MATH 538
ALGEBRAIC TOPOLOGY II

Note Title 24.06.2021

METU Math. Dept.

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Textbook: Algebraic Topology by

Allen Hatcher

(Textbook is available at his Webpage)

Video 1

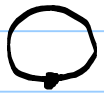
Euler Characteristic:

If X is a finite CW-complex then the Euler characteristic of X is defined to be the integer

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n c_n \, , \text{ where } c_n \text{ is the}$$

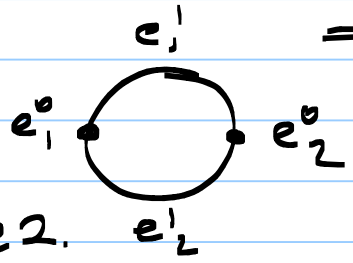
number of n-cells in X.

Example: 1) $X = S^1$, $S^1 = e^0 \cup e^1$



$c_0 = 1$, $c_1 = 1$, $c_n = 0$ if $n \geq 2$. So, $\chi(S^1) = c_0 - c_1$
$= 1 - 1$
$= 0.$

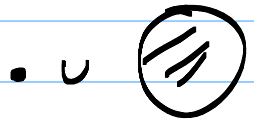
S^1: $e_1^0 \cup e_2^0 \cup e_1^1 \cup e_2^1$



$c_0 = 2$, $c_1 = 2$, $c_n = 0$, $n \geq 2$.

$\chi(S^1) = 2 - 2 = 0.$

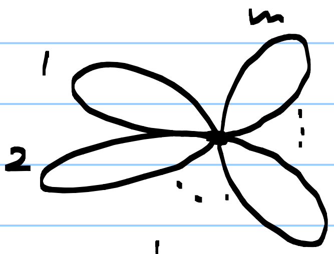
2) $S^m = e^0 \cup e^m$



$c_0 = 1$, $c_m = 1$
$c_n = 0$ if $n \neq 0$ and $n \neq m$.

$$\chi(S^m) = c_0 + (-1)^m c_m = 1 + (-1)^m = \begin{cases} 2 & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases}$$

3) $X = \bigvee_m S^1$



$= e^0 \cup e_1^1 \cup \dots \cup e_m^1$, $c_0 = 1$, $c_1 = m$
$c_n = 0$ if $n \geq 2$.

$\chi(X) = c_0 - c_1 = 1 - m$

4) $X = \mathbb{C}\mathbb{P}^n = e^0 \cup e^2 \cup e^4 \cup \dots \cup e^{2n}$

$$c_n = \begin{cases} 1 & \text{if } n=2k, \quad 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n c_n = 1 + \dots + 1 = n+1.$$

<u>Theorem:</u> $\chi(X) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(X)$

$H_n(X)$ is a finitely generated abelian group and thus

$$H_n(X) = \text{Free } H_n(X) \oplus \text{Tor } H_n(X)$$
$$\simeq \mathbb{Z}^k \oplus \text{Tor } H_n(X)$$

$\operatorname{rank} H_n(X) = k$.

Or equivalently, $H_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}^k$ $\mathbb{Q}$-vector space

$\operatorname{rank} H_n(X) = k = \dim_{\mathbb{Q}} H_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

<u>Proof</u>: <u>Exercise</u>: If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups then

$$\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C.$$

(Atiyah-MacDonald: Commutative Algebra)

X finite CW complex. Let's write cellular chain complex of X:

Say $\dim X = k$ as CW-complex

$$0 \to C_k(X) \xrightarrow{d_k} C_{k-1}(X) \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} C_1(X) \xrightarrow{d_1} C_0(X) \to 0$$

$C_i(X)$ = free abelian on the i-dimensional cells.

$$= \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{c_i\text{-copies}}$$

$Z_n = \ker d_n$, $B_n = \operatorname{Im} d_{n+1}$

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to$$

$H_n(X) = \dfrac{Z_n}{B_n}$. So we have exact sequences

$$0 \to Z_n \xrightarrow{i} C_n \xrightarrow{d_n} B_{n-1} \to 0$$ and

$$0 \to B_n \to Z_n \to H_n(X) \to 0$$

By the above exercise

$c_n = \operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1}$ and

$\operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n(X)$

So, $\chi(X) = \sum_{n=0}^{k} (-1)^n c_n = \sum_{n=0}^{k} (-1)^n \left[\operatorname{rank} Z_n + \operatorname{rank} B_{n-1}\right]$

$$= \sum_{n=0}^{k} (-1)^n \left[\operatorname{rank} B_n + \operatorname{rank} H_n(X) + \operatorname{rank} B_{n-1}\right]$$

$$= \sum_{n=0}^{k} (-1)^n \operatorname{rank} H_n(X) + \sum_{n=0}^{k} (-1)^n \left[b_n + b_{n-1}\right],$$

where $\sum_{n=0}^{k} (-1)^n (b_n + b_{n-1}) = (-1)^0 (\cancel{b_0} + 0) - (\cancel{b_1} + \cancel{b_0})$

$+ (\cancel{b_2} + \cancel{b_1})$

$\vdots$

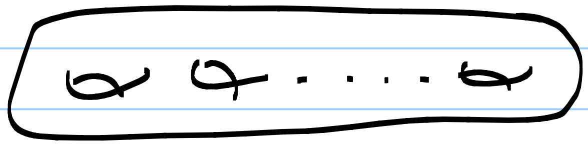
$+ (-1)^k (b_k + \cancel{b_{k-1}})$

$= b_k = 0$

because $B_k = \operatorname{Im}(d_{k+1} : \underset{\overset{\|}{0}}{C_{k+1}(X)} \to C_k(X)) = 0$

since X is k-dimensional. ■

Example: Σ_g: Orientable surface of genus g.



$H_0(\Sigma_g) \simeq \mathbb{Z}$

$H_1(\Sigma_g) \simeq \mathbb{Z}^{2g}$

$H_2(\Sigma_g) \simeq \mathbb{Z}$

$\chi(\Sigma_g) = 1 - 2g + 1$
$= 2 - 2g.$

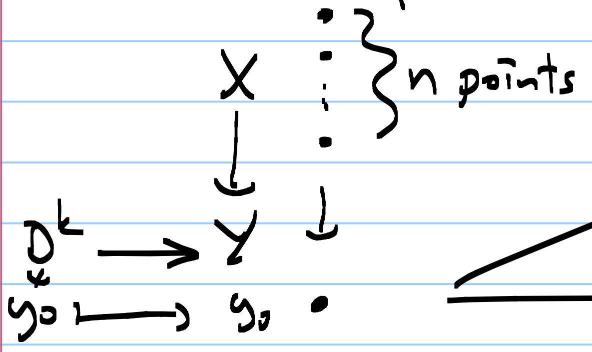
$H_k(\Sigma_g) = 0$ if $k \geq 3$.

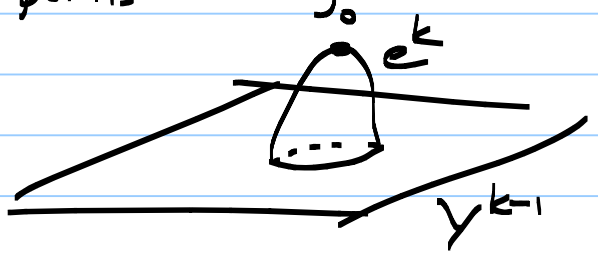
Euler Characteristic and Covering Spaces:

Let $p : X \to Y$ be a finite covering, say n-fold, of finite CW-complexes.

Proposition: $\chi(X) = n\,\chi(Y)$.

Proof: It is enough to show that the number of k-cells of X is exactly n times the number of k-cells of Y.

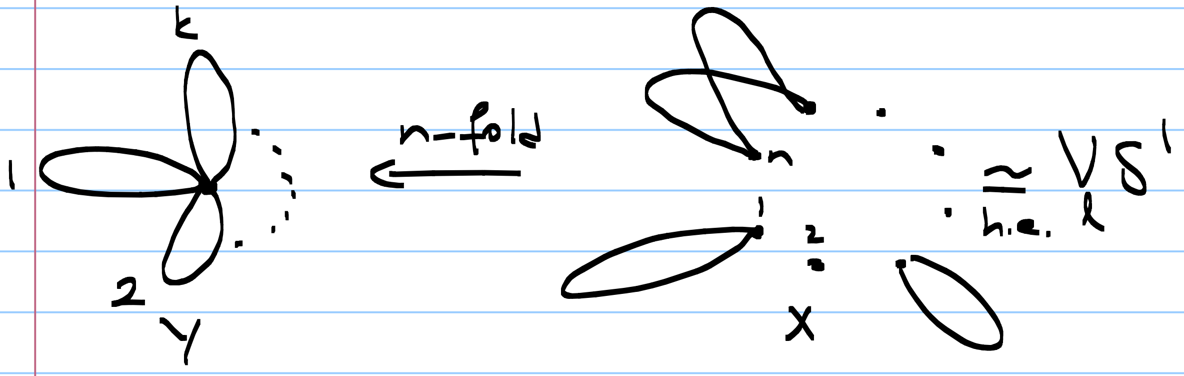




So, $\chi(X) = \sum_{k=0}^{\infty} (-1)^k c_k(X) = \sum_{k=0}^{\infty} (-1)^k n c_k(Y) = n\chi(Y)$. ■

Video 2

Example: $Y = \bigvee_k S^1$, $p: X \to Y$ n-fold covering of connected spaces



$$\chi(X) = n\,\chi(Y) \Rightarrow 1-\ell = n(1-k)$$

$$Y = \bigvee_k S^1 \Rightarrow \pi_1(Y) \simeq Fr_k$$

$$X = \bigvee_\ell S^1 \Rightarrow \pi_1(Y) \simeq Fr_\ell$$

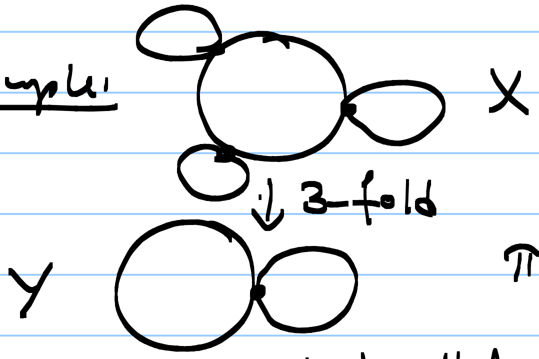
$$\pi_1(X) \xrightarrow{p_\#} \pi_1(Y) \text{ injection.}$$

$$Fr_\ell \overset{n}{\leq} Fr_k \text{ and } n = \frac{\ell-1}{k-1}$$

Proposition: If H is a finite index subgroup of the free on k-letters then H is a free group on ℓ-letters, where

$$(\ell-1) = n(k-1).$$

Example:



$\pi_1(Y) \simeq Fr_4$

$\pi_1(Y) \simeq Fr_2$

Note that $4-1 = 3\cdot(2-1)$

Homology with Coefficients:

Let G be an abelian group and consider the group $C_n(X;G) = \{ \sum n_i \sigma_i \mid \sigma_i$ singular simplex in X, $n_i \in G \}$

$\partial_n : C_n(X;G) \longrightarrow C_{n-1}(X;G)$ by

$\partial_n(\sum n_i \sigma_i) = \sum n_i \, \partial_n(\sigma_i)$.

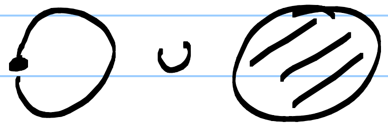
Clearly, $\partial_n \circ \partial_{n+1} = 0$ and thus we define its homology as

$$H_n(X;G) \doteq \frac{\ker \partial n}{\mathrm{Im}\, \partial_{n+1}},$$ the n^{th}

singular homology of X with G-coefficients.

All the theory we developed for $\mathbb{Z}$-coefficient will hold also for G-coefficients.

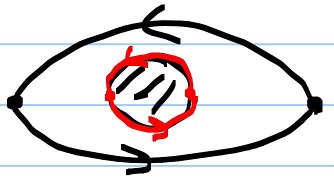
Example: $\mathbb{RP}^2 = e^0 \cup e^1 \cup e^2$



$e^0 \cup e^1 = S^1$

$\partial D^2 = S^1 \longrightarrow S^1$

$z \longmapsto z^2$



Cellular Homology for $\mathbb{Z}$-coefficients:

$$\begin{tikzcd} 0 \arrow[r] & C_2(\mathbb{RP}^2) \arrow[r, "d_2"] \arrow[d, phantom, "\cong"] & C_1(\mathbb{RP}^2) \arrow[r, "d_1 = 0"] \arrow[d, phantom, "\cong"] & C_0(\mathbb{RP}^2) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\ & \mathbb{Z} \arrow[r, "\times 2"'] & \mathbb{Z} \arrow[r, "\times 0"'] & \mathbb{Z} & \end{tikzcd}$$

$H_0(\mathbb{RP}^2) \simeq \mathbb{Z}$, $H_1(\mathbb{RP}^2) \simeq \frac{\ker d_1}{\mathrm{Im}\, d_2} = \frac{\mathbb{Z}}{2\mathbb{Z}} \simeq \mathbb{Z}_2$

and $H_2(\mathbb{RP}^2) \simeq \frac{\ker d_2}{\operatorname{Im} d_3} = \frac{0}{0} \simeq 0$.

What about homologies with $\mathbb{Z}_2$-coefficients

$$\begin{tikzcd}
0 \arrow[r] & C_2(\mathbb{RP}^2;\mathbb{Z}_2) \arrow[r, "d_2"] \arrow[d, phantom, sloped, "\cong"] & C_1(\mathbb{RP}^2;\mathbb{Z}_2) \arrow[r, "d_1"] \arrow[d, phantom, sloped, "\cong"] & C_0(\mathbb{RP}^2;\mathbb{Z}_2) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & 0 \\
& \mathbb{Z}_2 \arrow[r, "\times 2", "=0"'] & \mathbb{Z}_2 \arrow[r, "0"] & \mathbb{Z}_2 &
\end{tikzcd}$$

$H_0(\mathbb{RP}^2;\mathbb{Z}_2) \simeq \mathbb{Z}_2$, $H_1(\mathbb{RP}^2;\mathbb{Z}_2) \simeq \frac{\ker d_1}{\operatorname{Im} d_2} = \frac{\mathbb{Z}_2}{(0)} \simeq \mathbb{Z}_2$

$H_2(\mathbb{RP}^2;\mathbb{Z}_2) \simeq \frac{\ker d_2}{\operatorname{Im} d_3} = \frac{\mathbb{Z}_2}{(0)} \simeq \mathbb{Z}_2$.

(Universal Coefficient Theorem)

Lemma: If $f: S^k \to S^k$ has degree m, then $f_*: H_k(S^k;G) \to H_k(S^k;G)$ is multiplication by m.

Proof: If $\varphi: G_1 \to G_2$ is a homomorphism of abelian groups then φ induces a homomorphism on chain groups and homology:

$$\varphi_\#: C_n(X,A;G_1) \longrightarrow C_n(X,A;G_2)$$

$$\sum n_i \sigma_i \longmapsto \sum \varphi(n_i)\sigma_i$$

$n_i \in G_1$

$\partial \circ \varphi_\# = \varphi_\# \circ \partial$ and thus we get a homomorphism on homology level:

$$\varphi_* : H_n(X,A;G_1) \longrightarrow H_n(X,A;G_2).$$

Back to the lemma: $f : S^k \longrightarrow S^k$ degree m map.

$$\begin{tikzcd}
1 \arrow[rrr, mapsto, bend left] \arrow[dddd, mapsto] & \mathbb{Z} \arrow[r, "\times m"] \arrow[d, phantom, sloped, "\cong"] & \mathbb{Z} \arrow[d, phantom, sloped, "\cong"] & m \arrow[dddd, mapsto] \\
 & H_k(S^k;\mathbb{Z}) \arrow[r, "f_*"] \arrow[d, "\varphi_*"] & H_k(S^k;\mathbb{Z}) \arrow[d, "\varphi_*"] & \\
 & H_k(S^k;G) \arrow[r, "f_*"'] \arrow[d, phantom, sloped, "\cong"] & H_k(S^k;G) \arrow[d, phantom, sloped, "\cong"] & \\
 & G \arrow[r] \arrow[d, phantom, sloped, "\ni"] & G & \\
g \arrow[rrrd, mapsto, bend right] & g \arrow[r, mapsto] & ? & mg \arrow[d, phantom, sloped, "="] \\
 & & & f_*(g)
\end{tikzcd}$$

$$\boxed{\begin{aligned} \varphi &: \mathbb{Z} \to G \\ 1 &\longmapsto g \end{aligned}}$$

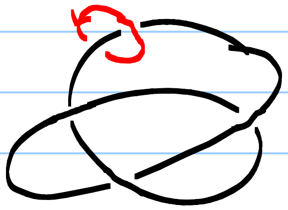
Video 3

Classical Applications of Homology

Proposition: a) For any embedding $h: D^k \to S^n$, $\tilde{H}_i(S^n \setminus h(D^k)) = 0$, for all i.

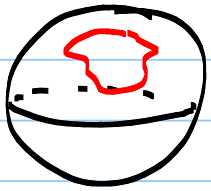
b) For any embedding $h: S^k \to S^n$ with $k<n$, $\tilde{H}_i(S^n \setminus h(S^k))$ is $\mathbb{Z}$ for $i=n-k-1$ and zero otherwise.

Remark: Special cases for b) 1) $h: S^1 \to S^3$



$\tilde{H}_1(S^3 \setminus h(S^1)) \simeq \mathbb{Z} = \langle \gamma \rangle$

2) $h: S^1 \to S^2$



$\tilde{H}_0(S^2 \setminus h(S^1)) \simeq \mathbb{Z}$

$\Rightarrow H_0(S^2 \setminus h(S^1)) \simeq \mathbb{Z} \oplus \mathbb{Z}$

$S^2 \setminus h(S^1)$ has two connected components. Since $h(S^1)$ is (compact) closed $S^2 \setminus h(S^1)$ is open and thus $S^2 \setminus h(S^1)$ has two path connected components. (Jordan Closed Curve Theorem!)

Proof: a) Induction on k.

$k=0$, $h: D^k = D^0 \to S^n$ embedding. $D^0 = \{pt\}$ and thus $S^n \setminus h(D^0) \simeq \mathbb{R}^n$ and therefore $\tilde{H}_i(S^n \setminus h(D^0)) = \tilde{H}_i(\mathbb{R}^n) = 0$, for all i.

Assume that $k \geq 1$ and the result holds for $k-1$.

Since D^k is homeomorphic to $I^k = \overbrace{I \times \cdots \times I}^{k\text{-times}}$ we may work with I^k instead of D^k.

Let $h: I^k \to S^n$ be an embedding and let

$A = S^n \setminus h(I^{k-1} \times [0, 1/2])$ and
$B = S^n \setminus h(I^{k-1} \times [1/2, 1])$.

Now, $A \cap B = S^n \setminus h(I^k)$ and
$$A \cup B = S^n \setminus h(\underbrace{I^{k-1} \times \{1/2\}}_{\substack{\text{is}\\ I^{k-1}}})$$

So by the induction hypothesis $\tilde{H}_i(A \cup B) = 0$ for all i.

Apply M–V sequence to $A \cup B$:

$$\to \underset{\substack{\text{is}\\ 0}}{\tilde{H}_{i+1}(A \cup B)} \to \tilde{H}_i(A \cap B) \xrightarrow[\substack{\uparrow\\ \cong}]{} \tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \underset{\substack{\text{is}\\ 0}}{\tilde{H}_i(A \cup B)}$$

Hence, the map $\tilde{H}_i(A \cap B) \to \tilde{H}_i(A) \oplus \tilde{H}_i(B)$ is an isomorphism for all i.

Assume on the contrary that there is some cycle α in $A \cap B$ so that $[\alpha] \neq 0$ in $\tilde{H}_i(A \cap B)$. Hence α is not a boundary in A or B. By playing the same inductively, we get a sequence of nested intervals

$$I_1 = [0, 1] \supset I_2 = [0, 1/2] \supseteq I_3 = [1/4, 1/2] \supseteq \cdots \supseteq I_m \supseteq$$

so that α is not a boundary in

$\tilde{H}_i(S^n \setminus h(I^{k-1} \times I_m))$, for all m.

However, we know that $\alpha = \partial\beta$ in $S^n \setminus h(\underbrace{I^{k-1} \times \{p\}}_{I^{k-1}})$
where $\{p\} = \bigcap_{m=1}^{\infty} I_m$

Since the support of β is compact and

$$B \subseteq S^n \setminus h(I^{k-1}\times\{p\}) = S^n \setminus h\Big(\bigcap_{m=1}^{\infty}(I^{k-1}\times I_m)\Big)$$

(h is an embedding) $$= \bigcup_{m=1}^{\infty} \overbrace{S^n \setminus \underbrace{h(\underbrace{I^{k-1}\times I_m}_{\text{compact}})}_{\text{compact}}}^{\text{open in } S^n}$$

Since the support of B is compact, it lies in some $S^n \setminus h(I^{k-1}\times I_m)$ for some m.

Hence, $\alpha = \partial \beta$ in $S^n \setminus h(I^{k-1}\times I_m)$ so that $[\alpha]=0$ in $\tilde{H}_i(S^n\setminus h(I^{k-1}\times I_m))$, a contradiction.

So $\tilde{H}_i(A\cap B) = \tilde{H}_i(S^n \setminus h(I^k)) = 0$.

This finishes the proof of part (a).

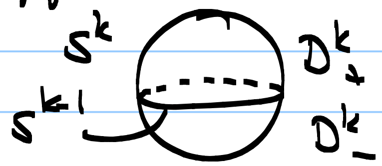
b) <u>Induction on k.</u>

→ $k=0$, $$S^n\setminus h(S^0) = S^n \setminus h(\overset{\text{2 points}}{\overset{\downarrow}{\pm p}}) = \mathbb{R}^n \setminus \{p\}$$
$$= S^{n-1}\times\mathbb{R}$$
$$\underset{h.e.}{\simeq} S^{n-1}$$

$$\tilde{H}_i(S^n\setminus h(S^0)) \simeq \tilde{H}_i(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } i=n-1 \\ 0 & \text{otherwise.}\end{cases}$$

Assume now $k\geq 1$ and suppose the result hold for $k-1$.

Write $S^k = D^k_+ \cup D^k_-$



Set, $A = S^n\setminus h(D^k_+)$, $B = S^n \setminus h(D^k_-)$.

$$A\cap B = S^n \setminus h(S^k) \text{ and } A\cup B = S^n \setminus \underset{\overset{\cong}{S^{k-1}}}{(D^k_+\cap D^k_-)}$$

Write down the M-V sequence:

Video 4

by Part (a)

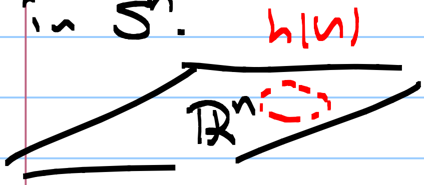
$$\overset{0}{\overset{\cong}{\tilde H_i(A)}} \oplus \overset{0}{\overset{\cong}{\tilde H_i(B)}} \to \tilde H_i(A\cup B) \xrightarrow{\simeq} \tilde H_{i-1}(A\cap B) \to \overset{0}{\overset{\cong}{\tilde H_{i-1}(A) \oplus H_{i-1}(B)}}$$

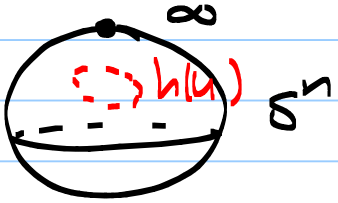
Hence, $\tilde H_i(S^n \setminus h(S^{k-1})) \simeq \tilde H_{i-1}(S^n \setminus h(S^k))$, which finishes the proof. ■

Theorem: If U is an open set in $\mathbb{R}^n$, then for any embedding $h: U \to \mathbb{R}^n$ the image $h(U)$ is also open in $\mathbb{R}^n$.

This result is called as the Invariance of Domain.

Proof: $S^n = \mathbb{R}^n \cup \{+\infty\}$, the one point compactification of $\mathbb{R}^n$ and thus it is enough to prove that $h(U)$ is open in S^n, because $\mathbb{R}^n$ is open in S^n.





