L]$ be as above. Then for any oriented compact submanifold K^k $(k = m-l)$ we have $\mathrm{Int}(L,K) = \int_K \nu_L$.

Proof: $K^k \subseteq M$ compact oriented submanifold.

Step1: Construct a form $\mu_K \in \Omega^{m-k}(M)$ so that

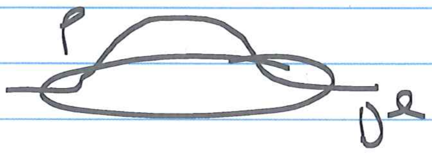
$$\mathrm{Int}(K,L) = \int_L \mu_K.$$



Tubular neighborhood of K in M.

$M: x_1, \dots, x_k, x_{k+1}, \dots, x_m$
$K: x_{k+1} = 0, \dots, x_m = 0$

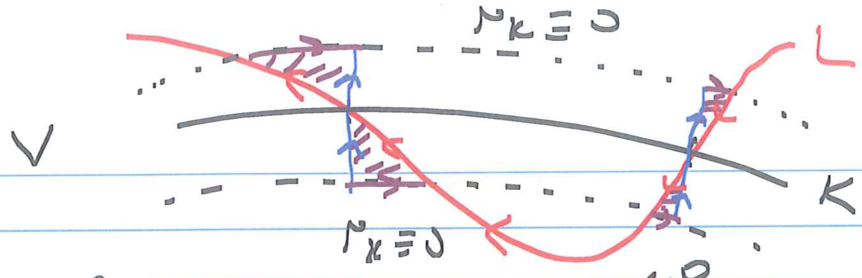
$$\int_{D^{m-k}} \rho(x_{k+1}, \dots, x_m)\, dx_{k+1} \wedge \dots \wedge dx_m = 1$$



$\mu_K \in \Omega^{m-k}(M)$

so that $\int_{D^{m-k}} \mu_K = 1$, where D^{m-k} is any transverse disk to K.

Claim: $\mathrm{Int}(K,L) = \int_L \mu_K$



$$\int_{🟥 - 🟦} \mu_K = \int_{\partial W} \mu_K = \int_W \overbrace{d\mu_K}^{=0} = 0.$$

$$\int_{🟥} \mu_K = \int_{🟦} \mu_K = \pm 1$$

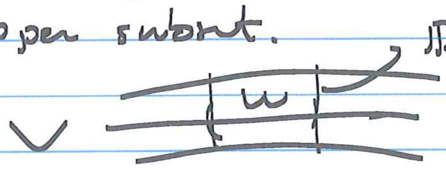
This proves the claim. ■

So we have $\int_L \mu_K = \mathrm{Int}(K,L)$.

On the other hand,

$$\mathrm{Int}(K,L) = \int_L \mu_K = \int_M \mu_K \wedge \nu_L = \int_V \mu_K \wedge \nu_L .$$

Locally we can write V as $W \times \mathbb{R}^l$, $W \subseteq K$ an open subset.



Also we have

$$\int_{\{x\} \times \mathbb{R}^{m-k} \cap V} \mu_K = 1.$$

Hence, $\int_V \mu_K \wedge \nu_L = (-1)^{kl} \int_K \nu_L$

So $\mathrm{Int}(L,K) = (-1)^{kl}\, \mathrm{Int}(K,L) = \int_K \nu_L$.

This finishes the proof. ■

Last time: Theorem: $L^\ell \subseteq M^m$ closed submanifold $[\nu_L] \in H^{m-\ell}_{DR}(M)$ Poincaré dual of L. Then for any compact K^k $(k = m-\ell)$ oriented submanifold we have

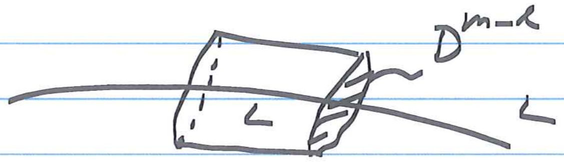
$$\text{Int}(L, K) = \int_K \nu_L .$$

Recall: $D_M([\nu_L]) \in (H^\ell_c(M))^*$

$$D_M([\nu_L])(\omega) = \int_M \nu_L \wedge \omega .$$

For the form μ_L we can see that

$$\phi([\omega]) = \int_L \omega = \int_M \omega \wedge \mu_L = \int_{L \times D^{m-\ell}} \omega \wedge \mu_L = \left(\int_L \omega\right) \cdot \overbrace{\left(\int_{D^{m-\ell}} \mu_L\right)}^{=1}$$



$$= \int_L \omega .$$

So we have

$$\phi([\omega]) = \int_L \omega = \int_M \omega \wedge \mu_L = \int_M \omega \wedge \nu_L$$ for all $[\omega] \in H^\ell_c(M)$.