So we'll prove that $h(U)$ is open in $S^n = \mathbb{R}^n \cup \{\infty\}$. For any $x \in U$ there is a disk $D^n_x \subset U$ so that x is its center.

Note that it is enough to show that $h(D^n_x \setminus \partial D^n_x)$ is open in S^n, because

$$U = \bigcup_{x\in U} (D^n_x \setminus \partial D^n_x).$$

must show: $h(D^n \setminus \partial D^n)$ is open in S^n.

$\partial D^n = S^{n-1}$, $S^n \setminus h(D^n \setminus \partial D^n) = S^n \setminus h(S^{n-1})$ is open and has exactly two connected components.

Note that these components are $h(D^n \setminus \partial D^n)$ and $S^n \setminus h(D^n)$, because they are both connected and

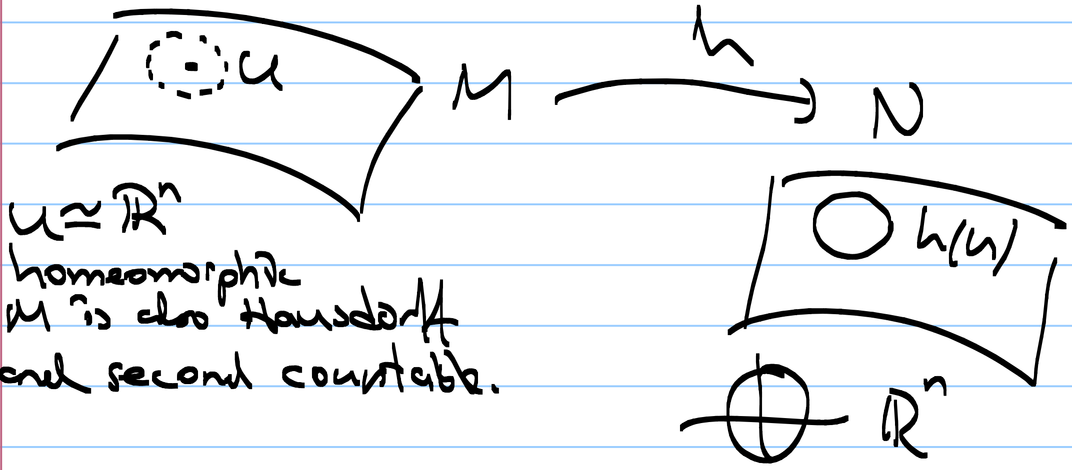
$$(S^n \setminus h(D^n)) \cup (h(D^n \setminus \partial D^n)) = S^n \setminus h(\partial D^n)$$
$$= S^n \setminus h(S^{n-1})$$

Since $S^n \setminus h(\partial D^n)$ is open in S^n, the path components are the same as its components. The components of a space with finitely many components are open, so $h(D^n \setminus \partial D^n)$ is open in $S^n \setminus h(\partial D^n)$ and hence also in S^n.

Proof: $X = E_1 \cup \dots \cup E_n$, E_i topological components. Then $\bar{E_i} = E_i$ because $\bar{E_i}$ is also connected. So each E_i is closed and thus open.

Corollary If M is a compact n-fold and N is a connected n-manifold, then an embedding $h: M \to N$ must be surjective, hence a homeomorphism.

Proof: $h(M)$ is compact in N and thus it is a closed subset.



$U \simeq \mathbb{R}^n$
homeomorphic
M is also Hausdorff and second countable.

Since M is a union of open sets each of which is homeomorphic to $\mathbb{R}^n$, by the Invariance of Domain Theorem, $h(M)$ is open in N. Thus, $h(M)$ is both closed and open in the connected space N. Then $h(M)=N$. Since $h: M \to N$ is an embedding we see that h is a homeomorphism.

Remark: If M^n is compact n-dim'l manifold then there is no embedding of M^n into $\mathbb{R}^n$: Indeed, if $h: M^n \to \mathbb{R}^n$ is an embedding then by the above corollary h must be onto, because $\mathbb{R}^n$ is connected. Then $\mathbb{R}^n = h(M^n)$ must be compact, a clear contradiction. ∎

Division Algebras:

Theorem: $\mathbb{R}$ and $\mathbb{C}$ are the only finite-dimensional division algebras over $\mathbb{R}$, which are commutative and have an identity.

An algebra structure on $\mathbb{R}^n$ is a bilinear multiplication map

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad (a,b) \mapsto ab, \text{ so that}$$

$a(b+c) = ab+ac$ and $(a+b)c = ac+bc$, for all $a,b,c \in \mathbb{R}^n$. This algebra is called a division algebra if the equations

$$ax = b \quad \text{and} \quad xa = b$$

are always solvable whenever $a \neq 0$.

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{H}$ Quaternions, which form a noncommutative division algebra.

$\mathbb{C} \simeq \mathbb{R}^2$, $\mathbb{H} \simeq \mathbb{R}^4$ $\quad$ $i \cdot j = k$, $j \cdot k = i$, $k \cdot i = j$

$1, i$ $\quad$ $1, i, j, k$ $\quad$ $i^2 = j^2 = k^2 = -1$

Proof of the theorem: Suppose that $\mathbb{R}^n$ has a commutative division algebra structure. Consider the map

$$f: S^{n-1} \to S^{n-1}, \; f(x) = \frac{x^2}{|x^2|}.$$

$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

Note that since $x \in S^{n-1}$, $x \neq 0$ and thus $x^2 \neq 0$. $x^2 \neq 0$ because $y \cdot x = 1$ has a solution and thus if $x^2 = 0$ then $0 = y \cdot x^2 = y \cdot x \cdot x = 1 \cdot x = x$, a contradiction.

Also note that $f(-x) = \frac{(-x)\cdot(-x)}{|(-x)^2|} = \frac{(-1)^2 \cdot x^2}{|x^2|} = f(x)$ for all $x \in S^{n-1}$.

$(a(\alpha b) = \alpha(ab) = (\alpha a)b$, for all $a, b \in \mathbb{R}^n, \alpha \in \mathbb{R})$

$$\begin{tikzcd}
S^{n-1} \arrow[r, "f"] \arrow[d, "p"'] & S^{n-1} \\
S^{n-1}/_{x\sim -x} & \cong \mathbb{RP}^{n-1} \arrow[u, "\bar{f}"']
\end{tikzcd}$$

So, f induces a map $\bar{f}: \mathbb{RP}^{n-1} \to S^{n-1}$ so that

$$f = \bar{f} \circ p.$$

Claim: $\bar{f}$ is injective.

Proof: If $f(x) = f(y)$ then $\frac{x^2}{|x^2|} = \frac{y^2}{|y^2|}$ and thus

$x^2 = \alpha^2 y^2$, $\alpha = \left(\frac{|x^2|}{|y^2|}\right)^{1/2} > 0$.

So, $0 = x^2 - \alpha^2 y^2 = (x + \alpha y) \cdot (x - \alpha y)$ (note that α is a scalar and multiplication is linear. Since we are in a division algebra

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$x = \pm \alpha y$. Also x, y are unit vectors and $\alpha \in \mathbb{R}$ we see that $x = \pm y$. So $\mathcal{I}(x) = \mathcal{I}(y)$ so that $\bar{f}$ is injective.

$\bar{f}: \mathbb{RP}^{n-1} \to S^{n-1}$ one to one continuous, where $\mathbb{RP}^{n-1} = S^{n-1}/_{x\sim -x}$ is compact and S^{n-1} is Hausdorff. So by the Homeomorphism theorem $\bar{f}$ is a homeomorphism onto its image. Moreover, by the previous corollary $\bar{f}$ is onto and $\bar{f}: \mathbb{RP}^{n-1} \to S^{n-1}$ is a homeomorphism, which is a contradiction unless $n \leq 2$, because

$$\pi_1(\mathbb{RP}^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } n=2 \\ \mathbb{Z}_2 & \text{if } n>2 \end{cases} \quad \text{and}$$

$$\pi_1(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } n=2 \\ 0 & \text{if } n>2 \end{cases}.$$

If $n=1$ then we have $\mathbb{R}^n = \mathbb{R}$ and since $\mathbb{R}^n$ is an $\mathbb{R}$-algebra we see that $\mathbb{R}^n = \mathbb{R}$ has its standard algebra structure.

For the $n=2$ case we need to show that the only possible commutative division algebra structure on $\mathbb{R}^2$, having identity element, then it must be $\mathbb{C}$:
If $J \in \mathbb{R}^2$ and is not a scalar multiple of 1: $J \neq x \cdot 1$, for any $x \in \mathbb{R}$.

Now $J^2 = a \cdot 1 + bJ$, for some $a, b \in \mathbb{R}$.

Then $(J - b/2)^2 = J^2 - bJ + \frac{b^2}{4}$

$$= a + \cancel{bJ} - \cancel{bJ} + \frac{b^2}{4}$$
$$= a + b^2/4$$

So by rechoosing j we may assume that $j^2 = a \in \mathbb{R}$. If $a \geq 0$, say $j^2 = c^2$, for some $c \in \mathbb{R}$, $c \geq 0$.

Now $\bar{j}^2 - c^2 = 0$ and thus $(j-c)(j+c) = 0$. Since we are in a division algebra we see that $\bar{j} = c$ or $\bar{j} = -c$, which is a contradiction since $c \in \mathbb{R}$ and j is not. Hence, $\bar{j}^2 = a < 0$. So, $\bar{j}^2 = -c^2$ for some $c > 0$. Now rescaling j we see that $j^2 = -1$ (just replace j by j/c).

$\mathbb{R}^2$ has an $\mathbb{R}$-basis $\{1, \bar{j}\}$, where $\bar{j}^2 = -1$. So, with this division algebra structure $\mathbb{R}^2$ becomes $\mathbb{C}$. ∎

The Borsuk–Ulam Theorem:

Proposition: Any odd map $f : S^n \to S^n$, satisfying $f(-x) = -f(x)$, for all x, must have odd degree.

For the proof we'll need so called transfer homomorphism for covering spaces:

Say $p : \tilde{X} \to X$ be a two-sheeted covering space.

$$\begin{tikzcd} \tilde{X} \arrow[d, "p"'] \\ X \end{tikzcd}$$

Then we have a short exact sequence of chain groups

$$0 \to C_n(X;\mathbb{Z}_2) \xrightarrow{\tau} C_n(\tilde{X};\mathbb{Z}_2) \xrightarrow{p_\#} C_n(X;\mathbb{Z}_2) \to 0$$

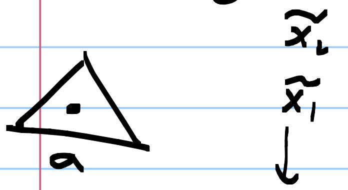
where $p_\#$ is induced by the projection map $p: \tilde{X} \to X$ and τ is defined as follows:

$$\tau : C_n(X;\mathbb{Z}_2) \to C_n(\tilde{X};\mathbb{Z}_2)$$

$i=1,2$
$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \Delta^n \arrow[ur, "\tilde{\sigma}_i"] \arrow[r, "\sigma"'] & X \end{tikzcd}$$

A typical element of $C_n(X;\mathbb{Z}_2)$ has the form

$\sum a_\sigma \sigma_j \quad a_\sigma \in \mathbb{Z}_2$



$\sigma : \Delta^n \to X$

$\sigma(a) = x$

$\tilde{\sigma}_1 : \Delta^n \to \tilde{X}$, $\tilde{\sigma}_1(a) = \tilde{x}_1$ and $\tilde{\sigma}_2 : \Delta^n \to \tilde{X}$ $\tilde{\sigma}_2(a) = \tilde{x}_2$.

Now define $\tau(\sigma) = \tilde{\sigma}_1 + \tilde{\sigma}_2$ and then extend to $C_n(X;\mathbb{Z}_2)$ linearly.

Why short exact?

$$0 \to C_n(X;\mathbb{Z}_2) \xrightarrow{\tau} C_n(\tilde{X};\mathbb{Z}_2) \xrightarrow{p_\#} \underline{\underline{C_n(X;\mathbb{Z}_2)}} \to 0$$

<u>must show:</u>

1) τ is injective: $\sigma : \Delta^n \to X$, $\tilde{\sigma}_i : \Delta^n \to X$

$$\sum_\alpha \sigma_\alpha \xmapsto{\tau} \sum_\alpha \tilde{\sigma}_{\alpha 1} + \tilde{\sigma}_{\alpha 2}$$

2) $p_\#$ is surjective

3) $(p_\# \circ \tau)(\sum_\alpha \sigma_\alpha) = \sum_\alpha p_\#(\tilde{\sigma}_{\alpha_1} + \tilde{\sigma}_{\alpha_2})$

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$$= \sum_\alpha (\sigma_\alpha + \sigma_\alpha)$$

$$= \sum_\alpha 2\sigma_\alpha = 0 \text{ in } \mathbb{Z}_2\text{-coeff.}$$

$\Rightarrow \operatorname{Im}\tau \subseteq \ker p_\#$

4) $\ker p_\# \subseteq \operatorname{Im}\tau$: $\quad p_\# : C_n(\tilde{X};\mathbb{Z}_2) \to C_n(X;\mathbb{Z}_2)$

$\sum_\alpha \tilde{\sigma}_\alpha \in C_n(\tilde{X};\mathbb{Z}_2)$, $p_\#(\sum_\alpha \tilde{\sigma}_\alpha) = \sum_\alpha \sigma_\alpha = 0$

Hence, each σ_α is cancelled by some $\sigma_{\bar{\alpha}}$ for some $\bar{\alpha}$. It follows that $\sigma_\alpha = \sigma_{\bar{\alpha}}$. In other words $\sum_\alpha \sigma_\alpha$ in $C_n(\tilde{X};\mathbb{Z}_2)$ has the form

$$\sum_\alpha \sigma_\alpha = \sum_\beta \tilde{\xi}_\beta + \tilde{\eta}_\beta, \text{ where}$$

$\tilde{\xi}_\beta$ and $\tilde{\eta}_\beta$ are the two disjoint lifts of the same simplex δ_β in $C_n(X,\mathbb{Z}_2)$.

So, $\sum_\alpha \sigma_\alpha = \tau(\sum_\beta \delta_\beta)$.

Hence, $0 \to C_n(X;\mathbb{Z}_2) \xrightarrow{\tau} C_n(\tilde{X};\mathbb{Z}_2) \xrightarrow{p_\#} C_n(X;\mathbb{Z}_2) \to 0$ is a short exact sequence of chain groups.

$$\begin{tikzcd}
 & {} \arrow[d, "\partial"] & {} \arrow[d, "\partial"] & {} \arrow[d, "\partial"] & \\
0 \arrow[r] & C_n(X;\mathbb{Z}_2) \arrow[r, "\tau"] \arrow[d, "\partial"] & C_n(\tilde{X};\mathbb{Z}_2) \arrow[r, "p_\#"] \arrow[d, "\partial"] & C_n(X,\mathbb{Z}_2) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{n-1}(X;\mathbb{Z}_2) \arrow[r, "\tau"] \arrow[d, "\partial"] & C_{n-1}(\tilde{X};\mathbb{Z}_2) \arrow[r, "p_\#"] \arrow[d, "\partial"] & C_{n-1}(X;\mathbb{Z}_2) \arrow[r] \arrow[d, "\partial"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

This induces a long exact sequence of homology groups

$$\to H_n(X;\mathbb{Z}_2) \xrightarrow{\tau_*} H_n(\tilde{X};\mathbb{Z}_2) \xrightarrow{p_*} H_n(X;\mathbb{Z}_2) \xrightarrow{\partial} H_{n-1}(X;\mathbb{Z}_2)$$

τ and τ_* are called transfer homomorphism.

Proof of the Proposition:

$f: S^n \to S^n$, $f(-x) = -f(x)$ and thus f induces a map $\bar{f}: \mathbb{RP}^n \to \mathbb{RP}^n$

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "p"'] & S^n \arrow[d, "p"] \\ \mathbb{RP}^n \arrow[r, "\bar{f}"] & \mathbb{RP}^n \end{tikzcd}$$

, where $p \circ f = \bar{f} \circ p$.

$[-x] = [x]$

We'll make use the transfer homomorphism for the double cover $p: S^n = \tilde{X} \to X = \mathbb{RP}^n$:

$$\begin{tikzcd} 0 \arrow[r] & H_n(\mathbb{RP}^n) \arrow[r, "\tau_*"] \arrow[d, phantom, sloped, "\cong"] & H_n(S^n) \arrow[r, "p_*"] \arrow[d, phantom, sloped, "\cong"] & H_n(\mathbb{RP}^n) \arrow[r, "\partial"] \arrow[d, phantom, sloped, "\cong"] & H_{n-1}(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & 0 \\ & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \end{tikzcd}$$

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & H_i(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_{i-1}(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & 0 \arrow[r] & {} \\ & & \mathbb{Z}_2 & \mathbb{Z}_2 & & \end{tikzcd}$$

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & H_1(\mathbb{RP}^n) \arrow[r, "\cong"'] \arrow[d, phantom, sloped, "\cong"] & H_0(\mathbb{RP}^n) \arrow[r, "= 0"'] \arrow[d, phantom, sloped, "\cong"] & H_0(S^n) \arrow[r, "p_*", "\cong"'] \arrow[d, phantom, sloped, "\cong"] & H_0(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & 0 \\ & & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \end{tikzcd}$$

The odd map $f: S^n \to S^n$ and $\bar{f}: \mathbb{RP}^n \to \mathbb{RP}^n$ induce a map from the transfer sequence to itself as follows:

$$\begin{tikzcd} 0 \arrow[r] & C_i(\mathbb{RP}^n) \arrow[r, "\tau"] \arrow[d, "\bar{f}_\#"] & C_i(S^n) \arrow[r, "p_\#"] \arrow[d, "f_\#"] & C_i(\mathbb{RP}^n) \arrow[r] \arrow[d, "\bar{f}_\#"] & 0 \\ 0 \arrow[r] & C_i(\mathbb{RP}^n) \arrow[r, "\tau"'] & C_i(S^n) \arrow[r, "p_\#"'] & C_i(\mathbb{RP}^n) \arrow[r] & 0 \end{tikzcd}$$

This diagram is commutative.

The above diagram induces a ladder of commutative diagrams

We can see that all the maps f_* and $\bar{f}_*$ in the commutative diagram below are isomorphism by induction on dimension, using the fact that if three maps in a commutative square are isomorphisms, so is the fourth. The induction starts with the trivial fact that f_* and $\bar{f}_*$ are isomorphisms in dimension zero.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, phantom, "\Rightarrow"] & H_1(\mathbb{RP}^n) \arrow[r, "\cong"] \arrow[d, "\cong"', "\bar{f}_*"] & H_0(\mathbb{RP}^n) \arrow[r, "0"] \arrow[d, "\cong"', "\bar{f}_*"] & H_0(S^n) \arrow[r, "\cong", "p_*"'] \arrow[d, "f_*"', "\cong"] & H_0(\mathbb{RP}^n) \arrow[r] \arrow[d, "\cong"', "f_*"] & 0 \\
\arrow[r] & 0 \arrow[r] & H_1(\mathbb{RP}^n) \arrow[r, "\cong"'] & H_0(\mathbb{RP}^n) \arrow[r, "0"'] & H_0(S^n) \arrow[r, "\cong", "p_*"'] & H_0(\mathbb{RP}^n) \arrow[r] & 0
\end{tikzcd}$$

$\vdots$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, phantom, "\Rightarrow"] & H_i(\mathbb{RP}^n) \arrow[r, "\cong"] \arrow[d, "\cong"', "\bar{f}_*"] & H_{i-1}(\mathbb{RP}^n) \arrow[r] \arrow[d, "\cong"', "\bar{f}_*"] & 0 \\
0 \arrow[r] & H_i(\mathbb{RP}^n) \arrow[r, "\cong"'] & H_{i-1}(\mathbb{RP}^n) \arrow[r] & 0
\end{tikzcd}$$

Here all f_* and $\bar{f}_*$ are isomorphism.

In particular, $f_*: \underset{\substack{\cong \\ \mathbb{Z}_2}}{H_n(S^n;\mathbb{Z}_2)} \to \underset{\substack{\cong \\ \mathbb{Z}_2}}{H_n(S^n;\mathbb{Z}_2)}$ is an isomorphism.

Hence, the $\mathbb{Z}_2$-degree of f is one.

$$\begin{aligned} \mathbb{Z} &\to \mathbb{Z}_2 \\ n &\longmapsto 1 \end{aligned}$$

Finally, since the $\mathbb{Z}_2$-degree of f is the mod 2 reduction of the integer degree of f we see that the integer degree of f

is an odd integer.

This finishes the proof. ∎

Corollary: For every map $g: S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ with $g(x) = g(-x)$.

Proof: Let $f(x) = g(x) - g(-x)$, so f is odd.

Want: $f(x) = 0$ for some $x \in S^n$.

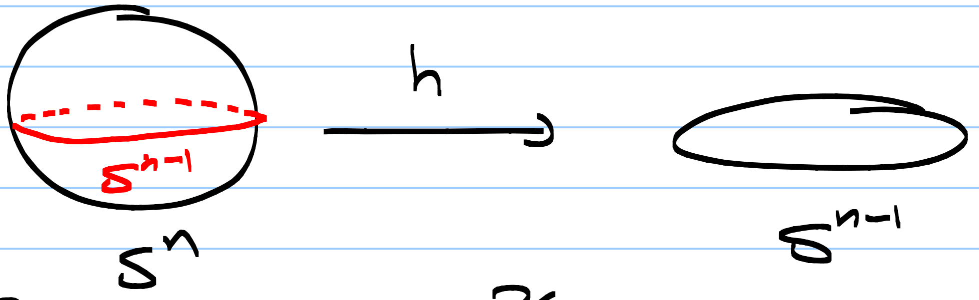
Suppose that such $x \in S^n$ does not exist. Then, $f(x) \neq 0$ for all $x \in S^n$. Consider the function

$$h: S^n \to S^{n-1}, \quad h(x) = \frac{f(x)}{\|f(x)\|}.$$

Note that this function is still odd:

$$h(-x) = -h(x)$$

The restriction of h to the equator S^{n-1} of S^n has odd degree by the above result:



$$\begin{tikzcd}
\mathbb{Z} & & \mathbb{Z} \\
H_{n-1}(S^{n-1}) \arrow[u, phantom, sloped, "\cong"] \arrow[rr, "h_*", "\times(2k-1)"'] \arrow[d, "i_*"'] & & H_n(S^{n-1}) \arrow[u, phantom, sloped, "\cong"] \\
H_{n-1}(S^n) \arrow[urr, "h_*"'] & & \\
0 \arrow[u, phantom, sloped, "\cong"] & &
\end{tikzcd}$$

$(n \geq 2)$

$h_* = h_* \circ i_*$

$\times(2k-1) = 0$

$\Rightarrow$ contradiction! ∎

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Simplicial Approximation:

Theorem: If X is a finite simplicial complex, or more generally retract of a finite simplicial complex, and $f: X \to X$ is a map with $T(f) \neq 0$, then f has a fixed point.

This is known as Lefschetz Fixed Point Theorem.

$$T(f) = \sum_{k=0}^{\infty} (-1)^k \, tr(f_*: H_k(X) \to H_k(X))$$

$H_i(X) \cong \mathbb{Z}^{b_k} \oplus Tor$, β basis for $\mathbb{Z}^{b_k}$.

$$[f_*]_\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ & a_{22} & \\ & & \ddots \end{bmatrix}_{b_k \times b_k} \qquad tr(f_*) = a_{11} + a_{22} + \cdots + a_{b_k b_k}$$

$$\beta = \{e_1, \ldots, e_{b_k}\} \qquad f_*(e_j) = \sum_{i=1}^{b_k} a_{ij} e_i$$

Example: $f: D^n \to D^n$ any continuous map, when we regard D^n as Δ^n. Then

$$H_k(D^n) = \begin{cases} \mathbb{Z} & k=0 \\ 0 & k>0 \end{cases}$$

$f_*: H_0(D^n) \to H_0(D^n)$, $f_* = id$, $tr(f_*) = 1$

If $k > 0$, then $H_k(D^n) = 0$ and thus $tr(f_*) = 0$.

$$T(f) = \sum_{k=0}^{\infty} (-1)^k \, tr(f_*: H_k(D^n) \to H_k(D^n))$$

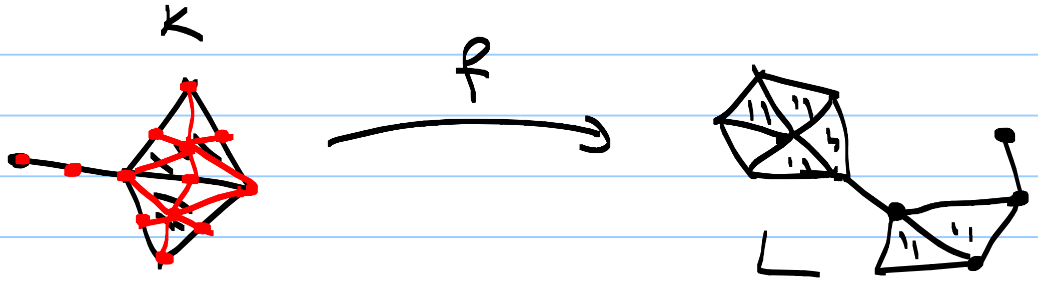
$$= 1 \neq 0$$

So by the Theorem above, f must have a fixed point.

To prove this we need some preparation:

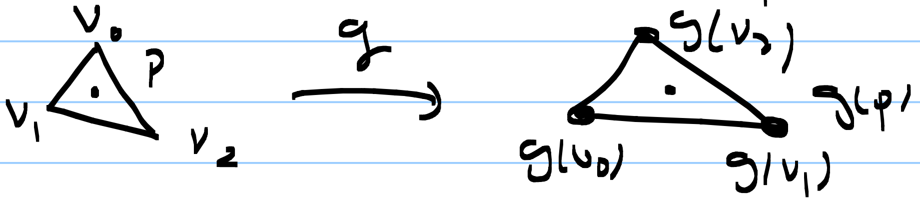
Theorem (Simplicial Approximation Theorem)

If K is a finite simplicial complex and L is an arbitrary simplicial complex, then any map $f: K \to L$ is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K.



$f \sim g$ $\quad g(v) \in L^0$, for any $v \in (K')^0$

g is linear on each simplex of K'.



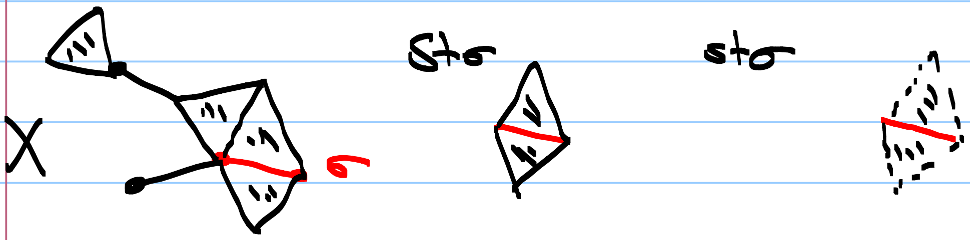
$$p = \sum_{i=0}^{2} a_i v_i, \quad a_i \geq 0, \quad \sum a_i = 1$$

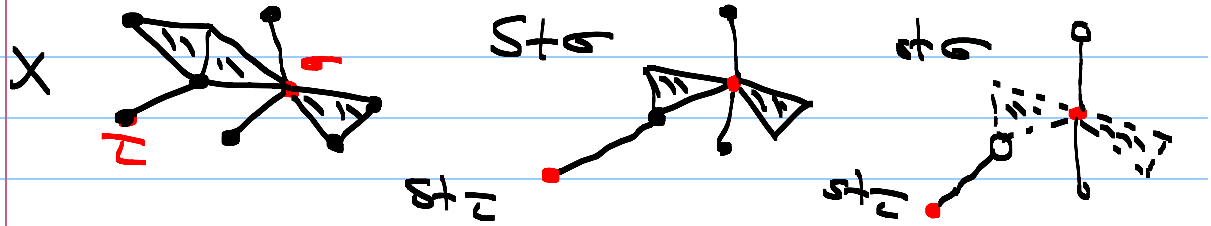
$$g(p) = \sum_{i=0}^{2} a_i (g(v_i))$$

Before proving this theorem we need some <u>definitions</u>:

If X is a simplicial complex and σ is a simplex of X then $\mathrm{St}\,\sigma$ is defined to be the subcomplex consisting of all the simplices of X that contain σ.

Similarly, the open star st σ of σ is defined to be the union of the interiors of all simplices containing σ, where the interior of a simplex τ is by definition $\tau - \partial\tau$. Thus st σ is an open subset of X, whose closure is St σ.





<u>Lemma</u>: For vertices $v_1, \dots, v_n$ of a simplicial complex X, the intersection $st\, v_1 \cap \dots \cap st\, v_n$ is empty unless $v_1 \dots v_n$ are vertices of a simplex σ of X, in which case $st\, v_1 \cap \dots \cap st\, v_n = st\, \sigma$.

<u>Proof</u>: By the definition $st\, v_1 \cap \dots \cap st\, v_n$ is the union of interiors of all simplices τ containing $v_1, \dots, v_n$. If $st\, v_1 \cap \dots \cap st\, v_n$ is nonempty, such τ exists and τ contains the simplex $\sigma = [v_1, \dots, v_n] \subset X$. The simplices τ containing $\{v_1, \dots, v_n\}$ are just those simplices containing the simplex σ and thus,

$$st\, \sigma = st\, v_1 \cap \dots \cap st\, v_n. \qquad \blacksquare$$

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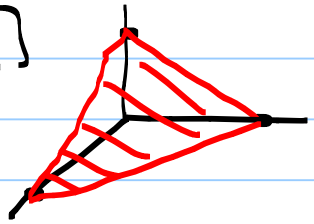
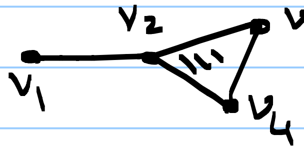
Proof of the Simplicial Approximation Theorem:

Put a metric on the simplicial complex K. If $v_1, \ldots, v_n$ are the vertices of K then $[v_1, \ldots, v_n]$ is a simplex in $\mathbb{R}^n$.

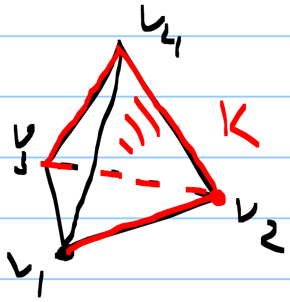
$n=3$, $[v_1, v_2, v_3]$



$K =$



$\Delta^3 \subset \mathbb{R}^3$:



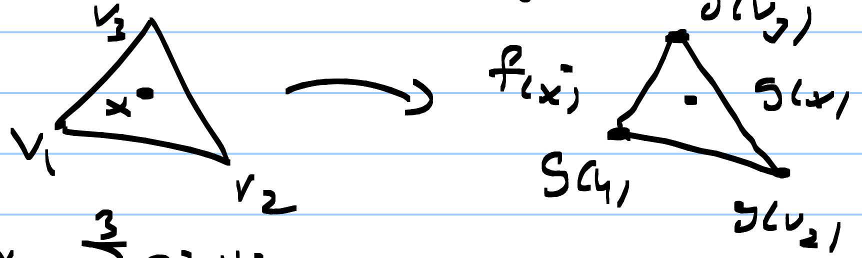
$$f : K \longrightarrow L$$

The collection $\{st\, w \mid w \text{ is a vertex of } L\}$ is an open cover for L. Then $\{f^{-1}(st\, w) \mid w \text{ is a vertex of } L\}$ is an open cover for K. Since K is a finite complex it is compact and thus this open cover has a Lebesgue number, say $\epsilon > 0$. Taking iterated barycentric subdivisions we may assume that the diameter of each simplex of K is less than $\epsilon/2$. It follows that $st\, v$ of a vertex v of K has diameter less than ϵ. So for any vertex v of K $st\, v \subseteq f^{-1}(st(w = g(v)))$ for some vertex

$w = g(v)$ of L. In particular, we have a map $g: K^0 \to L^0$ such that

$$f(st\, v) \subseteq st(g(w)).$$

Next step is to extend g to the simplices of K. Take any simplex $[v_1, \dots, v_n]$ of K. An interior point x of this simplex lies in $st\, v_i$ for each i, and thus $f(x)$ is contained in $st(g(v_i))$ for each i, because $f(st\, v_i) \subseteq f(St\, v_i) \subseteq st(g(v_i))$. Thus, $st\, g(v_1) \cap \dots \cap st\, g(v_n) \neq \emptyset$ and hence $[g(v_1), \dots, g(v_n)]$ is a simplex of L by the lemma. In particular, we can extend g to $[v_1, \dots, v_n]$ linearly.



$$x = \sum_{i=1}^{3} a_i v_i$$

$$g(x) \doteq \sum a_i g(v_i)$$

$a_i \geq 0, \ \Sigma a_i = 1$

$$\begin{tikzcd} K \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & L \end{tikzcd}$$

Both $f(x)$ and $g(x)$ lie in a single simplex of L because $g(x)$ lies in $[g(v_1) \dots g(v_n)]$ and $f(x)$ lies in the star of this simplex.

Now taking the linear homotopy

$$(1-t) f(x) + t g(x), \quad t \in [0,1]$$

we obtain a homotopy from $f(x)$ to $g(x)$. This homotopy is continuous because its restriction to each simplex $[v_1 \dots v_n]$ of K is continuous, since both $f(x)$ and $g(x)$ are continuous functions of x on $[v_1 \dots v_n]$.

This finishes the proof of the Simplicial Approximation Theorem. ■

Remarks 1) If f already sends some vertices of K to vertices of L then we can choose g so that $g=f$ on these vertices.

2) The proof shows that we can choose g as close to f as we want by taking subdivisions of L as well.

Proof of the Lefschetz Fixed Point Theorem

First observe that the general case reduces the simplicial case as follows: Suppose that $r: K \to X$ is a retraction of a finite simplicial complex K onto X. If $f: X \to X$ is a map then $f\circ r: K \to X \subseteq K$ has the same fixed point set as f.

$$x = (f\circ r)(x) = f(\underset{\overset{\uparrow}{K}}{r(x)}) \in X \quad , \; r(x)=x \quad f(x)=x$$

On the other hand,

$$r_* \circ i_* = (r\circ i)_*$$

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[rr, bend right, "id_X"'] & K \arrow[r, "r"] & X \end{tikzcd}$$

$$\begin{tikzcd} H_n(X) \arrow[r, "i_*"] \arrow[rr, bend right, "id"'] & H_n(K) \arrow[r, "r_*"] & H_n(X) \end{tikzcd}$$

$$\begin{tikzcd} \Rightarrow H_n(K) \simeq H_n(X) \oplus A \arrow[r, "r_*"] & H_n(X) \arrow[d, "f_*"] \\ & H_n(X) \end{tikzcd}$$

$$tr(f\circ r)_* = tr\, f_*$$

$$\Rightarrow \tau(f\circ r) = \tau(f)$$

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So, we may assume that X is finite simplicial complex and $f: X \to X$ is any map with $\tau(f) \neq 0$.

must show: f has a fixed point.

Assume on the contrary that f has no fixed points.

$$\begin{tikzcd} f: X \arrow[r] \arrow[d, equal] & X \arrow[d, equal] \\ K & L \end{tikzcd}$$

Claim: There is a subdivision L of X, a further subdivision K of L, and a simplicial map $g: K \to L$ homotopic to f such that $g(\sigma) \cap \sigma = \emptyset$ for each simplex σ of K.

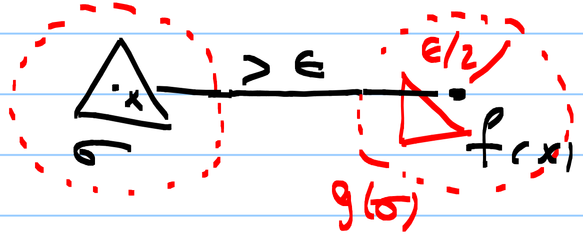
Proof: Put a metric d on K. Since K is compact there is some $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$, for all $x \in K$ ($\epsilon = 1/n$, $x_n \in K$ s.t. $d(x_n, f(x_n)) < 1/n$. Since K is compact w.l.o.g. we may assume that $\lim x_n = x_0$ exists. Taking limit of $d(x_n, f(x_n)) < 1/n$ as $n \to \infty$ we see that $d(x_0, f(x_0)) \leq 0$, which implies that $f(x_0) = x_0$. This is a contradiction since we assumed that f has no fixed points). Choose a subdivision L of X so that each simplex of L has diameter less than $\epsilon/2$. Applying simplicial approximation theorem to $f: L \to L$ we obtain a subdivision K of L and a simplicial map $g: K \to L$, homotopic to f. By construction, g has the property that for each simplex σ of K,

$f(\sigma) \subseteq St\, g(\sigma)$.

Claim: $g(\sigma) \cap \sigma = \emptyset$ for each simplex σ of K.

Proof: For any choice of $x \in \sigma$, $d(x, f(x)) > \epsilon$.

Also we know that $g(\sigma)$ lies within $\epsilon/2$ distance of $f(x)$ and σ lies within $\epsilon/2$ distance of x.



The Lefschetz numbers $\tau(f)$ and $\tau(g)$ are equal because f is homotopic to g and thus $f_* = g_*$ on $H_i(X;\mathbb{Z})$ for all i.

Since g is simplicial, it takes n-skeleton K^n of K to the n-skeleton L^n of L, for each n: $g(K^n) \subseteq K^n$, for each n.

Then g induces a chain map of the cellular chain complex

$$\to H_{n+1}(K^{n+1},K^n) \xrightarrow{d_{n+1}} H_n(K^n,K^{n-1}) \xrightarrow{d_n} H_{n-1}(K^{n-1},K^{n-2}) \to$$

Claim: $\tau(g) = \sum_n (-1)^n tr(g_*: H_n(K^n,K^{n-1}) \to H_n(K^n,K^{n-1}))$

Assuming the claim we see that $tr\, g_*$ is zero because

$$H_n(K^n,K^{n-1}) \simeq H_n(K^n/K^{n-1}) \simeq \bigoplus_\sigma \underset{\overset{\parallel}{\langle\sigma\rangle}}{\mathbb{Z}}$$

where the sum is over all n-simplices of K. Hence, $\tau(g)$ is zero, which is a contradiction because $\tau(g) = \tau(f) \neq 0$ by assumption.

Proof of the claim: This is purely algebraic

First we have an algebraic fact:

Fact: Suppose we have a short exact sequence of finitely generated abelian groups

$$0 \to A \to B \to C \to 0$$

and maps α, β and γ so that the diagram below is commutative:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

(Proof is left as an exercise!)

Then $\mathrm{tr}\, B = \mathrm{tr}\, A + \mathrm{tr}\, C$.

$$0 \to \mathrm{Im}\, d_{n+1} \to \ker d_n \to H_n(X) \to 0$$

$$0 \to \ker d_n \to H_n(K^n, K^{n-1}) \xrightarrow{d_n} \mathrm{Im}\, d_n \to 0$$

$$\underline{\qquad} \supset \underline{\qquad} \cup \underline{\qquad} \cup \underline{\qquad}$$

(*)
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}\, d_{n+1} \arrow[r] \arrow[d, "g_*"] & \ker d_n \arrow[r] \arrow[d, "g_*"] & H_n(X) \arrow[r] \arrow[d, "g_*"] & 0 \\
0 \arrow[r] & \mathrm{Im}\, d_{n+1} \arrow[r] & \ker d_n \arrow[r] & H_n(X) \arrow[r] & 0
\end{tikzcd}$$

$$\underline{\qquad} \supset \underline{\qquad} \cup \underline{\qquad} \cup \underline{\qquad}$$

$$\begin{tikzcd}
H_{n+1}(K^{n+1}, K^n) \arrow[r, "d_{n+1}"] \arrow[d, "g_*"'] & H_n(K^n, K^{n-1}) \arrow[r, "d_n"] \arrow[d, "g_*"] & H_{n-1}(K^{n-1}, K^{n-2}) \arrow[d, "g_*"'] \\
H_{n+1}(K^{n+1}, K^n) \arrow[r, "d_{n+1}"] & H_n(K^n, K^{n-1}) \arrow[r, "d_n"] & H_{n-1}(K^{n-1}, K^{n-2})
\end{tikzcd}$$

$$\alpha \longmapsto \beta \qquad g_*(\alpha) \longmapsto g_*(\beta)$$

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$g_*(\mathrm{Im}\, d_{n+1}) \subseteq \mathrm{Im}\, d_{n+1}$

(**)
$$\begin{tikzcd}
0 \arrow[r] & \ker d_n \arrow[r] \arrow[d, "g_*"] & H_n(K^n,K^{n-1}) \arrow[r, "d_n"] \arrow[d, "g_*"] & \mathrm{Im}\, d_n \arrow[r] \arrow[d, "g_*"] & 0 \\
0 \arrow[r] & \ker d_n \arrow[r] & H_n(K^n,K^{n-1}) \arrow[r, "d_n"'] & \mathrm{Im}\, d_n \arrow[r] & 0
\end{tikzcd}$$

By the algebraic fact applied to (*) and (**)

$$\mathrm{tr}(g_*|\ker d_n) = \mathrm{tr}(g_*|\mathrm{Im}\, d_{n+1}) + \mathrm{tr}(g_*|H_n(X))$$

and $\mathrm{tr}(g_*|H_n(K^n,K^{n-1})) = \mathrm{tr}(g_*|\mathrm{Im}\, d_n) + \mathrm{tr}(g_*|\ker d_n)$

So, $\mathrm{tr}(g_*|H_n(X)) = \mathrm{tr}(g_*|\ker d_n) - \mathrm{tr}(g_*|\mathrm{Im}\, d_{n+1})$

$$= \mathrm{tr}(g_*|\ker d_n) - \mathrm{tr}(g_* H_{n+1}(K^{n+1},K^n)) + \mathrm{tr}(g_*|\ker d_{n+1})$$

$$T(g) = \sum_{n=-1} (-1)^n \mathrm{tr}(g_*|H_n(X))$$

$$= -\mathrm{tr}(\overset{0}{\cancel{g_*|\ker d_{-1}}}) - \mathrm{tr}(\cancel{g_*|\ker d_0}) + \underline{\mathrm{tr}}(g_*|H_0(K^0,K^{-1}))$$

$$+ \mathrm{tr}(\cancel{g_*|\ker d_0}) + \mathrm{tr}(\cancel{g_*|\ker d_1}) - \underline{\mathrm{tr}}(g_*(H_1(K^1,K^0))$$

$$- \mathrm{tr}(\cancel{g_*|\ker d_1}) + \mathrm{tr}(g_*|\ker d_2) + \underline{\mathrm{tr}}(g_*|H_2(K^2,K^1))$$

$$= \sum_{n=0}^{\infty} (-1)^n \mathrm{tr}(g_*|H_n(K^n,K^{n-1}))$$

Example:

$\partial S = \alpha_2 - \alpha_2' = 0$

$\alpha_2 = \alpha_2'$

$f_*(\alpha_2) = \alpha_2'$ $\quad \alpha_2 - \alpha_2' = 0$

$f : \Sigma_3 \longrightarrow \Sigma_3$ π-radian rotation about the axis.

$$\begin{tikzcd} H_0(\Sigma_3) \arrow[r, "f_*"] \arrow[d, phantom, "\wr\wr"] & H_0(\Sigma_3) \arrow[d, phantom, "\wr\wr"] \\ \mathbb{Z} \arrow[r, "id"] & \mathbb{Z} \end{tikzcd}$$

$\operatorname{tr} f_*|_{H_0(\Sigma_3)} = 1.$

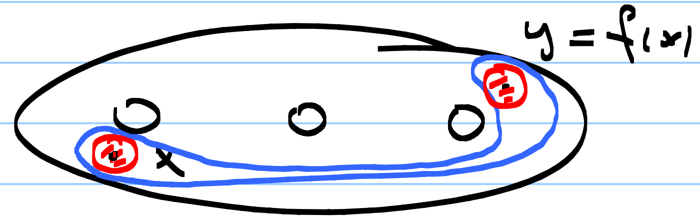
$H_1(\Sigma_3) \simeq \mathbb{Z}^6 = \langle \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \rangle$

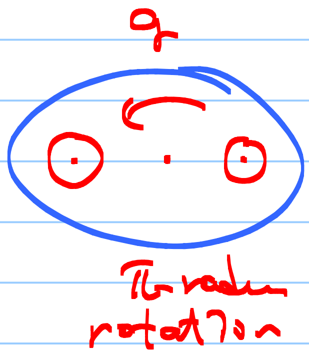
$f_*(\alpha_1) = \pm\alpha_3$, $f_*(\beta_1) = \pm\beta_3$, $f_*(\alpha_3) = \pm\alpha_1$

$f_*(\beta_3) = \pm\beta_3$, $f_*(\alpha_2) = \underline{\alpha_2}$, $f_*(\beta_2) = \underline{\beta_2}$

$\operatorname{tr}(f_*|H_1(\Sigma_3)) = 2$

$\operatorname{tr}(f_*|H_2(\Sigma_3)) = 1$





$$\begin{tikzcd} \mathbb{Z} \simeq H_2(X) \arrow[r, "f_*"] \arrow[d, "\simeq"] & H_2(X) \simeq \mathbb{Z} \arrow[d, "\simeq"] \\ H_2(X, X\setminus\{x\}) \arrow[r] \arrow[d, phantom, "\wr\wr"] & H_2(X, X\setminus\{y\}) \arrow[d, phantom, "\wr\wr"] \\ \mathbb{Z} \arrow[r, phantom, "\overset{f_*}{=}"] & \mathbb{Z} \\ 1 \arrow[r, mapsto] & 1 \end{tikzcd}$$

$H_2(D^2, D^2\setminus\{x\})$ $\quad H_2(D^2, D^2\setminus\{y\})$

g is homotopic to id and thus $g_* = id$.

$\mathrm{tr}(f_*|H_2(\Sigma_2)) = 1.$

$\tau(f) = 1-2+1 = \underline{\underline{0}} \longleftarrow$

Thus, f may <u>not</u> have fixed points.

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CHAPTER 3: Cohomology

Cohomology can be regarded as the dual of homology.

Consider the case vector spaces: V, $\mathbb{R}$ vector space.

$$V^* = \text{Hom}(V, \mathbb{R})$$

$f \in V^*$, $f : V \longrightarrow \mathbb{R}$ linear map.

V^* is isomorphic to V (if $\dim V < +\infty$), however it carries a kind natural multiplication:

$f, g \in V^*$, $\quad f \cdot g : V^* \longrightarrow \mathbb{R}$, $(f \cdot g)(v) = f(v) g(v)$.

Using this idea we'll define a product so called "cup product" on cohomology rings.

Definition: Let X be a topological space and G an abelian group. Define

$$C^n(X; G) = \text{Hom}(C_n(X), G), \text{ for any } n = 0, 1, 2, \dots$$

$C^n(X; G)$ is called the group of n-cocycles on X.

Recall that we have the singular chain complex

$$\longrightarrow C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \longrightarrow \dots$$

We know how to take dual of $C_n(X)$.
What about the dual of $\partial : C_{n+1}(X) \longrightarrow C_n(X)$?
We define the dual of ∂ as usual

$\partial^* = \delta : C^n(X; G) \longrightarrow C^{n+1}(X; G)$ defined by

$\delta(\varphi)(\alpha) \doteq \varphi(d\alpha)$, for any $\varphi \in C^n(X;G)$

and $\alpha \in C_{n+1}(X)$.

Since $d^2 = 0$, we see that for any $\varphi \in C^n(X;G)$

$$\begin{aligned}\delta^2(\varphi)(\alpha) &= \delta(\delta\varphi(\alpha)) \\ &= \delta(\varphi(d\alpha)) \\ &= \varphi(\underset{\overset{\shortparallel}{0}}{d(d\alpha)}) \\ &= 0.\end{aligned}$$

Hence

$$\to C^{n-1}(X;G) \xrightarrow{\delta^{n-1}} C^n(X;G) \xrightarrow{\delta^n} C^{n+1}(X;G) \to \cdots$$

is a **cochain** complex.