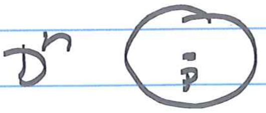
$$\Rightarrow \int_M \omega \wedge (\mu_L - \nu_L) = 0 \quad \forall \omega .$$

$\Rightarrow [\mu_L - \nu_L] = 0$ since the bilinear form is non-degenerate by the Poincaré duality.

Example: 1) $M = \mathbb{R}^n, S^n$, $1 \le k < n$, $H^k_{DR}(M) = 0$ and thus $\mathrm{Int}(K, L) = 0$ for any two submanifolds K and L of dimensions k and $n-k$.

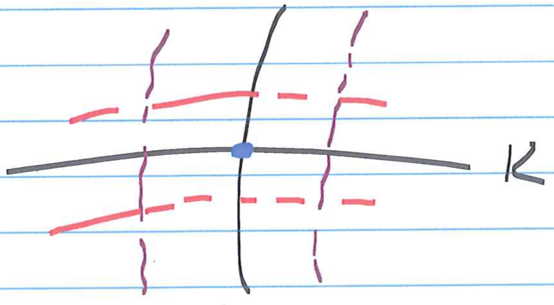
2) M any oriented manifold, $p \in M$, $K = \{p\}$ 0-dim'l submanifold.

$$\mathrm{Int}(M, \{p\}) = \pm 1 = \int_M \mu_K = \pm 1.$$



Theorem: Let $K^k, L^l \subseteq M^m$ be two transversally intersecting submanifolds, which are both closed and oriented. Let μ_K and μ_L be the Poincaré duals of K and L, respectively. Then the Poincaré dual of the oriented closed submanifold $K^k \cap L^l$ is $\mu_K \wedge \mu_L$.

Remark $\dim K^k \cap L^l = k + l - m$, and hence its Poincaré dual lies in $H^{m-(k+l-m)}_{DR}(M) = H^{2m-k-l}_{DR}(M)$.

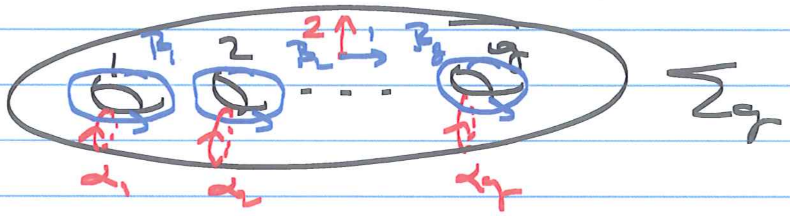


$$\mu_K = \rho_K \, dy \qquad \int_{\mathbb{R}} \rho_K = 1$$

$$\mu_L = \rho_L \, dx \qquad \int_{\mathbb{R}} \rho_L = 1$$

$$\int_{\mathbb{R}^2} \mu_K \wedge \mu_L = \int_{\mathbb{R}^2} \rho_K \rho_L \, dx \wedge dy = 1.$$

Examples 1) Σ_g: Compact connected Oriented surface of genus g.



$\alpha_i, \beta_i \subseteq \Sigma_g$ compact oriented submanifolds

$\mathrm{Int}(\alpha_i, \beta_i) = 1$, $\mathrm{Int}(\alpha_i, \beta_j) = 0$ if $i \neq j$.

a_i = Poincaré dual of α_i and b_i = Poincaré dual of β_i

$$\int_{\Sigma_g} a_i \wedge b_j = \delta_{ij}, \quad \int_{\Sigma_g} a_i \wedge a_j = 0 = \int_{\Sigma_g} b_i \wedge b_j$$

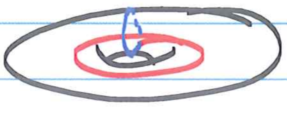
$[a_i \wedge b_i] \in H^2_{DR}(\Sigma_g)$, $\int_{\Sigma_g} a_i \wedge b_i = 1$.