Since $\delta^2 = 0$ we see that $\operatorname{Im}\delta^{n-1} \subseteq \ker \delta^n$ and therefore we may the quotient group $\ker\delta^n / \operatorname{Im}\delta^{n-1}$.

This quotient group will be called the nth cohomology group of X with G-coefficients and is denoted by

$$H^n(X;G) \doteq \frac{\ker \delta^n}{\operatorname{Im}\delta^{n-1}} = \frac{n\text{-cocycles}}{n\text{-coboundaries}}$$

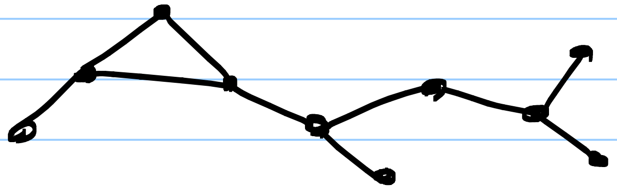
Elements of $\ker\delta^n$ are called n-cocycles and those of $\operatorname{Im}\delta^{n-1}$ are called n-coboundaries.

Examples: X Δ-complex

$\Delta_n(X)$: free abelian group generated by n-simplicies of X.

$\Delta^n(X;G) = \mathrm{Hom}(\Delta_n(X), G)$

a) Let X be a 1-dimensional Δ-complex.



$\varphi \in \Delta^0(X;G) = \mathrm{Hom}(\Delta_0(X), G)$ is a just a function from the vertex set of X to G.

$\phi \in \Delta^1(X;G) = \mathrm{Hom}(\Delta_1(X), G)$ is a function from the set of edges of X to G.

Suppose $\varphi \in \Delta^0(X;G)$ is a 1-cocycle. Then, $\delta\varphi = 0$ in $\Delta^1(X;G)$.

Let $\underset{v_1}{\bullet}\!\!-\!\!-\!\!\underset{v_2}{\bullet}$ is an edge in X, then

$$0 = \delta\varphi(\underset{v_1}{\bullet}\!\!\longrightarrow\!\!\underset{v_2}{\bullet}) = \varphi(\partial(\underset{v_1}{\bullet}\!\!\longrightarrow\!\!\underset{v_2}{\bullet}))$$

$$= \varphi(v_2 - v_1)$$

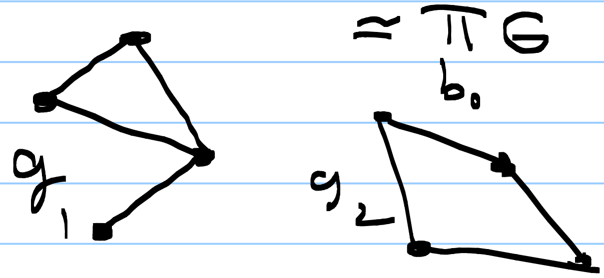
$$= \varphi(v_2) - \varphi(v_1)$$

$\Longrightarrow \varphi(v_2) = \varphi(v_1)$ on every edge.

$$H^0_\Delta(X;G) = \frac{\ker(\delta : C^0(X;G) \to C^1(X;G))}{(0)}$$

Thus $H^0_\Delta(X;G) = \ker(\delta : C^0(X;G) \to C^1(X;G))$

$\simeq \prod_{b_0} G$



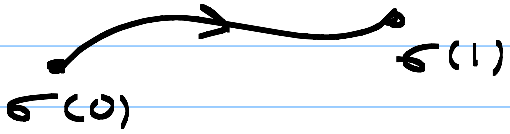
$b_0 = \operatorname{rank} H_0(X)$
$= \#$ of connected components of X.

2) Let $X = \mathbb{R}^2$, $\varphi \in C^1(X;\mathbb{R})$, where

$\varphi : C^1(X) \longrightarrow \mathbb{R}$, defined by

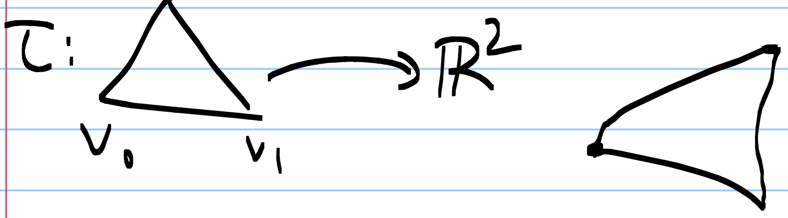
$$\varphi(\sigma) = \int_\sigma f(x,y)\,dx + g(x,y)\,dy$$

$\sigma : [0,1] \longrightarrow \mathbb{R}^2$ (σ smooth)



Assume that φ is a cycle. So if τ is a two simplex in $\mathbb{R}^2$ then

$$0 = \delta\varphi(\tau) = \varphi(\delta\tau) = \int_{\partial\tau} f(x,y)\,dx + g(x,y)\,dy$$



So $\int_C f(x,y)\,dx + g(x,y)\,dy = 0$ for every closed curve C in $\mathbb{R}^2$.

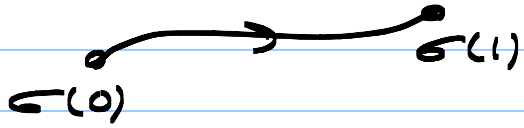
Video 12

In this case $f(x,y)dx+g(x,y)dy$ has a potential:

$$f(x,y)\,dx+g(x,y)\,dy = d\phi,$$ where

$\phi : \mathbb{R}^2 \to \mathbb{R}$ is a function. So $f=\phi_x$ and $g=\phi_y$.

In this case, $\int_\sigma f\,dx+g\,dy = \phi(\sigma(1))-\phi(\sigma(0))$



$\phi : \mathbb{R}^2 \to \mathbb{R}$ function

$\phi \in C^0(\mathbb{R}^2; \mathbb{R})$ and

$$\varphi(\sigma) = \phi(\underline{\underline{\sigma(1)}}) - \phi(\underline{\underline{\sigma(0)}})$$
$$= \phi(\partial\sigma)$$
$$= \partial\phi(\sigma)$$
$$\Rightarrow \varphi = \partial\phi$$

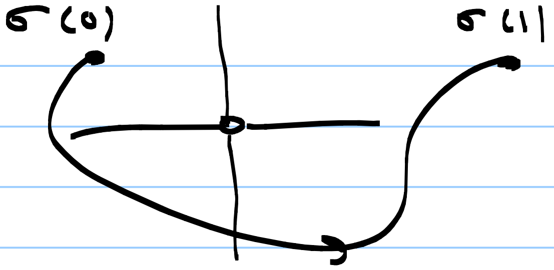
Hence, any 1-cocycle φ on $\mathbb{R}^2$ is a 1-coboundary and thus
$$H^1(\mathbb{R}^2; \mathbb{R}) = 0.$$

3) $X = \mathbb{R}^2 \setminus \{(0,0)\}$ $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2} = \pm d\theta$

$\theta : \mathbb{R}^2 \setminus (y\text{-axis}) \to \mathbb{R}$, $\theta = \tan^{-1} y/x$

$d\omega = 0$ and hence the 1-cochain

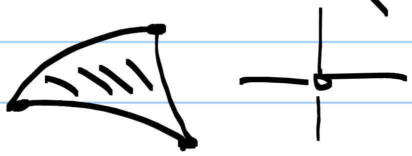
$\varphi(\sigma) = \int_\sigma \omega$, $\sigma : [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$



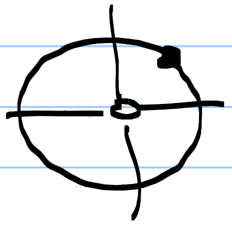
is indeed a cocycle:

$$\delta\varphi(\tau) = \varphi(\delta\tau) = \int_{\partial\tau}\omega = \int_{\tau}\overset{0}{\overset{\shortparallel}{\overbrace{d\omega}}} = 0.$$

τ : △ $\longrightarrow \mathbb{R}^2 \setminus \{(0\ 0)\}$



However, $\varphi(\sigma) = \int_{\sigma}\omega = 1$ of



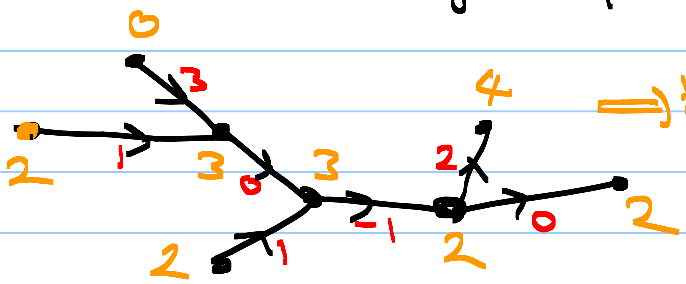
so that $\varphi \neq 0$.

This implies that $\omega = d\phi$ for some function $\phi : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$.

Hence, $H^1(\mathbb{R}^2 \setminus \{(0,0)\}, \mathbb{R}) \neq 0$.

4) Let X be Δ-complex of dimension 1.

$\varphi \overset{\shortparallel}{} \partial\phi$



$\Longrightarrow H^1(X;\mathbb{Z}) = 0$

If $\varphi \in \Delta^1(X;G)$, then $\delta\varphi = 0$ since $\Delta^2(X;G) = 0$.
Hence, φ is a 1-cocycle.

If $\varphi = \delta\phi$, for some $\phi \in \Delta^1(X;G)$, then

$$\varphi(\underset{v_0 \quad v_1}{\bullet\!\!\longrightarrow\!\!\bullet}) = \delta\phi(\underset{v_0 \quad v_1}{\bullet\!\!\longrightarrow\!\!\bullet}) = \phi(\partial(\underset{v_0 \quad v_1}{\bullet\!\!\longrightarrow\!\!\bullet}))$$
$$= \phi(v_1 + (-v_0))$$
$$= \underline{\underline{\phi}}(v_1) - \underline{\underline{\phi}}(v_0)$$

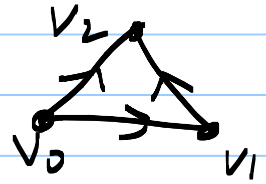
Hence, if $H_1(X;\mathbb{Z}) = 0$ then $H^1(X;\mathbb{Z}) = 0$.

Exercise: In general, $H^1(X;\mathbb{Z}) \simeq H_1(X;\mathbb{Z})$, if X is a 1-dimensional Δ-complex.

5) X Δ-complex, $G = \mathbb{Z}_2$

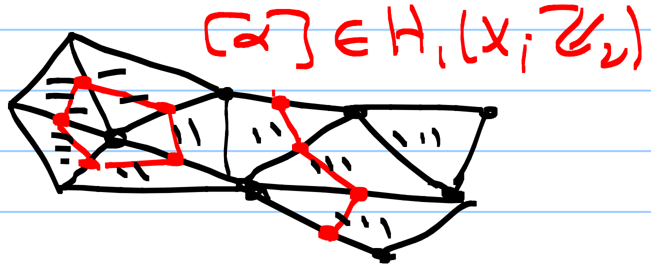
$\phi \in \Delta^1(X;\mathbb{Z}_2)$. Assume that $\delta\phi = 0$, i.e. ϕ is 1-cocycle. So for any 2-simplex

τ :



$$0 = \delta\phi(\tau) = \phi(\partial\tau)$$
$$= \phi([v_0,v_1]) + \phi([v_0,v_2])$$
$$+ \phi([v_1,v_2])$$

Hence, ϕ takes the value 1 on either two edges or no-edges of any simplex τ.



So, if X is a closed surface (with delta complex structure) then any 1-cocycle ϕ determines a 1-cycle in homology.

$$\begin{tikzcd}
H^1(X;\mathbb{Z}_2) \arrow[r, leftrightarrow] & H_1(X;\mathbb{Z}_2) \\
\phi \arrow[r, leftrightarrow] & \alpha
\end{tikzcd}$$

This is called Poincaré duality.

(Closed surface means compact without boundary)

Exercise: Try to go over the above example in $\mathbb{Z}$-coefficients.

Video 13

Cohomology Groups and The Universal Coefficient Theorem:

$\overset{\partial_n}{C_n(X)}$, $\overset{\partial_n}{\Delta_n(X)}$, $\overset{d_n}{H_n(X^n, X^{n-1})}$ singular, delta or cellular chain complexes.

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

G- abelian group, $C^n(X;G) = \mathrm{Hom}(C_n(X); G)$

$$\cdots \to C^{n-1}(X;G) \xrightarrow{\delta^{n-1}} C^n(X;G) \xrightarrow{\delta^n} C^{n+1}(X;G) \xrightarrow{\delta^{n+1}} \cdots$$

$$\delta^n : C^n(X;G) \longrightarrow C^{n+1}(X;G)$$
$$\varphi \longmapsto \delta^n(\varphi) : C_{n+1}(X) \longrightarrow G$$
$$\alpha \longmapsto \varphi(\partial\alpha)$$

$\varphi : C_n(X) \to G$

In other words, δ^n is the dual of ∂_n.

Since $\partial_n \circ \partial_{n+1} = 0$ we have $\delta^{n+1} \circ \delta^n = 0$, because,

$$\begin{aligned}(\delta^{n+1} \circ \delta^n)(\varphi) &= \delta^{n+1}(\delta^n(\varphi)) \\ &= \delta^{n+1}(\varphi \circ \partial_n) \\ &= (\varphi \circ \partial_n) \circ \partial_{n+1} \\ &= \varphi \circ (\underbrace{\partial_n \circ \partial_{n+1}}_{\overset{\shortparallel}{0}}) \\ &= 0.\end{aligned}$$

Hence, $(C^n(X;G), \delta^n)$ is a cochain complex so that we can define its cohomology as

$$H^n(X;G) \doteq \frac{\ker(\delta^n : C^n(X;G) \to C^{n+1}(X;G))}{\mathrm{Im}(\delta^{n-1} : C^{n-1}(X;G) \to C^n(X;G))}.$$

Aim: Show that $H^n(X;G)$ is determined only by $H_*(X)$ and G.

Remark: What we are going to do can be purely algebraically and thus it is not restricted to (co)homology theory of topological spaces.

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

$Z_n = \ker \partial$, the group of n-cycles on X
$B_n = \operatorname{Im} \partial_{n+1}$, the group of n-boundaries on X

$$H_n(X) \doteq \frac{Z_n}{B_n}.$$

First we'll construct a natural homomorphism

$$h : H^n(X;G) \to \operatorname{Hom}(H_n(X), G)$$
$$[\varphi] \longmapsto \bar{\varphi}$$

$[\varphi] \in H^n(X;G)$, $\varphi : C_n(X) \to G$ hom. so that

$\delta\varphi = 0$ (i.e., $\varphi \in \ker \delta$).

Note that φ is zero on $B_n(X)$: If $\alpha \in B_n(X)$ then $\alpha = \partial\beta$ for some $\beta \in C_{n+1}(X)$. Then $\varphi(\alpha) = \varphi(\partial\beta) = \delta(\varphi)(\beta) = 0$, since $\delta\varphi = 0$.

$$\begin{tikzcd}
 & C_n(X) \arrow[r, "\varphi"] & G & & Z_n \arrow[r, "\varphi_0"] \arrow[d] & G \\
\varphi_0 \doteq \varphi|_{Z_n} & Z_n \arrow[u, phantom, sloped, "\subseteq"] \arrow[ur, "\varphi|_{Z_n}"] & & \Rightarrow & Z_n/B_n \arrow[ur, "\bar{\varphi}_0"'] & \\
 & B_n \arrow[u, phantom, sloped, "\subseteq"] \arrow[uur, "\varphi|_{B_n} = 0"'] & & & &
\end{tikzcd}$$

Since φ is zero on B_n φ induces a homomorphism $\bar{\varphi}_0 : Z_n/B_n = H_n(X) \to G$.

Now we define $h([\varphi])$ as $h([\varphi]) \doteq \overline{\varphi_0}$.

Clearly, h is a homomorphism (Exercise!)

$$h: H^n(X;G) \longrightarrow \mathrm{Hom}(H_n(X), G)$$

Claim: h is surjective.
Proof: Consider the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r] & C_n(X) \arrow[r, "\partial"] \arrow[l, bend left, "p"] & B_{n-1} \arrow[r] & 0. \end{tikzcd}$$

Since $C_n(X)$ is free abelian all its subgroups are free abelian and thus the above sequence splits as

$$C_n(X) \simeq Z_n \oplus B_{n-1}$$

and we obtain a projection map $p: C_n(X) \to Z_n$ so that the composition

$$Z_n \to C_n(X) \xrightarrow{p} Z_n \text{ is identity on } Z_n.$$

Composing with p defines a way extending homomorphisms $\varphi_0 : Z_n \to G$ to $\varphi = \varphi_0 \circ p : C_n(X) \to G$. Note that the restriction of φ to Z_n is φ_0.

Moreover, if $\varphi_0 : Z_n \to G$ were already zero on B_n then its extension $\varphi = \varphi_0 \circ p : C_n(X) \to G$ would be still zero on B_n, because $B_n \subseteq Z_n$.

Hence, this procedure extends homomorphism $H_n(X) \to G$ to elements of $\ker \delta$.

$$\begin{aligned} \mathrm{Hom}(H_n(X), G) &\longrightarrow \ker(\delta : \mathrm{Hom}(C_n(X) \to G) \to \mathrm{Hom}(C_{n+1}(X), G)) \\ \overline{\varphi_0} &\longmapsto \varphi = \varphi_0 \circ p \end{aligned}$$

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$$B_n \subseteq \underline{\underline{Z_n}} \to \underline{\underline{C_n}} \xrightarrow{p} \underline{\underline{Z_n}}$$

<u>must check:</u> $\delta(\varphi_0 \circ p) = 0.$

$$\delta(\varphi_0 \circ p)(\alpha) = (\varphi_0 \circ p)(\underline{\underline{\partial\alpha}})$$
$$= \varphi_0(p(\partial\alpha))$$
$$= \varphi_0(\partial\alpha)$$

$= 0$, since φ_0 is chosen to be zero on B_n.

In particular, $[\varphi] \in H^n(X;G)$. So we get a homomorphism

$$\begin{tikzcd} \mathrm{Hom}(H_n(X);G) \arrow[r] & H^n(X;G) \arrow[l, bend right, "h"'] \\ \overline{\varphi_0} \arrow[r, mapsto] & {[\varphi]} \end{tikzcd}$$

So that the composition of h with this homomorphism is identity on $H^n(X;G)$.

Hence, h is onto and moreover we have split short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \ker h \arrow[r] & H^n(X;G) \arrow[r, "h"] & \mathrm{Hom}(H_n(C);G) \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$$

and $H^n(X;G) \simeq \ker h \oplus \mathrm{Hom}(H_n(C);G)$.

So, we just need to understand $\ker h$.

<u>Remark:</u> The above splitting is not natural.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "i"] & \mathbb{Z} \oplus \mathbb{Z}_2 \arrow[r, "p"] & \mathbb{Z} \arrow[r] & 0 \\ & a \arrow[r, mapsto] & (0,a) & & \\ & & (b,a) \arrow[r, mapsto] & b & \end{tikzcd}$$

The map $q_1 : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2$, $q_1(b) = (b,0)$

gives a splitting of the sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "i"'] & \mathbb{Z}\oplus\mathbb{Z}_2 \arrow[r, "p"'] & \mathbb{Z} \arrow[r] \arrow[l, bend right, "q_1"'] & 0 \end{tikzcd}$$

$p\circ q_1 = id_{\mathbb{Z}}: \ b \longmapsto (b,0) \longmapsto b.$

However, $q_2 : \mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}_2$ given by $q_2(b) = (b,\bar{1})$ is another splitting of the sequence.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "i"'] & \mathbb{Z}\oplus\mathbb{Z}_2 \arrow[r, "p"'] & \mathbb{Z} \arrow[r] \arrow[l, bend right, "q_2"'] & 0 \end{tikzcd}$$

$q_2(b) = (b,\bar{1}), \quad p(q_2(b)) = p(b,\bar{1}) = b$

$\Rightarrow p\circ q_2 = id_{\mathbb{Z}}.$

$G \simeq \mathbb{Z}\oplus\mathbb{Z}_2 = \langle (1,0), (0,\bar{1}) \rangle \quad \leftarrow q_1$

$G \simeq \mathbb{Z}\oplus\mathbb{Z}_2 = \langle (1,\bar{1}), (0,\bar{1}) \rangle \quad \leftarrow q_2$

These splitting are not natural means that we do not get commutative diagrams:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r, shift left, "p"] \arrow[d] & C \arrow[r] \arrow[l, shift left, "q"] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"'] & B' \arrow[r, shift right, "p'"'] & C' \arrow[r] \arrow[l, shift right, "q'"'] & 0 \end{tikzcd}$$

Is

$$\begin{tikzcd} B \arrow[d] & C \arrow[l, "q"'] \arrow[d] \\ B' & C' \arrow[l, "q'"] \end{tikzcd}$$

commutative?

Not in general! The above gives a counter example.

<u>Aim:</u> Study ker h, $h: H^n(X;G) \to \mathrm{Hom}(H_n(X), G)$

To do so we consider the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd} \qquad (*)$$

$\varphi = \varphi_0 \circ \underline{p}$

$\ulcorner$ $0 \to (\underline{Z_n}, \partial=0) \to (\underline{C_n(X)}, \partial) \to (\underline{B_n}, \partial=0) \to 0$

short exact sequences of chain complexes. This induces a long exact sequence in homology (of these complexes)

$\dots \to Z_n \to H_n(X) \to B_n \to Z_{n-1} \to \dots$ $\lrcorner$

Note that the dual of $(*)$ is as follows:

$$\begin{tikzcd}
0 & Z^*_{n+1} \arrow[l] & C^*_{n+1} \arrow[l] & B^*_n \arrow[l] & 0 \arrow[l] \\
0 & Z^*_n \arrow[l] \arrow[u, "0"] & C^*_n \arrow[l] \arrow[u, "\delta"] & B^*_{n-1} \arrow[l] \arrow[u, "0"] & 0 \arrow[l]
\end{tikzcd} \qquad (**) \quad \text{(commutative)}$$

$Z^*_n = \mathrm{Hom}(Z_n, G)$, $B^*_n = \mathrm{Hom}(B_n; G)$
and $C^*_n = \mathrm{Hom}(C_n(X); G)$.

Since Z_n and B_n's are free abelian the sequences in $(*)$ are split exact and the sequences in $(**)$ are also split exact.

$\ulcorner$ $C_{n+1}(X) \cong \underline{\underline{Z_n}} \oplus B_{n-1} \Rightarrow C^*_{n+1} \simeq Z_n^* \oplus B^*_{n-1}$ $\lrcorner$

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The commutative diagram in (**) induces a long exact sequence in cohomology as follows:

$$\cdots \leftarrow B_n^* \xleftarrow{i_n^*} Z_n^* \leftarrow H^n(X;G) \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow \cdots$$

Let $i_n : B_n \longrightarrow Z_n$ be the inclusion map. Then the above map $Z_n^* \longrightarrow B_n^*$ is just i_n^* (Just review the zig zag homomorphism we've studied last semester)

Remark: The map i_n^* is indeed the zig-zag homomorphism in the long exact sequence of cohomology groups. Hence, we see that this zig-zag homomorphism is equal to the dual of the inclusion map $i_n : B_n \longrightarrow Z_n$.

$i_{n-1}^* : Z_{n-1}^* \longrightarrow B_{n-1}^*$

$$\operatorname{coker} i_{n-1}^* = \frac{B_{n-1}^*}{\operatorname{Im}(i_{n-1}^*)}$$

$$0 \to \operatorname{coker} i_{n-1}^* \longrightarrow H^n(X;G) \xrightarrow{h} \ker i_n^* \longrightarrow 0$$

is clearly (short) exact.

$i_n^* : \operatorname{Hom}(Z_n(X), G) \longrightarrow \operatorname{Hom}(B_n(X), G)$

$\ker i_n^* = \operatorname{Hom}(H_n(X), G)$

So we may write the above sequence as

$$0 \to \operatorname{coker} i_{n-1}^* \to H^n(X;G) \xrightarrow{h} \operatorname{Hom}(H_n(X), G) \to 0.$$

We'll see that coker i^*_{n-1} depends only on $H_{n-1}(X)$ and G in a natural and functorial way.

Remark: Note that coker i^*_{n-1} would be zero if it were always true that the dual of a short exact sequence was exact:

Since $0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(X) \to 0$ is exact

$$0 \leftarrow B^*_{n-1} \xleftarrow{i^*_{n-1}} Z^*_{n-1} \leftarrow \mathrm{Hom}(H_{n-1}(X), G) \leftarrow 0$$

would be exact. Thus, coker $i^*_{n-1} = \dfrac{B^*_{n-1}}{\mathrm{Im}\, i^*_{n-1}} = \dfrac{B^*_{n-1}}{B^*_{n-1}} = (0)$.

However, the dual of an exact sequence may not be exact.

Example: $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}_n \to 0$ is exact.

Its dual w.r.t. $\mathbb{Z}$ ($\mathrm{Hom}(-, \mathbb{Z})$) is

$$\begin{tikzcd}
0 & \mathbb{Z}^* \arrow[l] \arrow[d, phantom, "\wr|"] & \mathbb{Z}^* \arrow[l, "\times n"'] \arrow[d, phantom, "\wr|"] & \mathbb{Z}_n^* \arrow[l] \arrow[d, phantom, "||"] & 0 \arrow[l] \\
 & \mathbb{Z} & \mathbb{Z} & \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}) = 0 &
\end{tikzcd}$$

$0 \leftarrow \mathbb{Z} \xleftarrow{\times n} \mathbb{Z} \leftarrow 0$ is not exact.

$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to \underline{\underline{H_{n-1}(X)}} \to 0$ is exact

but $0 \leftarrow B^*_{n-1} \xleftarrow{i^*_{n-1}} Z^*_{n-1} \leftarrow \mathrm{Hom}(H_{n-1}(X), G) \leftarrow 0$

is not exact in general.

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To study the $\operatorname{coker} i_{n-1}^{\alpha}$, we will use so called free resolutions of abelian groups:

A free resolution for an abelian group H is an exact sequence of abelian groups

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where each F_i is free abelian.

<u>Remark</u>: If H is an abelian group H clearly has a free resolution. Let $B=\{h_\alpha\}_{\alpha\in\Lambda}$ be be a generating set for H.
Let F_0 be the free abelian group on Λ.

$$F_0 = \langle \alpha \mid \alpha\in\Lambda \rangle$$

$$\begin{tikzcd}
0 \arrow[r] & \ker f_0 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & H \arrow[r] & 0 \\
 & & \alpha \arrow[r, mapsto] & h_\alpha &
\end{tikzcd}$$ is

short exact, where f_1 is the inclusion map. Since F_0 is free abelian so is its subgroup $\ker f_0$. Let $F_1 = \ker f_0$ so we get the required free resolution (of finite length)

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

<u>Remark</u>: Abelian groups are just $\mathbb{Z}$-modules. If one considers R-modules in general, the free resolutions may not be of finite length.

If we have a free resolution of H, say

(F) $\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$ then its dual (Hom$(-,G)$)

$$\cdots \leftarrow F_2 \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \leftarrow 0$$

may <u>not</u> be exact. Since $f_i \circ f_{i-1} = 0$ for all $i \geq 1$, $f_{i-1}^* \circ f_i^* = 0$, for all $i \geq 1$, so that $\mathcal{H}$ is a cochain complex.

Here, we define compute the cohomology this this cochain complex:

$$H^n(F;G) \doteq \frac{\ker f_{n+1}^*}{\operatorname{Im} f_n^*}.$$

<u>Lemma:</u> a) Given free resolutions F and F' of abelian group H and H', then every homomorphism $\alpha : H \to H'$ can be extended to a chain map from F to F':

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r, "f_2'"'] & F_1' \arrow[r, "f_1'"'] & F_0' \arrow[r, "f_0'"'] & H' \arrow[r] & 0
\end{tikzcd}$$

Furthermore, any two such chain maps extending α are chain homotopic.

b) For any two free resolutions F and F' of H there are isomorphisms $H^n(F;G) \approx H^n(F';G)$ for all n.

<u>Proof:</u> We construct α_i's inductively as follows.

$y \mapsto f_1(y) \mapsto \overset{0}{\overset{\|}{f_0(f_1(y))}}$

$x \mapsto f_0(x)$

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r, dashed] & F_2' \arrow[r, "f_2'"'] & F_1' \arrow[r, "f_1'"'] & F_0' \arrow[r, "f_0'"'] & H'
\end{tikzcd}$$

$y' \mapsto \alpha_0(f_1(y)) \mapsto f_0(\alpha_0(f_1(y))) = 0$

$x' \mapsto \alpha(f_0(x))$

Pick a basis for F_0 and if $x \in F_0$ is any element define $\alpha_0(x)$ as

$$\alpha_0(x) = x', \text{ where } f_0'(x') = \alpha(f_0(x)).$$

This uniquely determines α_0, once we choose x''s.

Similarly, define $\alpha_1(y)$, for any basis element y of F_2, as

$$\alpha_1(y) \doteq y', \text{ where } f_1'(\underline{y'}) = \alpha_0(f_1(y)).$$

Inductive we may define each $\alpha_n : F_n \to F_n'$ so that

$$f_n' \circ \alpha_n = \alpha_{n-1} \circ f_n.$$

$$\begin{tikzcd}
F_n \arrow[r, "f_n"] \arrow[d, "\alpha_n"'] \arrow[d, bend left, "\alpha_n'"] & F_{n-1} \arrow[d, "\alpha_{n-1}"] \arrow[d, bend right, "\alpha_{n-1}'"'] \\
F_n' \arrow[r, "f_n'"'] & F_{n-1}'
\end{tikzcd}$$

Need to construct a chain homotopy between two such extensions. Namely, if $\alpha_n' : F_n \to F_n'$ are another extension then we must construct maps

$$\lambda_n : F_n \to F_{n+1}'$$

defining a chain homotopy from $\beta_n = \alpha_n - \alpha_n'$

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to the zero map: $\beta_n = f'_{n+1}\lambda_n + \lambda_{n-1} f_n$.

$$\begin{tikzcd}
\cdots \arrow[r] & F_{n+1} \arrow[r] \arrow[d] & F_n \arrow[r, "f_n"] \arrow[d, "\alpha_n", "\alpha_n'"'] \arrow[dl, "\lambda_n"'] & F_{n-1} \arrow[r] \arrow[d] \arrow[dl, "\lambda_{n-1}"'] & \cdots \\
\cdots \arrow[r] & F'_{n+1} \arrow[r, "f'_{n+1}"'] & F'_n \arrow[r] & F'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$\beta_n = \alpha_n - \alpha_n' = f'_{n+1} \circ \lambda_n + \lambda_{n-1} \circ f_n$

Construction of λ_n:

$n=0$, $\alpha_0 - \alpha_0' = f_1' \circ \lambda_0 + \lambda_{-1} \circ f_0$

$\lambda_{-1} : F_{-1} \longrightarrow F_0'$ will be defined to be zero.
$\lambda_{-1} = 0$.
$\Rightarrow \beta_0 = f_1' \circ \lambda_0$

We'll define λ_n, $n \geq 0$, inductively.
$\lambda_0 : F_0 \to F_1'$. For any basis element $x \in F_0$ define $\lambda_0(x) \doteq x' \in F_1'$ so that $f_1'(x') = \beta_0(x)$. Such x' exists since $\mathrm{Im} f_1' = \ker f_0'$ and $f_0'(\beta_0(x)) = \beta f_0(x) = 0$.

$$\begin{tikzcd}
F_2 \arrow[r, "f_2"] \arrow[d] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1", "\alpha_1'"'] \arrow[dl, "\lambda_1"'] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0", "\alpha_0'"'] \arrow[dl, "\lambda_0"'] & H \arrow[r] \arrow[d, "\alpha", "\alpha'"'] \arrow[dl, "\lambda_{-1}"'] & 0 \\
F_2' \arrow[r, "f_2'"'] & F_1' \arrow[r, "f_1'"'] & F_0' \arrow[r, "f_0'"'] & H' \arrow[r] & 0
\end{tikzcd}$$

$\beta_i = \alpha_i - \alpha_i'$
$\beta = \alpha - \alpha'$

By induction we see that λ_n exists by similar arguments.

λ_n's are constructed to give chain homotopy between β_n and 0 homomorphisms. Taking dual of the above diagram we get
$\alpha_n : F_n \to F_n' \Rightarrow \alpha_n^* : F_n'^* \to F_n^*$
$(\mathrm{Hom}(-, G))$

$\beta_n^* = \alpha_n^* - \alpha_n'^*$ and $\beta_n^* = \alpha_n^* - \alpha_n'^* = \lambda_n^* f_{n+1}'^* + f_n^* \lambda_{n-1}^*$ which gives a cochain homotopy between cochain complexes.

This implies that the homomorphisms

$$\alpha_n^* = \alpha_n'^* : \underset{[\varphi]}{H^n(F';G)} \longrightarrow H^n(F;G).$$

$$(\alpha_n^* - \alpha_n'^*)(\varphi) = \lambda_n^* \underbrace{f_{n+1}'^*(\varphi)}_{\overset{\|}{0}} + \underbrace{f_n^* \underbrace{\lambda_{n-1}^*(\varphi)}}_{\text{coboundary}}$$

$$\Rightarrow [\alpha_n^*(\varphi)] = [\alpha_n'^*(\alpha)].$$

Next consider homomorphisms $H \xrightarrow{\alpha} H' \xrightarrow{\beta} H''$ with a resolution F'' of H'':

$$\cdots \longrightarrow F_1'' \rightarrow F_0'' \rightarrow H'' \rightarrow 0.$$

Similarly, we can construct β_n':. In particular, if α is an isomorphism taking $\beta : H'' = H' \rightarrow H$ as its inverse then $\alpha \circ \beta = id_{H'}$ and $\beta \circ \alpha = id_H$. Taking duals we get $\alpha^* \circ \beta^* = id$ and $\beta^* \circ \alpha^* = id$. Hence, α^* would be an isomorphism.

Finally, taking $\alpha = id_H$, which means F and F' are two resolutions for H,

$$\begin{tikzcd}
F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & H \arrow[r] \arrow[d, equal, "\alpha = id"] & 0 \\
F_2' \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r] & H \arrow[r] & 0
\end{tikzcd},$$

we obtain the isomorphism

$$Id^* : H^n(F';G) \rightarrow H^n(F;G).$$

Remark: If H is an abelian group then we have a free resolution for H as follows:

$$\begin{tikzcd} 0 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & H \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 & F_2^* \arrow[l, "f_2^*"'] & F_0^* \arrow[l, "f_1^* = i^*_{n-1}"'] & H^* \arrow[l, "f_0^*"'] & 0 \arrow[l] \end{tikzcd}$$

$$H^2(F;G) = \frac{\ker f_3^*}{\operatorname{Im} f_2^*} = \frac{(0)}{(0)} = 0. \quad \left(\text{Similarly, } H^n(F,G)=0 \ \forall\ n \geq 3\right)$$

$$H^1(F;G) = \frac{\ker f_2^*}{\operatorname{Im} f_1^*} = \frac{F_2^*}{\operatorname{Im} f_1^*} = \operatorname{Ext}(H;G)$$

$$H^0(F;G) = \frac{\ker f_1^*}{\operatorname{Im} f_0^*}$$

Theorem: If C is a chain complex C ($C = C_n(X)$, $\Delta_n(X)$, $H_n(X^n, X^{n-1})$) of free abelian groups with homology $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $\operatorname{Hom}(C_n, G)$ are determined by the split exact sequences

$$0 \to \operatorname{Ext}(H_{n-1}(C);G) \to H^n(C;G) \xrightarrow{h} \operatorname{Hom}(H_n(C);G) \to 0.$$

This is known as the Universal Coefficient Theorem for cohomology.

How to compute $\operatorname{Ext}(H_{n-1}(C);G)$?
We'll make use of the following properties:

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1) $\mathrm{Ext}(H \oplus H'; G) \simeq \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$

2) $\mathrm{Ext}(H, G) = 0$ if H is free.

3) $\mathrm{Ext}(\mathbb{Z}_n, G) \approx G/nG$

Proof of these properties:

1) $$\cdots \to F_1 \to F_0 \to H \to 0$$
$$\cdots \to F_1' \to F_0' \to H' \to 0$$

$$\Rightarrow \cdots \to F_1 \oplus F_1' \to F_0 \oplus F_0' \to H \oplus H' \to 0$$

$$\Rightarrow \cdots \leftarrow F_1^* \oplus F_1'^* \leftarrow F_0^* \oplus F_0'^* \leftarrow H^* \leftarrow H'^* \leftarrow 0$$

$$\Rightarrow H^n(F \oplus F', G) \simeq H^n(F, G) \oplus H^n(F', G)$$

$$\Rightarrow \mathrm{Ext}(H \oplus H'; G) \simeq \mathrm{Ext}(H; G) \oplus \mathrm{Ext}(H', G).$$

2) $0 \to H \xrightarrow{id} H \to 0 \Rightarrow 0 \leftarrow H^* \leftarrow H^* \leftarrow 0$

$\Rightarrow \mathrm{Ext}(H; G) = 0.$

3) $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$ free resolution of the cyclic group of order n.

$$\begin{tikzcd}
0 & \mathrm{Ext}(\mathbb{Z}_n, G) \arrow[l] & \mathrm{Hom}(\mathbb{Z}, G) \arrow[l] \arrow[d, phantom, "\wr\wr"] & \mathrm{Hom}(\mathbb{Z}; G) \arrow[l, "\times n"'] \arrow[d, phantom, "\wr\wr"] & \mathrm{Hom}(\mathbb{Z}_n, G) \arrow[l] & 0 \arrow[l] \\
 & & G & G \arrow[l, "\times n"'] & &
\end{tikzcd}$$

$\mathrm{Ext}(\mathbb{Z}_n, G) \simeq G/nG$

Corollary If the homology group H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups T_n of H_n and $T_{n-1} \subseteq H_{n-1}$, then

$$H^n(C;\mathbb{Z}) \approx (H_n/T_n) \oplus T_{n-1}$$

Proof: We have the split exact sequence

$$0 \to \text{Ext}(H_{n-1}(C),G) \to H^n(C,G) \xrightarrow{h} \text{Hom}(H_n(C),G) \to 0$$

$$G=\mathbb{Z} \Rightarrow$$

$$0 \to \text{Ext}(\underset{\text{IS}}{H_{n-1}},\mathbb{Z}) \to H^n(C;\mathbb{Z}) \xrightarrow{h} \text{Hom}(\underset{\text{IS}}{H_n}(C);\mathbb{Z}) \to 0$$

$$\underbrace{H_{n-1}/T_{n-1} \oplus T_{n-1}} \qquad\qquad \underbrace{H_n/T_n \oplus T_n}_{\substack{\text{IS} \\ H_n/T_n}}$$

$$\text{Ext}(H_{n-1};\mathbb{Z}) \approx \text{Ext}(T_{n-1};\mathbb{Z})$$

$$T_{n-1} \approx \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$$

$$\text{Ext}(T_{n-1};\mathbb{Z}) \approx \bigoplus_{i=1}^{k} \text{Ext}(\mathbb{Z}_{n_i};\mathbb{Z})$$

$$\approx \bigoplus_{i=1}^{k} \mathbb{Z}/n_i\mathbb{Z}$$

$$\approx \bigoplus_{i=1}^{k} \mathbb{Z}_{n_i}$$

$$\approx T_{n-1}$$

∎

Corollary 2A a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomomorphism on cohomology groups with any coefficient group G.

$$\begin{tikzcd}
{} \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & C_n \arrow[r] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
{} \arrow[r] & C'_{n+1} \arrow[r] & C'_n \arrow[r] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$\underline{f_{n*}} : H_n(C) \to H_n(C')$ isom.

$$\Rightarrow \begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(C), G) \arrow[r] & H^n(C;G) \arrow[r, "h"] & \mathrm{Hom}(H_n(C), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(C'), G) \arrow[r] \arrow[u, "f^*_{n-1}", "\cong"'] & H^n(C';G) \arrow[r, "h"'] \arrow[u, "f^*_n = \text{isom.}"'] & \mathrm{Hom}(H_n(C'), G) \arrow[r] \arrow[u, "f^*_n", "\cong"'] & 0
\end{tikzcd}$$

By naturality of the sequence $0 \to \mathrm{Ext} \cdots \to \quad \to \mathrm{Hom}(\ldots) \to 0$ and 5 lemma give the result.

Remark: 1) take $C_n = C_n(X)$ the singular chain complex of a topological space X.

$$0 \to \mathrm{Ext}(H_{n-1}(X); G) \to H^n(X;G) \xrightarrow{h} \mathrm{Hom}(H_n(X), G) \to 0$$

$$n=1 \Rightarrow 0 \to \mathrm{Ext}(\underset{\substack{\uparrow \\ \text{free abelian}}}{H_0(X)}; G) \to H^1(X;G) \xrightarrow{h} \mathrm{Hom}(H_1(X), G) \to 0$$

free abelian $\Rightarrow \mathrm{Ext} = 0$.

$h: H^1(X;G) \to \mathrm{Hom}(H_1(X), G)$ is an isomorphism.

$$\mathrm{Hom}(H_1(X), G) \cong \mathrm{Hom}(\pi_1(X), G) \text{ if } X \text{ is connected}$$

because, any homomorphism $\pi_1(X) \to G$ descends to a homomorphism $H_1(X) \to G$, since G is abelian.

2) $G = \mathbb{F}$ a field $\Rightarrow \mathrm{Ext}(H, \mathbb{F}) = 0$.
Read from the book.

$\Rightarrow H^n(X;F) \approx \mathrm{Hom}_{\mathbb{F}}(H_n(X;\mathbb{F}), \mathbb{F})$

$C_n(X)$ $\quad$ $C_n(X;\mathbb{F})$ $\quad$ $\underline{\underline{\textcircled{\mathbb{Z}}}}$

Abelian group $\to$ free $\mathbb{Z}$-module
$\to$ free $\mathbb{F}$-module

<u>Reduced Cohomology</u> $\tilde{H}^n(X;G)$

$(*)$ $\quad C_n(X) \to \cdots \to C_1(X) \to \underline{C_0(X)} \xrightarrow{\epsilon} \mathbb{Z} \to 0$

$\epsilon(\sum n_i \sigma_i) = \sum n_i$

Homology of $(*)$ is reduced homology $\tilde{H}_n(X)$.

Take $\mathrm{Hom}(-, G)$ dual of $(*)$:

$\cdots\cdots \leftarrow C_0(X)^* \leftarrow \mathbb{Z}^* \leftarrow 0$

The cohomology of this cochain complex is defined to be $\tilde{H}^n(X;G)$.

$\mathbb{Z}^* = \mathrm{Hom}(\mathbb{Z}; G) \simeq G$. $\tilde{H}^n(X;G) = H^n(X;G)$ if $n \geq 1$. Also, $\tilde{H}^0(X;G) = H^0(X,G)/G$: is the set of all functions $X \to G$ that are constant on path components modulo functions that are constant on X.

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Relative Groups and Long Exact Sequence of a Pair:

(X,A) topological pair

$$\Rightarrow 0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} \overset{\overset{C_n(X)/C_n(A)}{\text{IS}}}{C_n(X,A)} \to 0 \quad (**)$$

$\uparrow$ free abelian group generated by singular simplices whose image not contained in A.

$$\Rightarrow 0 \leftarrow C_n^*(A) \leftarrow C_n^*(X) \leftarrow C_n^*(X,A) \leftarrow 0$$ is

still exact since $C_n(X;A)$ is free abelian and thus the sequence $(**)$ is split exact.

In particular, $C_n^*(X,A) \simeq C_n^*(X)/C_n^*(A)$ and

hence $C_n^*(X,A)$ consists of cochains on X that vanish on chains contained in A.

The $\mathrm{Hom}(-,G)$ dual of the complex

$$\cdots \to C_{n+1}(X,A) \xrightarrow[\partial]{} C_n(X,A) \xrightarrow[\partial]{} C_{n-1}(X,A) \to \cdots$$

is $$\cdots \leftarrow C_{n+1}^*(X,A) \xleftarrow{\delta} C_n^*(X,A) \xleftarrow{\delta} C_{n-1}^*(X,A) \leftarrow \cdots$$

and its cohomology defined as

$$H^n(X,A;G) \doteq \frac{\ker(\delta: C_n^*(X,A) \to C_{n+1}^*(X,A))}{\mathrm{Im}(\delta: C_{n-1}^*(X,A) \to C_n^*(X,A))}.$$

The exact sequence of cochain complexes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & C^{n-1}(A;G) \arrow[l] \arrow[d, "\delta"] & C^{n-1}(X;G) \arrow[l, "i^*"'] \arrow[d, "\delta"] & C^{n-1}(X,A;G) \arrow[l, "j^*"'] \arrow[d, "\delta"] & 0 \arrow[l] \\
0 & C^{n}(A;G) \arrow[l] \arrow[d, "\delta"] & C^{n}(X;G) \arrow[l] \arrow[d, "\delta"] & C^{n}(X,A;G) \arrow[l] \arrow[d, "\delta"] & 0 \arrow[l] \\
0 & C^{n+1}(A;G) \arrow[l] \arrow[d] & C^{n+1}(X;G) \arrow[l] \arrow[d] & C^{n+1}(X,A;G) \arrow[l] \arrow[d] & 0 \arrow[l] \\
 & {} & {} & {} &
\end{tikzcd}$$

This gives a long exact sequence in cohomology

$$\to H^n(X,A;G) \to H^n(X;G) \to H^n(A;G) \to H^{n+1}(X,A;G) \to \cdots$$

Exact sequence for a triple: (X,A,B), $B \subseteq A \subseteq X$

$$\begin{tikzcd}
0 & C^n(A,B;G) \arrow[l] \arrow[d, phantom, sloped, "\cong"] & C^n(X,B;G) \arrow[l] \arrow[d, phantom, sloped, "\cong"] & C^n(X,A;G) \arrow[l] \arrow[d, phantom, sloped, "\cong"] & 0 \arrow[l] \\
 & C^n(A)/C^n(B) & C^n(X)/C^n(B) & C^n(X)/C^n(A) &
\end{tikzcd}$$

This is clearly exact and induces a long exact sequence in cohomology

$$\to H^n(X,A;G) \to H^n(X,B;G) \to H^n(A,B;G) \to H^{n+1}(X,A;G)$$

Remark:

$$\begin{tikzcd}
H^n(A;G) \arrow[r, "\delta"] \arrow[d, "h"] & H^{n+1}(X,A;G) \arrow[d, "h"] \\
\mathrm{Hom}(H_n(A),G) \arrow[r, "\partial^*"] & \mathrm{Hom}(H_{n+1}(X,A),G)
\end{tikzcd}$$

$$H_{n+1}(X,A) \xrightarrow{\partial} H_n(A)$$

$$h\delta = \partial^* h$$

Induced Homomorphisms:

$f: X \to Y$ continuous map.

$\Rightarrow$ $f_\#: C_n(X) \to C_n(Y)$ homomorphism

$\Rightarrow$ $f^\#: C^n(Y,G) \to C^n(X,G)$ homomorphism

$$\begin{tikzcd} C^n(Y,G) \arrow[d, phantom, "\shortparallel"] & C^n(X,G) \arrow[d, phantom, "\shortparallel"] \\ \mathrm{Hom}(C_n(Y),G) & \mathrm{Hom}(C_n(X),G) \end{tikzcd}$$

$\Rightarrow$ $f^*: H^n(Y;G) \to H^n(X;G)$

Similarly, for map of pairs $f:(X,A) \to (Y,B)$ we have
$f^*: H^n(Y,B;G) \to H^n(X,A;G)$.

$(f \circ g)^* = g^* \circ f^*$, $g: X \to Y$, $f: Y \to Z$

Also the diagram below is commutative:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X,A);G) \arrow[r] & H^n(X,A;G) \arrow[r, "h"] & \mathrm{Hom}(H_n(X,A),G) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Ext}(H_{n-1}(Y,B);G) \arrow[r] \arrow[u, "(\downarrow f_*)^* = f^*"] & H^n(Y,B;G) \arrow[r, "h"] \arrow[u, "f^*"] & \mathrm{Hom}(H_n(Y,B);G) \arrow[r] \arrow[u, "(\downarrow f_*)^* = f^*"] & 0 \end{tikzcd}$$

This is commutative by the naturality of Universal Coeff. Thm.