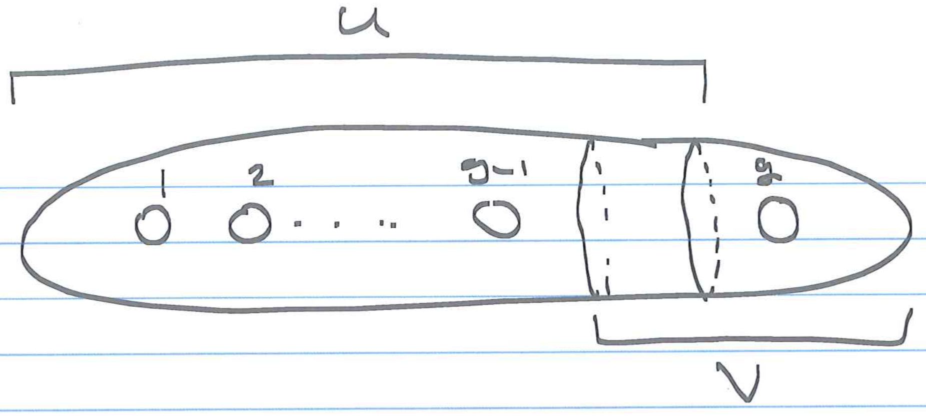
Using Meyer-Vietories Exact Sequence one can show that

$H^0_{DR}(\Sigma_g) \cong \mathbb{R}$, $H^2_{DR}(\Sigma_g) \cong \mathbb{R}$ and $H^1_{DR}(\Sigma_g) \cong \mathbb{R}^{2g}$

with basis $\{[a_i], [b_i] \mid i = 1 \to g\}$.

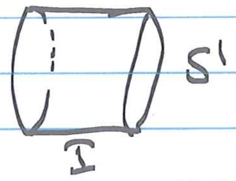
$$H^i_{DR}(T^2) = H^i_{DR}(\Sigma_g) = \begin{cases} \mathbb{R} & i = 0 \\ \mathbb{R}^2 & i = 1 \\ \mathbb{R} & i = 2 \end{cases}$$



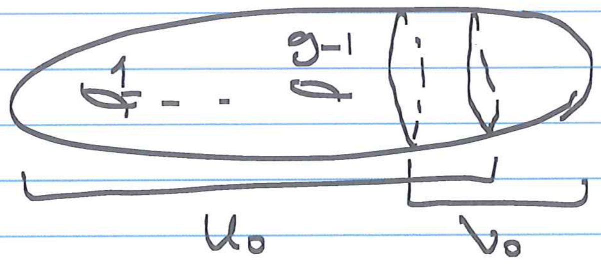


$U = \Sigma_{g-1} \setminus D^2$, $V = T^2 \setminus D^2$, $U \cup V = \Sigma_g$

$U \cap V = S^1 \times I$



Step 1 Use Mayer–Vietoris to compute $H^*(U)$ and $H^*(V)$



$V_0 = D^2$

$U_0 \cap V_0 = S^1 \times I$

Exercise: Finish this computation.

Summary $H^*_{DR}(\Sigma_g) \cong \mathbb{R}[x_i, y_i \mid i = 1, \dots, g] \Big/ \begin{pmatrix} x_i x_j, y_i y_j, \\ x_i y_j - \delta_{ij}, \\ x_i y_j + y_j x_i \end{pmatrix}$

$x_i = [a_i]$, $y_i = [b_i]$

2) $M = \mathbb{CP}^n$, $H = \{z_0 = 0\} = \mathbb{CP}^{n-1}$

$a \in H^2_{DR}(M)$ Poincaré dual of the submanifold H.

$H \pitchfork H \longleftrightarrow$ Poincaré dual $a^2 \in H^4_{DR}(M)$.

$\underbrace{H \pitchfork H \pitchfork H \pitchfork \cdots \pitchfork}_{k\text{-times}} H \longleftrightarrow$ Poincaré dual $a^k \in H^{2k}_{DR}(M)$

$k = n$, $\underset{= 1}{H \pitchfork \cdots \pitchfork H = \{pt\}} \longleftrightarrow$ Poincaré dual $a^n \in \underset{\underset{\mathbb{R}}{\mathrel{\text{IS}}}}{H^{2n}_{DR}(M)}$

$\int_{\mathbb{CP}^n} a^n = 1. \Rightarrow a^k \neq 0$ in $\underset{\underset{\mathbb{R}}{\mathrel{\text{IS}}}}{H^{2k}_{DR}(\mathbb{CP}^n)}$ $k = 0, \ldots, n.$

$\Rightarrow H^{2k}_{DR}(\mathbb{CP}^n) = \langle a^k \rangle.$

$H^*_{DR}(\mathbb{CP}^n) = \mathbb{R}[a] / (a^{n+1})$

$\deg a = 2$