Homotopy Invariance:

If $f \simeq g: (X,A) \to (Y,B)$ are homotopic maps

then we have $g_\# - f_\# = \partial P + P\partial$, which

implies that $g_* = f_*$ on homology, when

$\underline{P}$ is the "Prism Operator".

Taking dual we get

$g^\# - f^\# = P^* \delta + \delta P^*$, $\quad (\underline{\delta = \partial^*})$

a cochain homotopy from $f^\#$ to $g^\#$.

This cochain homotopy implies that $f^* = g^*$ on cohomology.

<u>Excision</u>: $Z \subseteq A \subseteq X$, when $\overline{Z} \subseteq \text{Int}(A)$.

$i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ inclusion map

$i_* : H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$ is an isomorphism for all n.

The proof of this fact uses the chain homomorphism

$$i : \underline{\underline{C_n(A+B)}} \to \underline{\underline{C_n(X)}}, \quad X = A \cup B, \text{ when}$$

$C_n(A+B)$ is the subcomplex of $C_n(X)$, when each simplex is contained either in A or B.

Recall that we had constructed a homomorphism

$\rho : C_n(X) \to C_n(A+B)$: subdivision operation,

which satisfies $\rho \circ i = id$, $id - i \circ \rho = \partial D + D \partial$, for a chain homotopy D.

Take $\text{Hom}(-, G)$ of this to get ρ^*, i^* between $C^n(X; G)$ and $C^n(A+B; G)$ and

$i^* \circ \rho^* = id^* = id$, $id^* - \rho^* \circ i^* = D^* \delta + \delta D^*$

which will yield isomorphism

$$H^n(X,A;G) \simeq H^n(X\setminus Z, A\setminus Z; G)$$

or $H^n(B, A\cap B; G)$.

Mayer – Vietoris for Cohomology

$X = A\cup B$, $(X = \mathrm{Int}\,A \cup \mathrm{Int}\,B)$

$$\to H_n(A\cap B) \to H_n(A)\oplus H_n(B) \to H_n(X) \to H_{n-1}(A\cap B) \to \cdots$$

Mayer – Vietoris for Homology.

$$0 \to C_n(A\cap B) \to C_n(A)\oplus C_n(B) \to C_n(A+B) \to 0$$

$$(a, b) \longmapsto a+b \quad \text{exact.}$$

$$c \longmapsto (c, -c)$$

For cohomology take Hom(−, G) of the above chain complexes so that we get

$$\to H^n(X;G) \to H^n(A;G)\oplus H^n(B,G) \to H^n(A\cap B;G) \to H^{n+1}(X,G).$$

Video 20

CUP PRODUCT:

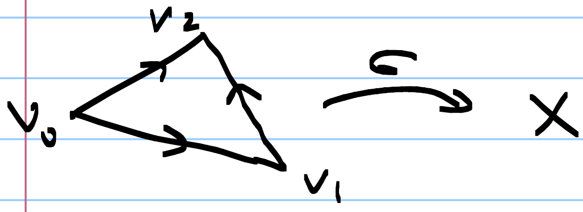
Let R be a ring most commonly $\mathbb{Z}, \mathbb{Z}_n, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$ the cup product, denoted as $\varphi \cup \psi \in C^{k+\ell}(X;R)$, is the cochain defined as

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_k,\dots,v_{k+\ell}]}),$$

where $\sigma : [v_0, \dots, v_{k+\ell}] \longrightarrow X$ is a singular $k+\ell$ simplex.

Example: $\varphi, \psi \in C^1(X;R)$, then for any two simplex $\sigma : [v_0, v_1, v_2]$ we have

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,v_1]}) \cdot \psi(\sigma|_{[v_1,v_2]})$$



Lemma: $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$, for $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$.

Proof: For a simplex $\sigma : \Delta^{k+\ell+1} \longrightarrow X$, we have

$$(\delta\varphi \cup \psi)(\sigma) = (\delta\varphi)(\sigma|_{[v_0,\dots,v_{k+1}]})\, \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$

$$= \varphi(\partial(\sigma|_{[v_0,\dots,v_{k+1}]}))\, \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$

$$= \varphi\Big(\sum_{i=0}^{k+1} (-1)^i \sigma|_{[v_0,\dots,\hat{v_i},\dots,v_{k+1}]}\Big)\, \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$

$$= \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]})\, \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$

$$(-1)^{k+1} \varphi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$

Similarly,

$$(-1)^k (\varphi \cup \delta\psi)(\sigma) = \sum_{i=k}^{k+l+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+l+1}]})$$

$$(-1)^k \varphi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$

The last term of the 1st sum cancels the first term of the 2nd sum. The remaining terms adds up to

$$(\varphi\cup\psi)(\partial\sigma) = (\varphi\cup\psi)\Big(\sum_{i=0}^{k+l+1} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+l+1}]}\Big)$$

$$= \sum_{i=0}^{k} (-1)^i \varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]})\, \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$

$$+ \sum_{i=k+1}^{k+l+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_{k+1},\dots,\hat{v}_i,\dots,v_{k+l+1}]})$$

This finishes the proof. ■

Using the lemma we define cup product on cohomology classes:

For classes $[\varphi] \in H^k(X;R)$, $[\psi] \in H^l(X;R)$

define $[\varphi] \cup [\psi] \doteq [\varphi\cup\psi]$.

must check 1) $\varphi\cup\psi$ is a cocycle.

Video 21

2) Product is well defined.

Proof of 1) $\delta(\varphi\cup\psi) = \underbrace{\delta\varphi}_{\overset{\shortparallel}{0}}\cup\psi + (-1)^k\varphi\cup\underbrace{\delta\psi}_{\overset{\shortparallel}{0}}$

$= 0.$

Proof of 2) Let $[\varphi]=[\varphi']$. Then $\varphi-\varphi'=\delta\eta$, for some $\eta\in C^{k-1}(X;R)$.

Then $\varphi\cup\psi-\varphi'\cup\psi = (\varphi-\varphi')\cup\psi$

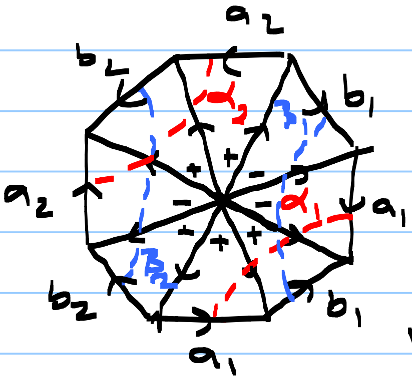
$= \delta\eta\cup\psi$

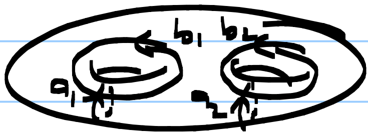
$= \delta(\eta\cup\psi)$, so that

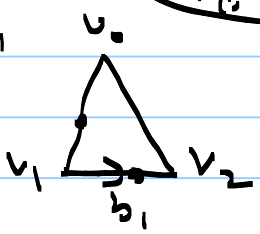
$[\varphi\cup\psi]=[\varphi'\cup\psi]$. Similar argument works for the second term of product. This finishes the proof. ∎

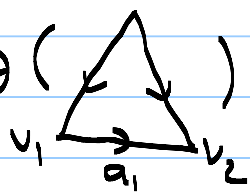
Example: 1)

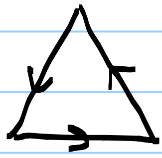
$M=\Sigma_2$







$\partial\Big($  $\Big) = [v_1,v_2]-[v_0,v_2]+[v_0,v_1]$



$H^1(M;\mathbb{Z}) = \text{Hom}(\overset{\mathbb{Z}^4}{\overset{\text{IS}}{H_1(M;\mathbb{Z})}},\mathbb{Z}) \oplus \underbrace{\text{Tor}(\overset{\mathbb{Z}}{\overset{\text{IS}}{H_0(M;\mathbb{Z})}})}_{\overset{\shortparallel}{0}}$

$\cong \mathbb{Z}^4$

$H_1(M;\mathbb{Z}) = \langle a_1, b_1, a_2, b_2 \rangle$

α_i dual to a_i, β_i dual to b_i.

$\alpha_1(a_1) = 1,\ \alpha_1(b_1) = 0,\ \alpha_1(a_2) = 0,\ \alpha_1(b_2) = 0$
$\beta_2(a_1) = 0,\ \beta_2(b_1) = 0,\ \beta_2(a_2) = 0,\ \beta_2(b_2) = 1$

$\alpha_i = [\varphi_i]$, $\beta_i = [\psi_i]$

φ_i takes the value 1 on any 1-simplex α_i intersects and takes the value 0 otherwise. Similarly, ψ_i takes the 1 on any 1-simplex β_i intersects and takes the value 0 otherwise.

$(\varphi_1 \cup \psi_1)[v_0, v_1, v_2] = \varphi_1([v_0, v_1])\psi([v_1, v_2]) = 1$ on only one 2-simplex and thus

$(\varphi_1 \cup \psi_1)[c] = 1$, when c is the sum of all 2-simplices.

$H_2(M;\mathbb{Z}) \cong \mathbb{Z} = \langle [c] \rangle$

$\varphi_1 \cup \psi_1 \in \mathrm{Hom}(H_2(M;\mathbb{Z}), \mathbb{Z}) \cong H^2(M;\mathbb{Z})$

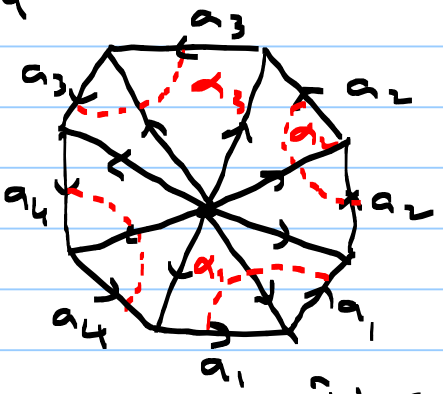
Hence, $[\varphi_1 \cup \psi_1] \in H^2(M;\mathbb{Z})$ is a generator.

$[\psi_1 \cup \varphi_1] = -[\varphi_1 \cup \psi_1]$, $[\varphi_1 \cup \varphi_2] = 0$

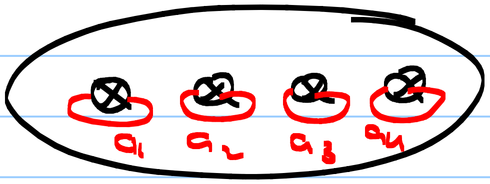
$[\psi_1, \psi_2] = 0$, $[\varphi_2 \cup \psi_2] = [\varphi_1 \cup \psi_1]$.

Example 2: N_4 Genus 4 nonorientable surface

$N_4 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 = T^2 \# KB$



$H_1(N_4;\mathbb{Z}_2) = \langle a_1, a_2, a_3, a_4 \rangle$



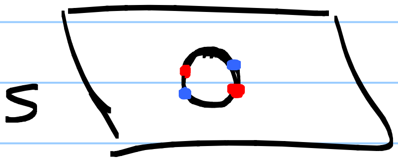
i ≠ j

$\alpha_i \in H^1(N_4;\mathbb{Z}_2) \simeq \mathbb{Z}_2^4$

$\wr\wr$

$\mathrm{Hom}(H_1(N;\mathbb{Z}),\mathbb{Z}_2)$

$\underbrace{\oplus\, \mathrm{Ext}(H_0(N,\mathbb{Z}))}_{\overset{\shortparallel}{0}}$



$(S \setminus D^2) \cup MB$

$\alpha_i \in H^1(N_4,\mathbb{Z}_2) = \mathrm{Hom}(H_1(N_4,\mathbb{Z});\mathbb{Z}_2) \simeq \mathbb{Z}_2^4$

$\alpha_i(a_j) = \delta_{ij}$

$\alpha_i \cup \alpha_j = 0$ if $i \neq j$.

$\alpha_1 \cup \alpha_1 \neq 0$, $[\alpha_1 \cup \alpha_2] \in H^2(N_4;\mathbb{Z}_2) \simeq \mathbb{Z}_2$ is a generator.

$H^2(N_4,\mathbb{Z}_2) \cong \mathrm{Hom}(\underset{\overset{\shortparallel}{0}}{H_2(N_4,\mathbb{Z})},\mathbb{Z}_2) \oplus \underbrace{\mathrm{Ext}(H_1(N_4,\mathbb{Z}),\mathbb{Z}_2)}_{\overset{\wr\wr}{\mathbb{Z}_2}}$

$H_1(N_4,\mathbb{Z}) \simeq \underline{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2}$

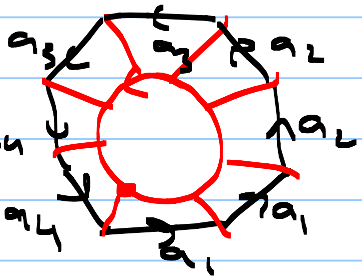
$H^1(N_4,\mathbb{Z}_2) \simeq \mathrm{Hom}(H_1(N_4,\mathbb{Z}),\mathbb{Z}_2) \oplus \underbrace{\mathrm{Ext}(\underset{\overset{\wr\wr}{\mathbb{Z}}}{H_0(N_4,\mathbb{Z})},\mathbb{Z}_2)}_{}=0$
$\simeq \mathbb{Z}_2^4 = \langle \alpha_1, \dots, \alpha_4 \rangle.$

Video 22

$$H^1(N_4,\mathbb{Z}) \simeq \text{Hom}(H_1(N_4,\mathbb{Z}),\mathbb{Z}) \oplus \overbrace{\text{Ext}(\underset{\underset{\mathbb{Z}}{\wr\wr}}{H_0(N_4,\mathbb{Z})},\mathbb{Z})}^{0}$$
$$\simeq \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}.$$

Cellular Homology Computation for N_4.

$N_4 = X$



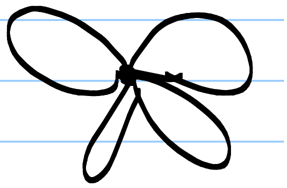
$X^0 = e^0,\quad X^1 = e^0 \cup e_1^1 \cup e_2^1 \cup e_3^1 \cup e_4^1 = \bigvee_4 S^1$

$X^2 = X = N_4 = X^1 \cup e^2$

$$\begin{tikzcd} 0 \arrow[r] & C_2(X) \arrow[r, "\partial"] \arrow[d, phantom, "\wr\wr"] & C_1(X) \arrow[r, "\partial"] \arrow[d, phantom, "\wr\wr"] & C_0(X) \arrow[r] \arrow[d, phantom, "\wr\wr"] & 0 \\ & \mathbb{Z} & \mathbb{Z}^4 & \mathbb{Z} & \end{tikzcd}$$

$e_i \xmapsto{\partial} 0$

$e^2 \longmapsto 2(e_1^1+\cdots+e_4^1)$



$$0 \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^4 \xrightarrow{0=\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$
$$1 \longmapsto (2,2,2,2)$$

$H_k(N^4;\mathbb{Z}) = 0$ for $k \geqslant 3$

$$H_2(N^4;\mathbb{Z}) = \frac{\ker\partial_2}{\text{Im}\,\partial_3} = \frac{(0)}{(0)} = (0).$$

$$H_1(N^4;\mathbb{Z}) = \frac{\ker\partial_1}{\text{Im}\,\partial_2} = \frac{\mathbb{Z}^4}{(2(1,1,1,1))} = \frac{\langle(1,0,0,0),(0,1,0,0),(0,0,0,1),(1,1,1,1)\rangle}{\langle 2(1,1,1,1)\rangle}$$

$\simeq \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$

$H_0(N;\mathbb{Z}) \simeq \mathbb{Z}.$

What about $\mathbb{Z}_2$-coefficient?

$$0 \to C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

$$0 \to \mathbb{Z}_2 \xrightarrow{\partial_2=0} \mathbb{Z}_2^4 \xrightarrow{\partial_1=0} \mathbb{Z}_2 \longrightarrow 0$$

$$1 \longmapsto \underset{\overset{\|}{(0,0,0,0)}}{(2,2,2,2)}$$

$H_3(N_4,\mathbb{Z}_2)=0$, $H_2(N_4;\mathbb{Z}_2)\simeq\mathbb{Z}_2$, $H_1(N_4,\mathbb{Z}_2)\simeq\mathbb{Z}_2^4.$

$H_0(N_4,\mathbb{Z}_2)=0.$

Cup Product Relative Cycles

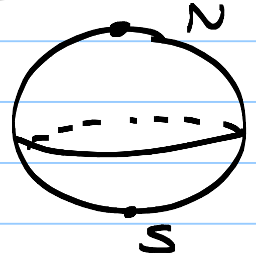
$H^k(X;R)\times H^\ell(X;R) \longrightarrow H^{k+\ell}(X;R)$

$H^k(X,A;R)$, $C^k(X,A;R)$: k-cochains that
$A\subseteq X$ subspace evaluates zero on simplicies
contained in A.

$$\begin{tikzcd}
C^k(X,A;R)\times C^\ell(X,B;R) \arrow[r, "\cup"] & C^{k+\ell}(X,\underline{\underline{A+B}};R) \\
(\varphi,\psi) \arrow[r, mapsto] \arrow[d, squiggly] & \varphi\cup\psi \\
 & H^{k+\ell}(X,A\cup B;R) \arrow[d, phantom, "\cong" rotate=90] \\
H^k(X,A;R)\times H^\ell(X,B;R) \arrow[r] & H^{k+\ell}(X,A+B;R)
\end{tikzcd}$$

Corollary 2 | X is a topological space so that $X = U_1 \cup U_2 \cup \cdots \cup U_n$, where each U_i is open and contractible then $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n = 0$ for any choice of cohomology classes $\alpha_i \in H^{k_i}(X;R)$, $k_i \geq 1$.

Ex $S^n = U \cup V$, $\quad \underset{\substack{\wr \\ \mathbb{R}^n}}{U = S^n \setminus \{N\}}$, $\quad \underset{\substack{\wr \\ \mathbb{R}^n}}{V = S^n \setminus \{S\}}$



both contractible

Hence, by the above corollary any product $\alpha \cup \beta = 0$ if $\alpha \in H^k(S^n;R)$ and $\beta \in H^l(S^n;R)$, $k, l \geq 1$.

Proof of the Corollary:

$H^i(X, U_k; R) \simeq H^i(X)$ if $i > 0$. Just consider the relative sequence of the pair (X, U_k):

$$\begin{tikzcd} H^{i-1}(X) \arrow[r] & H^{i-1}(U_k) \arrow[r, "\overset{0}{=}"] & H^i(X,U_k) \arrow[r, "\downarrow \simeq"'] & H^i(X) \arrow[r] & H^i(U_k) \arrow[d, phantom, "\wr"] \\ & & & & 0 \end{tikzcd}$$

$\alpha_i \in H^{k_i}(X) \simeq H^{k_i}(X, U_i)$

$$\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n \in H^{k_1+k_2+\cdots+k_n}(X, \underbrace{U_1 \cup U_2 \cup \cdots \cup U_n}_{\overset{\shortparallel}{X}}) = (0).$$

■

Proposition: For any map $f: X \to Y$ the induced maps $f^*: H^n(Y,R) \to H^n(X,R)$ satisfy

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta).$$

Hence, f^* is a ring homomorphism from $\bigoplus_{k=0}^{\infty} H^k(Y;R) \doteq H^*(Y;R)$ to $\bigoplus_{k=0}^{\infty} H^k(X;R) \doteq H^*(X;R)$

Proof: On the (co) chain level we have

$$\begin{aligned}(f^\# \varphi) \cup (f^\# \psi)(\sigma) &= (f^\# \varphi)(\sigma|_{[v_0,\dots,v_k]}) \cdot \\ &\quad (f^\# \psi)(\sigma|[v_k,\dots,v_\ell]) \\ &= \varphi(f_\#(\sigma|_{[v_0,\dots,v_k]})) \\ &\quad \cdot \psi(f_\#(\sigma|_{[v_k,\dots,v_\ell]})) \\ &= (\varphi \cup \psi)(f_\#(\sigma)) \\ &= f^\#(\varphi \cup \psi)(\sigma),\end{aligned}$$

for any $k+\ell$ simplex $\sigma: [v_0,\dots,v_{k+\ell}] \to X$ or X. Hence,

$$f^\#(\varphi) \cup f^\#(\psi) = f^\#(\varphi \cup \psi),$$ which

yields $f^*([\varphi]) \cup f^*([\psi]) = f^*([\varphi] \cup [\psi])$.

∎

Definition: Cross Product (or External Cup Product)

$$H^k(X;R) \times H^\ell(Y;R) \longrightarrow H^{k+\ell}(X \times Y; R)$$
$$(a \times b) \longmapsto p_1^*(a) \cup p_2^*(b),$$

where $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ are the projections.

Relative Version:

$$H^k(X,A;R) \times H^\ell(Y,B;R) \longrightarrow H^{k+\ell}(X \times Y,\ A \times Y \cup X \times B; R)$$
$$(a,\ b) \longmapsto \underline{\underline{p_1^*(a)}} \cup \underline{\underline{p_2^*(b)}}$$

$$\begin{tikzcd} \sigma \in C_k(A \times Y) \arrow[d, "p_{1\#}"] \\ C_k(A) \end{tikzcd}$$

$$p_1^{\#}(a)(\sigma) = a(\underline{\underline{p_{1\#}(\sigma)}})$$
$$= a.$$

Definition: The cohomology ring with R-coefficients of a topological space X is the direct sum
$$\bigoplus_{k=0}^{\infty} H^k(X;R)$$
whose multiplication is given by the cup product.

Theorem: $H^*(\mathbb{RP}^n; \mathbb{Z}_2) \simeq \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$,

$H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \simeq \mathbb{Z}_2[\alpha]$, where $\deg \alpha = 1$.
$(\alpha \in H^1(\mathbb{RP}^n; \mathbb{Z}))$

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Similarly, $H^*(\mathbb{CP}^n;\mathbb{Z}) \simeq \mathbb{Z}[\alpha]/(\alpha^{n+1})$

and $H^*(\mathbb{CP}^\infty;\mathbb{Z}) \simeq \mathbb{Z}[\alpha]$, where $\deg\alpha = 2$ ($\alpha \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$).

Definition: A topological manifold is a second countable Hausdorff topological space so that every point of the space has an open neighborhood homeomorphic to some $\mathbb{R}^n$.

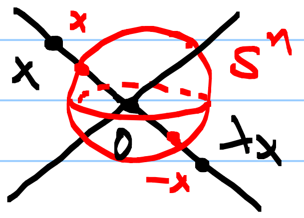
X top. space, Hausdorff, 2^{nd} countable
$x \in X$, $\exists\ x \in U \subseteq X$ open and $\exists\ \varphi: U \longrightarrow \mathbb{R}^n$
a homeomorphism.

Remark: Note that if X is a connected topological manifold then there is some $n \in \mathbb{N}$ s.t. for each $x \in X$ there is some $\varphi: U \longrightarrow \mathbb{R}^n$ homeomorphism for some open subset U of X with $x \in U$. The integer n is called the dimension of X.

The assumption that X is Hausdorff and second countable is required to embed manifolds into Euclidean spaces.

The Real Projective Space $\mathbb{RP}^n$:

$\mathbb{RP}^n$: the space of lines in $\mathbb{R}^{n+1}$ through the origin.



$\lambda \neq 0$

$\underline{\mathbb{RP}}^n = \mathbb{R}^{n+1} \setminus \{0\} \big/ x \sim \lambda x, \ \lambda \in \mathbb{R} \setminus \{0\}.$

$\underline{\mathbb{RP}}^n$ has the quotient topology. Note that

$S^n \big/ \substack{x \sim -x \\ x \in S^n}$ is homeomorphic to $\mathbb{RP}^n$

$$S^n \xrightarrow{p} S^n \big/_{x \sim -x} \doteq \underline{\mathbb{RP}}^n$$

S^n is compact and connected and thus $\mathbb{RP}^n$ is compact and connected.

$S^n \longrightarrow \underline{\mathbb{RP}}^n$ double covering and $\mathbb{RP}^n$ is a manifold of dimension n.

$(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, the equivalence class of this point in $\underline{\mathbb{RP}}^n$ is denoted as $[x_0 : \dots : x_n]$.

$[x_0 : \quad : x_n] = \{ \lambda(x_0, \_, x_n) \mid \lambda \neq 0 \}.$

$U_k = \{ [x_0 : \dots : x_n] \mid x_k \neq 0 \}.$

$U_k \longrightarrow \mathbb{R}^n, \ [x_0 : \dots : x_k : \dots x_n] \longmapsto (x_0/x_k, \dots, \widehat{\frac{x_k}{x_k}}, \dots, \frac{x_n}{x_k})$

is a homeomorphism.

$$\mathbb{RP}^n = U_0 \cup U_1 \cup \dots \cup U_n$$

$\underline{\underline{Ex}}$ $\mathbb{RP}^1 = U_0 \cup U_1$

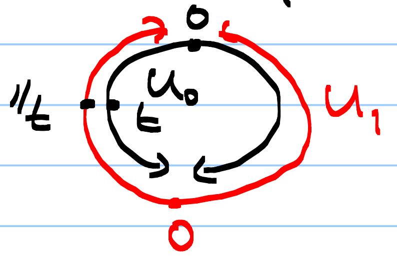
$U_0 = \{ [x_0 : x_1] \mid x_0 \neq 0 \} \longleftrightarrow \mathbb{R}$

$\qquad [x_0 : x_1] \longleftrightarrow x_1/x_0 = t$

$U_1 = \{[x_0 : x_1] \mid x_1 \neq 0\} \longleftrightarrow \mathbb{R}$

$[x_0 : x_1] \longleftrightarrow x_0/x_1 = 1/t$

$\mathbb{RP}^1 = U_0 \cup U_1 = \mathbb{R} \,\dot\cup\, \mathbb{R} / t \sim 1/t ,\ t \neq 0$

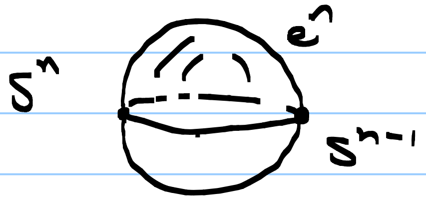


$\Rightarrow \mathbb{RP}^1 \simeq S^1.$

Homology of $\mathbb{RP}^n$:

$S^n = \underbrace{\underbrace{(e^0_1 \cup e^0_2)}_{S^0} \cup (e^1_1 \cup e^1_2)}_{S^1} \cup \dots\dots \cup (e^n_1 \cup e^n_2)$

$\mathbb{RP}^n = S^n / x \sim -x$

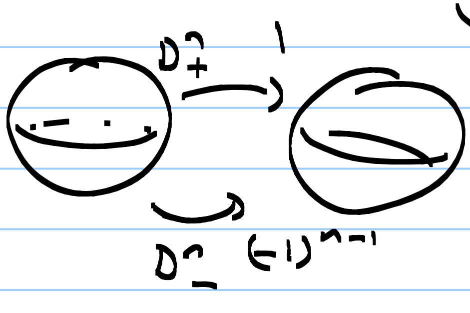


$\tau : S^n \to S^n,\ x \mapsto -x$

$\tau(e^i_1) = e^i_2$

$\mathbb{RP}^n = \underset{\mathbb{R}^0}{\underset{\shortparallel}{e^0}} \cup \underset{\mathbb{R}^1}{\underset{\shortparallel}{e^1}} \cup \underset{\mathbb{R}^2}{\underset{\shortparallel}{e^2}} \cup \dots \cup \underset{\mathbb{R}^n}{\underset{\shortparallel}{e^n}}$

$$\begin{tikzcd}
e^n \simeq D^n & \partial D^n = S^{n-1} \arrow[r] \arrow[rrrd, bend right, "2:1"] & S^{n-1} \arrow[r] \arrow[r, bend right, "2:1"'] & \mathbb{RP}^{n-1} \arrow[r] & \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \arrow[d, phantom, "\cong"] \\
& & & & S^{n-1}
\end{tikzcd}$$



$1 + (-1)^{n-1} = \begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$$\begin{tikzcd}
0 \arrow[r] & C_n(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & C_{n-1}(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \dots \arrow[r] & C_{k+1}(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & C_k(\mathbb{RP}^n) \arrow[d, phantom, "\cong" rotate=90] \\
& \mathbb{Z} & \mathbb{Z} & & \mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \arrow[r] & C_2(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & C_1(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & C_0(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
& \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "\times 0"] & \mathbb{Z}
\end{tikzcd}$$

$\underline{\underline{n=3}}$

$$\begin{tikzcd}
0 \arrow[r] & C_3 \arrow[r, "\partial_3"] \arrow[d, phantom, "\cong" rotate=90] & C_2 \arrow[r, "\partial_2"] \arrow[d, phantom, "\cong" rotate=90] & C_1 \arrow[r, "\partial_1"] \arrow[d, phantom, "\cong" rotate=90] & C_0 \arrow[r, "\partial_0"] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
& \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} &
\end{tikzcd}$$

$$H_3(\mathbb{RP}^3) = \frac{\mathbb{Z}}{(0)} \simeq \mathbb{Z}, \quad H_2(\mathbb{RP}^3) = \frac{\ker\partial_2}{\operatorname{Im}\partial_3} = (0)$$

$$H_1(\mathbb{RP}^3) = \frac{\ker\partial_1}{\operatorname{Im}\partial_2} = \frac{\mathbb{Z}}{2\mathbb{Z}} \simeq \mathbb{Z}_2.$$

$\underline{\underline{n=4}}$

$$\begin{tikzcd}
0 \arrow[r] & C_4 \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & C_3 \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & C_2 \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & C_1 \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & C_0 \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
& \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} &
\end{tikzcd}$$

$$H_4(\mathbb{RP}^4) = 0, \quad H_3(\mathbb{RP}^4) = \frac{\mathbb{Z}}{2\mathbb{Z}} \simeq \mathbb{Z}_2$$

$$H_2(\mathbb{RP}^4) = \frac{0}{(0)} = (0), \quad H_1(\mathbb{RP}^4) = \frac{\mathbb{Z}}{2\mathbb{Z}} \simeq \mathbb{Z}_2$$

What about $\mathbb{Z}_2$-coefficients?

$$\begin{tikzcd}
0 \arrow[r] & C_n(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & C_{n-1}(\mathbb{RP}^n) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \cdots \arrow[r] & C_2(\mathbb{RP}^n) \arrow[r] & C_1(\mathbb{RP}^n) & & \\
& \mathbb{Z}_2 \arrow[r, "\times 0"] & \mathbb{Z}_2 \arrow[r, "\times 0"] & & \arrow[r, "\times 0"] & C_0(\mathbb{RP}^n) \arrow[r, "\times 0"] & 0
\end{tikzcd}$$

$$H_k(\mathbb{RP}^n;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & k=0\ldots n \\ 0, & \text{otherwise} \end{cases}$$

$$H^k(\mathbb{RP}^n;\mathbb{Z}_2) = \operatorname{Hom}_{\mathbb{Z}_2}(H_k(\mathbb{RP}^n;\mathbb{Z}_2),\mathbb{Z}_2) \oplus \underbrace{\operatorname{Ext}\qquad\qquad}_{} = 0$$

$$\simeq \mathbb{Z}_2.$$

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Complex Projective Spaces

$\mathbb{CP}^n$ is the space of complex lines in $\mathbb{C}^{n+1}$ through the origin.

$$\mathbb{CP}^n = \mathbb{C}^{n+1}\setminus\{0\} \Big/ z \sim \lambda z, \quad \lambda \in \mathbb{C}^*$$

$$= \{[z_0 : \cdots : z_n] \mid z_i \neq 0, \text{ for some } i\}.$$

Again $\mathbb{CP}^n = S^{2n+1} \Big/ z \sim \lambda z \quad |\lambda| = 1 \ (\Leftrightarrow \lambda \in S^1)$

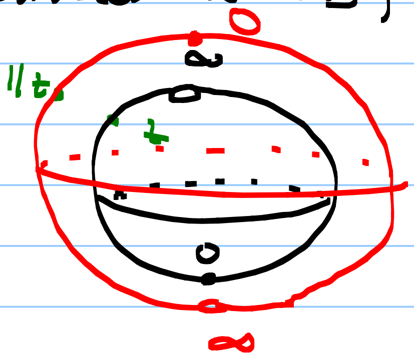
$$S^{2n+1} = \{(z_0, \dots, z_n) \mid z_i \in \mathbb{C}, \sum_{i=0}^{n} |z_i|^2 = 1\}.$$

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Similar to $\mathbb{RP}^1$, $\mathbb{CP}^1 = \mathbb{C} \dot{\cup} \mathbb{C} \Big/ z \sim 1/z, \ z \neq 0$

$\simeq S^2$

Riemann Sphere



$$\mathbb{CP}^n = e^0 \cup e^2 \cup e^4 \cup \dots \cup e^{2n}$$
$$\quad\ \ \overset{\shortparallel}{\mathbb{C}^0} \quad \overset{\shortparallel}{\mathbb{C}} \quad \overset{\shortparallel}{\mathbb{C}^2} \qquad \overset{\shortparallel}{\mathbb{C}^n}$$

$$0 \to \underset{\mathbb{Z}}{\underset{\shortparallel}{C_{2n}(\mathbb{CP}^n)}} \to \underset{0}{\underset{\shortparallel}{C_{2n-1}(\mathbb{CP}^n)}} \to \underset{\mathbb{Z}}{\underset{\shortparallel}{C_{2n-2}(\mathbb{CP}^n)}} \to \cdots$$

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to \mathbb{Z} \to 0$$

$$H_k(\mathbb{CP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0,2,\dots,2n \\ 0, & \text{otherwise.} \end{cases}$$

$$H^k(\mathbb{CP}^n;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & k=0,2,\dots,2n \\ 0, & \text{otherwise} \end{cases}$$

Quaternionic Projective Space!

$$\begin{tikzcd} \mathbb{R} \arrow[r, phantom, "\subseteq"] \arrow[d] & \mathbb{C} \arrow[r, phantom, "\subseteq"] \arrow[d] & \mathbb{H} \simeq \mathbb{R}^4 = \{a+bi+cj+dk \mid a,b,c,d \in \mathbb{R}\} \arrow[d] \\ \mathbb{RP}^n & \mathbb{CP}^n & \mathbb{HP}^n \end{tikzcd}$$

$$\mathbb{HP}^n = e^0 \cup e^4 \cup e^8 \cup \dots \cup e^{4n}$$

$$\begin{tikzcd} H_k(\mathbb{HP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0,4,8,\dots,4n \\ 0, & \text{otherwise} \end{cases} \arrow[d, phantom, sloped, "\cong"] \\ H^k(\mathbb{HP}^n;\mathbb{Z}). \end{tikzcd}$$

Theorem $H^*(\mathbb{HP}^n;\mathbb{Z}) \simeq \mathbb{Z}[\alpha]/(\alpha^{n+1})$

$\deg \alpha = 4$, $H^*(\mathbb{HP}^\infty;\mathbb{Z}) \simeq \mathbb{Z}[\alpha]$.

Theorem: If R is a commutative ring then

$\alpha \cup \beta = (-1)^{k\ell} \beta \cup \alpha$, for any classes

$\alpha \in H^k(X, A; R)$, $\beta \in H^\ell(X, A; R)$, for any pair of spaces (X, A).

Proof: consists of several steps.

First take $A = \emptyset$. Let $\varphi \in C^k(X; R)$, $\psi \in C^\ell(X; R)$

$\sigma : [v_0, \dots, v_{k+\ell}] \longrightarrow X$

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0, \dots, v_k]}) \cdot \psi(\sigma|_{[v_k, \dots, v_{k+\ell}]})$$

$$\psi \cup \varphi\, (\sigma) = \psi(\sigma|_{[v_0, \dots, v_\ell]})\, \varphi(\sigma|_{[v_\ell, \dots, v_{k+\ell}]}).$$

Thus $\varphi \cup \psi$ and $\psi \cup \varphi$ differ by a permutation of the vertices of the simplex σ.

For any n-simplex $\sigma : [v_0, \dots, v_n] \to X$, let $\bar{\sigma}$ denote the n-simplex given by
$\bar{\sigma}(v_i) = \sigma(v_{n-i})$

$\bar{\sigma} = \sigma \circ \tau$, $\tau : [v_0, \dots, v_n] \to [v_0, \dots, v_n]$
linear map s.t. $\tau(v_i) = v_{n-i}$.

Note that τ is a composition of $1 + 2 + \dots + n$ $= n(n+1)/2$ transpositions.

Let $\epsilon_n = (-1)^{n(n+1)/2}$ and define

$\rho : C_n(X) \to C_n(X)$, by $\rho(\sigma) = \epsilon_n \bar{\sigma}$.

<u>Aim:</u> Show that ρ is a chain map, chain homotopic to the identity, so that ρ induces identity on homology.

Note that the theorem is a consequence of the above fact:

$$(\rho^*\varphi \cup \rho^*\psi)(\sigma) = (\rho^*\varphi)(\sigma|_{[v_0,\dots,v_k]}) \cdot (\rho^*\psi)(\sigma|_{[v_k,\dots,v_{k+l}]})$$

$$= \varphi(\rho\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\rho\sigma|_{[v_k,\dots,v_{k+l}]})$$

$$= \epsilon_k \varphi(\sigma|_{[v_k,\dots,v_0]})\, \epsilon_l \psi(\sigma|_{[v_{k+1},\dots,v_k]})$$

Also, $\rho^*(\psi \cup \varphi)(\sigma) = (\psi \cup \varphi)(\rho\sigma)$

$$= \epsilon_{k+l}(\psi \cup \varphi)(\sigma|_{[v_{k+l},\dots,v_0]})$$

$$= \epsilon_{k+l}\, \psi(\sigma|_{[v_{k+l},\dots,v_k]})\, \varphi(\sigma|_{[v_k,\dots,v_0]})$$

$$\Rightarrow \epsilon_k \epsilon_l (\rho^*\varphi \cup \rho^*\psi) = \epsilon_{k+l}\, \rho^*(\psi \cup \varphi).$$

Moreover, $\epsilon_{k+1} = (-1)^{kl} \epsilon_k \epsilon_l$ and hence

$$\rho^*(\psi \cup \varphi) = (-1)^{kl} (\rho^*\varphi \cup \rho^*\psi)$$

Since ρ is chain homotopic to the identity

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ρ^* is cochain homotopic to the identity and thus ρ^* will disappear when we pass to the cohomology:

$$[\psi]\cup[\varphi] = (-1)^{kl}[\varphi]\cup[\psi].$$

ρ is a chain map: $\partial\rho = \rho\partial$.

For any n-simplex $\sigma : [v_0, \ldots, v_n] \to X$,

$$\begin{aligned}\partial\rho(\sigma) &= \epsilon_n \partial(\bar{\sigma}) \\ &= \epsilon_n \partial(\sigma : [v_n, \ldots, v_0] \to X) \\ &= \epsilon_n \sum_i \underline{(-1)^i}\, \sigma|_{[v_n, \ldots, \hat{v}_{n-i}, \ldots, v_0]}\end{aligned}$$

On the other hand,

$$\begin{aligned}\rho\partial(\sigma) &= \rho\Big(\sum_i (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}\Big) \\ &= \epsilon_{n-1} \sum_i (-1)^i \sigma|_{[v_n, \ldots, \hat{v}_i, \ldots, v_0]} \\ &= \epsilon_{n-1} \sum_i \underset{\parallel}{\underline{(-1)^{n-i}}}\, \sigma|_{[v_n, \ldots, \hat{v}_{n-i}, \ldots, v_0]}\end{aligned}$$

$$(-1)^n(-1)^{-i} = \underline{\underline{(-1)^n}}\,(-1)^i$$

Note that $\epsilon_n = \epsilon_{n-1}(-1)^n$ (Exercise) and thus the proof finishes.

ρ is chain homotopic to the identity:

To construct the chain homotopy we'll use the

prism operator which we used before:
The prism operator P is a subdivision of $\Delta^n \times I$ into $(n+1)$-simplices with vertices v_i in $\Delta^n \times \{0\}$ and w_i in $\Delta^n \times \{1\}$.

Let $\pi : \Delta^n \times I \to \Delta^n$ be the projection and define $P : C_n(X) \to C_{n+1}(X)$ by

$$P(\sigma) = \sum_i (-1)^i \epsilon_{n-i} (\sigma\pi)|_{[v_0, \dots, v_i, w_n, \dots, w_i]}.$$

Note that the vertices w_i's are written in reverse order and to compensate this we plug the sign ϵ_{n-i}.

Claim: $\partial P + P \partial = \rho - \mathrm{id}$

Proof: For simplicity we leave out σ's and π's.

$$\partial P = \sum_{j \le i} (-1)^i (-1)^j \epsilon_{n-i} [v_0, \dots, \hat{v}_j, \dots, v_i, w_n, \dots, w_i]$$
$$+ \sum_{j \ge i} (-1)^i (-1)^{i+1+n-j} \epsilon_{n-i} [v_0, \dots, v_i, w_n, \dots, \hat{w}_j, \dots, w_i]$$

In the above sums the terms with $i = j$ gives

$$\epsilon_n [v_0, \dots, w_0] + \sum_{i > 0} \epsilon_{n-i} [v_0, \dots, v_{i-1}, w_n, \dots, w_i]$$
$$+ \sum_{i < n} (-1)^{n+i+1} \epsilon_{n-i} [v_0, \dots, v_i, w_n, \dots, w_{i+1}]$$
$$- \epsilon_0 [v_0, \dots, v_n].$$

Note that the two sums above cancels each other, which can be seen by replacing i

with $i-1$ the second summation, which produces a new sign $(-1)^{n+i}\epsilon_{n-i+1} = -\epsilon_{n-i}$.

The remaining two terms $\epsilon_n[w_n, \ldots w_0]$ and $-[v_0, \ldots, v_n]$ represent $\rho(\sigma) - \sigma$.

So in order to prove $\partial P + P\partial = \rho - id$, we need to show that the terms in ∂P with $i \neq j$ corresponds to $-P\partial$.

Finally, $$P\partial = \sum_{i<j} (-1)^i (-1)^j \epsilon_{n-i-1} [v_0, \ldots, v_i, w_n, \ldots, \hat{w}_j, \ldots w_i]$$

$$+ \sum_{i>j} (-1)^{i-1} (-1)^j \epsilon_{n-i} [v_0, \ldots, \hat{v}_j, \ldots, v_i, w_n, \ldots, w_i].$$

Noting that $\epsilon_{n-i} = (-1)^{n-i}\epsilon_{n-i-1}$ finishes the proof in case $\Delta = \emptyset$.

For general case, just note that the same proof works in the case $A \neq \emptyset$.

Finally, one needs to work with P^* instead P, to obtain the cochain homotopy from ρ^* to the identity. ▬

A Künneth Formula:

Aim: Express $H^*(X\times Y;R)$ in terms of $H^*(X;R)$ and $H^*(Y;R)$.

$$H^*(X;R)\otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X\times Y;R)$$

$$a\otimes b \longmapsto a\times b \doteq p_1^*(a)\cup p_2^*(b),$$

where $p_1 : X\times Y \to X$ and $p_2 : X\times Y\to Y$ are projections.

Remark: $H^*(X;R)=\bigoplus_{k=0}^{\infty} H^k(X;R)$ is a graded R-algebra.

Definition: For pure elements $a,c\in H^*(X;R)$, $b,d\in H^*(Y;R)$ define the product $(a\otimes b)\cdot(c\otimes d)$ as follows:

$$(a\otimes b)\cdot(c\otimes d) \doteq (-1)^{|b||c|}\, ac \otimes bd$$

Note that

$$(-1)^{|b||c|}\, ac\times bd = (-1)^{|b||c|}\, p_1^*(ac)\cup p_2^*(bd)$$

$$= (-1)^{|b||c|}\, p_1^*(a\cup c)\cup p_2^*(b\cup d)$$

$$= (-1)^{|b||c|}\, (p_1^*(a)\cup p_1^*(c))\cup(p_2^*(b)\cup p_2^*(d))$$

$$= p_1^*(a)\cup p_2^*(b)\cup p_1^*(c)\cup p_2^*(d)$$

$$= (a\times b)\cdot(c\times d)$$

Hence, $(a\otimes b)\cdot(c\otimes d) = (a\times b)(c\times d)$

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Theorem (Künneth Formula)

The cross product $H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$ is an isomorphism of rings if X and Y are CW-complexes and $H^k(Y;R)$ is a finitely generated free R-module for all k.

Idea of the proof: We'll define two cohomology functors defined for a fixed CW-complex Y, from the category pairs of CW-complexes (X,A) to the category of R-algebras:

$h^n(X,A) = \bigoplus_i \left(H^i(X,A;R) \otimes_R H^{n-i}(Y;R) \right)$, and

$k^n(X,A) = H^n(X \times Y, A \times Y; R)$.

We have a natural map $\mu : h^n(X,A) \to k^n(X,A)$ defined by the relative version of cross product.

We'll show that μ is an isomorphism, when X is a CW-complex and $A = \emptyset$ in several steps:

1) h^* and k^* are cohomology theories on the category of CW-pairs.

2) μ is a natural transformation: It commutes with induced homomorphisms and with coboundary homomorphisms in long exact sequences of pairs.

Remark: $\mu : h^n(X) \to k^n(X)$ is an isomorphism when X is a point.

$$X = \{pt\},\ h^n(X) = \bigoplus_i \big(\underset{\substack{\| \\ 0 \text{ if } i \neq 0}}{H^i(X;R)} \otimes_R H^{n-i}(Y;R)\big)$$

$$= H^n(Y;R)$$

$k^n(X) = H^n(Y;R)$ and $\mu : h^n(X) \to k^n(X)$ is an isomorphism.

The Künneth Formula (Theorem) is a consequence of the proposition below:

Proposition: If a natural transformation between **unreduced** cohomology theories on the category of CW-pairs is an isomorphism when the pair is (point, $\emptyset$), then it is an isomorphism for all CW-pairs.

Before we pass to the proof of the above proposition let's review Axioms for a cohomology theory:

Axioms for Cohomology: A reduced cohomology theory is a sequence of functors $\tilde{h}^n$ from CW-complexes to abelian groups, together with natural coboundary homomorphisms

$$\delta : \tilde{h}^n(A) \to \tilde{h}^{n+1}(X/A) \text{ for CW-pairs}$$
(X, A) satisfying the following axioms:

1) If $f \simeq g : X \to Y$ are homotopic maps then $f^* = g^* : \tilde{h}^n(Y) \to \tilde{h}^n(X)$.

2) For each CW-pair (X, A) there is a long

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exact sequence

$$\xrightarrow{\delta} \tilde{h}^n(X/A) \xrightarrow{q^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \xrightarrow{q^*} \cdots,$$

where $q: X \to X/A$ is the quotient map and $i: A \to X$ is the inclusion map.

3) For a wedge sum $X = \bigvee_\alpha X_\alpha$ with inclusion function $i_\alpha : X_\alpha \to X$ the product map

$$\prod_\alpha i_\alpha^* : \tilde{h}^n(X) \longrightarrow \prod_\alpha \tilde{h}^n(X_\alpha) \text{ is an}$$

isomorphism for each n.

$$\left[\begin{array}{l} \tilde{h}^n(X) \longrightarrow \tilde{h}^n(X_1) \times \tilde{h}^n(X_2) \\ \quad a \longmapsto (i_1^*(a), i_2^*(a)) \end{array}\right]$$

Recall that the cohomologies such as singular cohomology, cellular cohomology, simplicial cohomology and Δ-cohomology all satisfy the above axioms.

Proof of the Proposition.

The general case (X, A) will follow from the absolute case and 5-lemma.
So let's take $A = \emptyset$.

Induction on $n = \dim X$.
If $n = 0$ then X is a discrete space, i.e., $X = \bigcup_{\lambda \in \Lambda} \{x_\lambda\}$. The results hold for a single point. So the general case follows from the axiom for disjoint unions:

$$h^n\left(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha ; G\right) \simeq \prod_\alpha h^n(X_\alpha, A_\alpha ; G)$$

So, $h^n\left(\coprod_\lambda \{x_\lambda\}\right) \simeq \prod_\lambda h^n(\{x_\lambda\}) \simeq \prod_\lambda k^n(\{x_\lambda\})$

$$\simeq k^n\left(\coprod_\lambda \{x_\lambda\}\right).$$

Assume the result for all dimensions $0, \dots, n-1$. For $\dim X = n$, consider the long exact sequence of the pair (X^n, X^{n-1}).

$$\begin{tikzcd}
\arrow[r] & h^m(X^n) \arrow[r] \arrow[d, "\wr ?"] & h^m(X^{n-1}) \arrow[r] \arrow[d, "\wr S"] & h^{m+1}(X^n/X^{n-1}) \arrow[r] & h^{m+1}(X^n) \arrow[d, "\wr ?"] \\
\arrow[r] & \tilde{k}^m(X^n) \arrow[r] & \tilde{k}^m(X^{n-1}) \arrow[r] & \tilde{k}^{m+1}(X^n/X^{n-1}) \arrow[r] & \tilde{k}^{m+1}(X^n)
\end{tikzcd}$$

The proof finishes by 5-lemma once we prove the result for X^n/X^{n-1}, which corresponds $\coprod_\alpha (D^n_\alpha, \partial D^n_\alpha)$. The axiom for disjoint unions further reduces this to the case $(D^n, \partial D^n)$. For the pair $(D^n, \partial D^n)$ the result follows from 5-lemma since D^n is contractible.

$$\begin{tikzcd}
0 \arrow[d, phantom, "\wr"] & & & 0 \arrow[d, phantom, "\wr"] \\
h^m(D^n) \arrow[r] \arrow[d, "\simeq"] & h^m(\partial D^n) \arrow[r] \arrow[d, "\wr S"] & h^{m+1}(D^n, \partial D^n) \arrow[r] \arrow[d, "\simeq"] & h^{m+1}(D^n) \arrow[d, "\simeq"] \\
k^m(D^n) \arrow[r] \arrow[d, phantom, "\shortparallel"] & k^m(\partial D^n) \arrow[r] \arrow[d] & h^{m+1}(D^n, \partial D^n) \arrow[r] & k^{m+1}(D^n) \arrow[d, phantom, "\wr"] \\
0 & \text{since } \dim \partial D^n = \dim S^{n-1} = n-1. & & 0
\end{tikzcd}$$

Hence, the proof finishes if X is finite dimensional.

For X infinite dimensional we need so called Telescoping Argument (Lemma 2.34). This finishes the proof of the Proposition. ∎

Proof of the Künneth Formula (theorem)

We just need to show that h^* and k^* are cohomology theories:

$h^n(X,A) = \bigoplus_i H^i(X,A;R) \otimes_R H^{n-i}(Y;R)$

$k^n(X,A) = H^n(X\times Y; A\times Y; R)$.

Axiom 1 Homotopy invariance: If $f \simeq g$ where $f,g : (X,A) \to (Z,B)$ are maps, then $f^* = g^*$.

This is clear since singular cohomology satisfies this axiom.

Axiom 2 Excision.

$X = A \cup B$, A, B subcomplexes of X.

$h^*(X,A) \simeq h^*(B, A\cap B)$ is clear since singular cohomology satisfy excision.

For k^*, note that

$$(*) \begin{cases} (A\times Y) \cup (B\times Y) = (A\cup B)\times Y \text{ and} \\ (A\times Y) \cap (B\times Y) = (A\cap B)\times Y. \end{cases}$$

$k^n(X,A) = H^n(X\times Y, A\times Y; R)$

$k^n(B, A\cap B) = H^n(B\times Y, (A\cap B)\times Y, R)$

By $(*)$ the R.H.S. are isomorphic since

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Singular cohomology satisfy Excision Axiom.

3) Long exact sequence for pairs.

$$h^n(X,A) \to h^n(X) \to h^n(A) \to h^{n+1}(X,A) \to \cdots$$

$$\bigoplus_i (\underline{H^i(X,A;R)} \otimes_R \underline{H^{n-i}(Y;R)}) \to \bigoplus_i \underline{H^i(X;R)} \otimes_R \underline{H^{n-i}(Y;R)}$$

$$\to \bigoplus_i \underline{H^i(A;R)} \otimes_R \underline{H^{n-i}(Y;R)} \to \bigoplus_i \underline{H^{i+1}(X,A;R)} \otimes \underline{H^{n-i}(Y,R)}$$

Since each $H^k(Y;R)$ is finitely generated R-module tensoring with $H^{n-i}(Y;R)$ does not spoil exactness of the sequence

$$H^i(X,A;R) \to H^i(X;R) \to H^i(A,R) \to H^{i+1}(X,A;R) \to$$

For k^* this is trivial!

4) Disjoint Union: Trivial for k^*.

$$(X,A) = \coprod_\alpha (X_\alpha, A_\alpha)$$

$$h^n(X,A) = \bigoplus_i H^i(X,A;R) \otimes_R H^{n-i}(Y,R)$$

$$= \bigoplus_i H^i\left(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha; R\right) \otimes_R H^{n-i}(Y;R)$$

$$= \bigoplus_i \left(\prod_\alpha H^i(X_\alpha, A_\alpha; R) \otimes_R H^{n-i}(Y;R)\right)$$

$$= \bigoplus_i \left(\prod_\alpha \left(H^i(X_\alpha, A_\alpha; R) \otimes_R H^{n-i}(Y;R)\right)\right)$$

$$= \prod_\alpha \left(\bigoplus_i H^i(X_\alpha, A_\alpha; R) \otimes_R H^{n-i}(Y; R) \right)$$
$$= \prod_\alpha h^n(X_\alpha, A_\alpha).$$

Here we used the following algebraic fact: If M_α are R-modules and N is finitely generated free R-module then

$$\left(\prod_\alpha M_\alpha\right) \otimes_R N \simeq \prod_\alpha (M_\alpha \otimes_R N)$$

Hence, $\tilde{h}^*$ and $\tilde{k}^*$ are cohomology theories. Naturality of μ comes from naturality of cup products and naturality of coboundary maps:

$$f : (X, A) \to (Y, B) \Rightarrow f^* : H^n(Y, B) \to H^n(X, A)$$

$$\alpha \in H^k, \beta \in H^l \Rightarrow f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta).$$

This finishes the proof of the Künneth formula.

An Application:

Theorem If $\mathbb{R}^n$ has the structure of a division algebra over $\mathbb{R}$ then n must be a power of 2.

Proof Assume that $\mathbb{R}^n$ has a division algebra structure $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(x, y) \mapsto x \cdot y$.
Consider the map $\underline{\underline{g}} : \underline{\underline{S^{n-1} \times S^{n-1}}} \to S^{n-1}$ given by

$g(x,y) = \frac{xy}{|xy|}$. g is continuous by bilinearity of the multiplication.

Note that $g(-x,y) = g(x,-y) = -g(x,y)$ and $g(-x,-y) = g(x,y)$ and thus g induces a map

$$\begin{array}{cccccc} h: & S^{n-1}/_{x\sim -x} & \times & S^{n-1}/_{x\sim -x} & \longrightarrow & S^{n-1}/_{x\sim -x} \\ & \wr| & & \wr| & & \wr| \\ & \mathbb{RP}^{n-1} & \times & \mathbb{RP}^{n-1} & & \mathbb{RP}^{n-1} \\ & \alpha & & \beta & & \gamma \end{array}$$

$H^{n-1}(\mathbb{RP}^{n-1};\mathbb{Z}_2) \simeq \mathbb{Z}_2[\alpha]/(\alpha^n)$,

$H^{n-1}(\mathbb{RP}^{n-1};\mathbb{Z}_2) \simeq \mathbb{Z}_2[\beta]/(\beta^n)$

$H^{n-1}(\mathbb{RP}^{n-1};\mathbb{Z}_2) \simeq \mathbb{Z}_2[\gamma]/(\gamma^n)$

By Künneth Formula,

$$H^*(\mathbb{RP}^{n-1}\times\mathbb{RP}^{n-1};\mathbb{Z}_2) \simeq H^*(\mathbb{RP}^{n-1};\mathbb{Z}_2)\otimes_{\mathbb{Z}_2} H^*(\mathbb{RP}^{n-1};\mathbb{Z}_2)$$
$$\simeq \mathbb{Z}_2[\alpha,\beta]/(\alpha^n,\beta^n).$$

$$\begin{array}{ccc} g^*: H^{n-1}(\mathbb{RP}^{n-1};\mathbb{Z}_2) & \longrightarrow & H^{n-1}(\mathbb{RP}^{n-1}\times\mathbb{RP}^{n-1};\mathbb{Z}_2) \\ \wr| & & \wr| \\ \mathbb{Z}_2[\gamma]/(\gamma^n) & & \mathbb{Z}_2[\alpha,\beta]/(\alpha^n,\beta^n) \end{array}$$

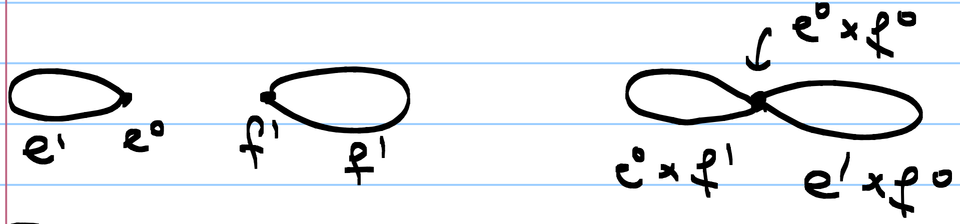
Claim $g^*(\gamma) = \alpha + \beta$.

Proof $X = X^0 \cup X^1 \cup X^2 \cup \cdots, Y = Y^0 \cup Y^1 \cup \cdots$

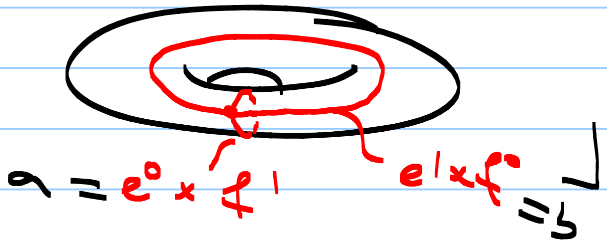
$X\times Y = (X^0\times Y^0) \cup (X^1\times Y^0 \cup X^0\times Y^1) \cup (X^2\times Y^0 \cup X^1\times Y^1 \cup \ldots)$

$\mathbb{RP}^{n-1} = e^0 \cup e^1 \cup e^2 \cup \cdots \cup e^{n-1}$
$\mathbb{RP}^{n-1} = f^0 \cup f^1 \cup f^2 \cup \cdots \cup f^{n-1}$

$\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} = (e^0 \times f^0) \cup (e^0 \times f^1 \cup e^1 \times f^0) \cup \cdots$



[$S^1 \times S^1 = T^2$

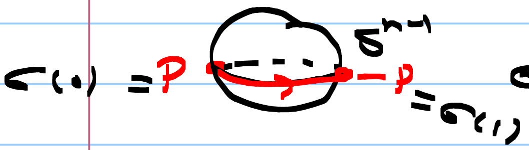


$H_1(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}) \simeq \langle a, b \rangle \longrightarrow H_1(\mathbb{RP}^{n-1})$

$a \overset{?}{\longmapsto} c$

(a, b, c generators) $b \overset{?}{\longmapsto} c$

? $H_1(\mathbb{RP}^{n-1}) = \langle a \rangle = \pi_1(\mathbb{RP}^{n-1}) \simeq \mathbb{Z}_2$



$\sigma : I \longrightarrow S^{n-1}$

If $q \in \mathbb{RP}^{n-1}$, then $\tau : I \to \mathbb{RP}^{n-1}$

$\tau(s) = g(\sigma(s), q)$ is a loop in $\mathbb{RP}^{n-1}$ generating the fundamental group, because

$\tau(0) = g(\sigma(0), q) = g(p, q)$ and
$\tau(1) = g(\sigma(1), q) = g(-p, q) = -g(p, q)$

So, if $\gamma \in H^1(\mathbb{RP}^{n-1}; \mathbb{Z}/2)$ is a generator then $g^*(\gamma)(a) = \gamma(g_*(a)) = \gamma(c) = 1$.
Similarly, $g^*(\gamma)(b) = \gamma(g_*(b)) = \gamma(b) = 1$

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and thus $g^*(\gamma) = \alpha+\beta$.

$\gamma^n = 0, \alpha^n = 0, \beta^n = 0$

$\Rightarrow (\alpha+\beta)^n = 0 \Rightarrow 0 = \sum \binom{n}{k}\alpha^k \beta^{n-k} \pmod 2$

$\mathbb{Z}_2[\alpha,\beta]/(\alpha^n,\beta^n)$ $\Downarrow$

$\alpha^n + \binom{n}{1}\alpha^{n-1}\beta + \cdots + \binom{n}{i}\alpha^{n-i}\beta^i + \cdots + \beta^n = 0 \pmod 2$

$\Rightarrow \binom{n}{k} = 0$, for all $1 \le k \le n-1$.

Claim n is a power of 2.

Proof: $\mathbb{Z}_2[x]$, $(1+x)^2 = 1+x^2$, $(1+x)^4 = 1+x^4$

$\Rightarrow (1+x)^k = 1+x^k$ if k is a power of 2.

$n = n_1+n_2+\cdots+n_k$, n_i is a power of 2 and $n_1 < n_2 < \cdots < n_k$.

[$n = 27 = 16+8+2+1$, $n_1 = 1$, $n_2 = 2$, $n_3 = 8$, $n_4 = 16$

$k=4$

$$(1+x)^{27} = (1+x)(1+x)^2(1+x)^8(1+x)^{16}$$
$$= (1+x)(1+x^2)(1+x^8)(1+x^{16})$$
$$= 1+x+x^2+x^3+ \cdots + x^{27}$$]

Note that since $n_i \ge 2n_{i-1}$ no terms combine or cancel each other and thus the resulting polynomial has 2^k terms.

In our case $(1+x)^n = 1+x^n$ so that it has only two terms. Hence $2^k = 2 \Rightarrow k = 1$ so that $n = n_1$ is a power of 2. ∎

Relative Version of Künneth Formula

$$H^*(X,A;R) \otimes_R H^*(Y,B;R) \longrightarrow H^*(X\times Y, X\times B \cup A\times Y \,;\, R)$$

$$a \otimes b \longmapsto p_1^*(a) \cup p_2^*(b)$$

$p_1 : X\times Y \to X$, $p_2 : X\times Y \to Y$ projections.

The cross product is an isomorphism if $H^k(Y,B;R)$ is a finitely generated free R-module for each k.

Example Cohomology of n-torus: $T^n = \underbrace{S^1\times S^1\times \cdots \times S^1}_{n\text{-times}}$

$$H^*(S^1;R) \simeq R[\alpha]/(\alpha^2) = \{a+b\alpha \mid a,b \in R\}.$$

$$H^*(T^n;R) = \bigotimes_n H^*(S^1;R) \qquad \deg \alpha_i = 1$$

$$\simeq R[\alpha_1,\alpha_2,\dots,\alpha_n]/(\alpha_1^2,\alpha_2^2,\dots,\alpha_n^2)$$

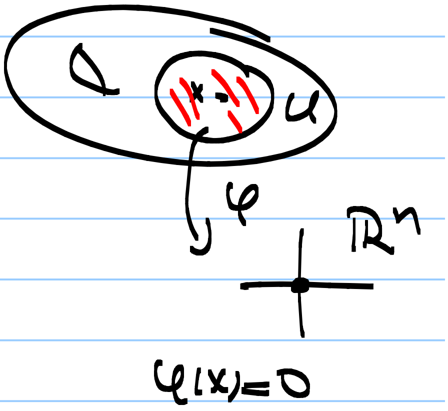
$$\alpha_i \cdot \alpha_j = -\alpha_j \cdot \alpha_i$$

Poincaré Duality

Poincaré Duality hold for topological manifolds.

Let M be a topological manifold of dimension n. Hence, every point $x \in M$ has an open neighborhood homeomorphic to $\mathbb{R}^n$.

$$x \in U \cong \mathbb{R}^n \subseteq M$$



$$H_i(M, M\setminus\{x\}) \overset{\text{Excision}}{\approx} H_i(U, U\setminus\{x\})$$

$$\approx H_i(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})$$

$$\approx \tilde{H}_{i-1}(\mathbb{R}^n\setminus\{0\})$$

$$\begin{tikzcd}
\tilde{H}_i(\mathbb{R}^n\setminus\{0\}) \arrow[r] & \tilde{H}_i(\mathbb{R}^n) \arrow[r] \arrow[d, phantom, "\wr"] & H_i(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[r, "\cong"'] & \tilde{H}_{i-1}(\mathbb{R}^n\setminus\{0\}) \arrow[d] \\
 & 0 & & 0 = \tilde{H}_{i-1}(\mathbb{R}^n)
\end{tikzcd}$$

$$\simeq \tilde{H}_{i-1}(S^{n-1}) \qquad (\mathbb{R}^n\setminus\{0\} \underset{h.e.}{\simeq} S^{n-1})$$

$$\simeq \begin{cases} \mathbb{Z} & \text{if } i=n \\ 0 & \text{otherwise} \end{cases}$$

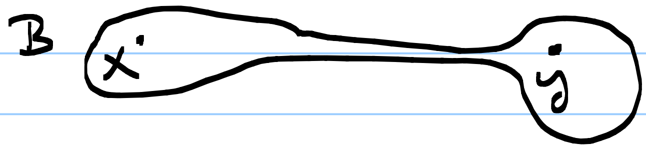
So, $H_i(M, M\setminus\{x\}) \simeq \begin{cases} \mathbb{Z} & \text{if } i=n \\ 0 & \text{otherwise.} \end{cases}$

Definition: A compact manifold without boundary is called closed.

Orientations and Homology

An orientation of $\mathbb{R}^n$ at a point x is a choice of generator of the infinite cyclic group $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \simeq \mathbb{Z}$.

Remark: An orientation at a point x of $\mathbb{R}^n$ determines an orientation at all points. Choose a ball B in $\mathbb{R}^n$ containing both x and y.



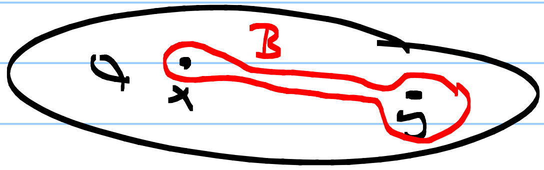
$$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \xleftarrow[i_*^x]{\simeq} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \xrightarrow[i_*^y]{\simeq} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$$

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(\mathbb{R}^n \setminus B) \arrow[r] \arrow[d, "\simeq"', "i_*"] & H_k(\mathbb{R}^n) \arrow[r] \arrow[d, "\simeq"', "i_*"] & H_k(\mathbb{R}^n, \mathbb{R}^n \setminus B) \arrow[r] \arrow[d, "\rightarrow\ \simeq"', "i_*\ (\text{5-lemma})"] & \cdots \\
\arrow[r] & \underline{H_k(\mathbb{R}^n \setminus \{x\})} \arrow[r] & H_k(\mathbb{R}^n) \arrow[r] & \underline{H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})} \arrow[r] & - \ -
\end{tikzcd}$$

Here, the isomorphism $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$ is the canonical isomorphism induced by the inclusion maps.

Note that if M is a topological n-manifold then for any $x \in M$, $H_n(M, M \setminus \{x\}) \simeq \mathbb{Z}$ is infinite cyclic. A choice of a generator μ_x of the infinite cyclic group $H_n(M, M \setminus \{x\})$ is called a local orientation of M at x.

Remark: In a manifold a local orientation may not give a canonical orientation at all points.



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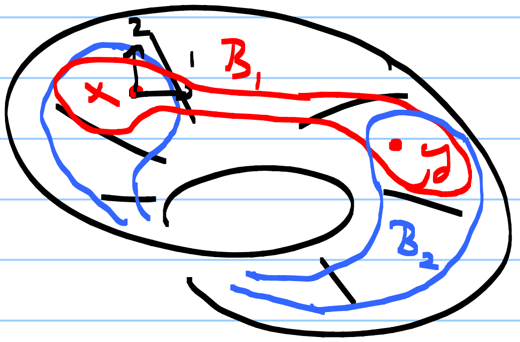
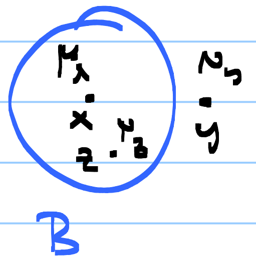
Example Möbius Band MB is <u>not</u> orientable.



Notation: $H_n(X|A) \doteq H_n(X, X \setminus A)$ local homology of X at A.

$$H_n(X|x) = H_n(X, X \setminus \{x\})$$

Definition: An orientation of an n-dimensional manifold M is a function $x \longmapsto \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M|x)$ satisfying the "local consistency" condition that each $x \in M$ has a neighborhood $\mathbb{R}^n \subseteq M$ containing an open ball B of finite radius about x such that all the local orientations μ_y at points $y \in B$ are the images of one generator μ_B of $H_n(M|B) \simeq H_n(\mathbb{R}^n|B)$, under the natural maps $H_n(M|B) \to H_n(M|y)$. If an orientation exists then we say that M is orientable.



$$\begin{tikzcd}
& H_n(M|B) \arrow[dl, "r^x_*"', "\simeq"] \arrow[dr, "r^y_*", "\simeq"'] & \\
\mu_x \in H_n(M|x) \arrow[rr, bend left=60, dash, "="] & & H_n(M|y) \ni \mu_y
\end{tikzcd}$$

Every manifold M has an orientable two-sheeted covering space. The general construction of this covering is as follows:

$$\begin{tikzcd}
\widetilde{M} \arrow[d, "\pi"'] & = \{ \mu_x \arrow[d, mapsto] & \mid x \in M,\ \mu_x \text{ a local orientation at } x \} \\
M & x &
\end{tikzcd}$$

Clearly $\pi : \widetilde{M} \to M$ is a 2 to 1 map.

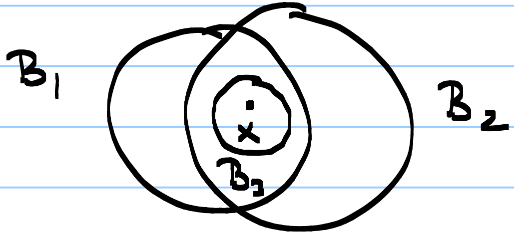
Let's put a topology on $\widetilde{M}$ so that $\pi : \widetilde{M} \to M$ becomes a covering space: Given a ball $B \subseteq \mathbb{R}^n \subseteq M$ of finite radius and a generator $\mu_B \in H_n(M|B)$, let $U(\mu_B) = \{ \mu_x \in \widetilde{M} \mid x \in B,\ \mu_x$ is the image of μ_B under the map $H_n(M|B) \to H_n(M|x) \}$

<u>Claim:</u> The collection $\{ U(\mu_B) \mid B \subseteq \mathbb{R}^n \subseteq M,\ \mu_B \in H_n(M|B) \}$ form a basis for a topology on $\widetilde{M}$.

<u>Proof:</u> i) $\mu_x \in \widetilde{M}$. Clearly there is a finite radius ball B and an orientation $\mu_B \in H_n(M|B)$ so that $x \in B \subseteq \mathbb{R}^n \subseteq M$ and

$$\begin{tikzcd}
H_n(M|B) \arrow[r, "\cong"] & H_n(M|x). \\
\mu_B \arrow[r, mapsto] & \mu_x
\end{tikzcd}$$

ii) If $\underline{\underline{\mu_x}} \in \underline{\underline{U}}(\mu_{B_1}) \cap \underline{\underline{U}}(\mu_{B_2})$ then



$\mu_{B_1} \mapsto \mu_x$

$\mu_{B_2} \mapsto \mu_x$

Choose a ball of finite radius B_3 so that $x \in B_3 \subseteq B_1 \cap B_2$.

$$\begin{tikzcd}
\mu_{B_1} \arrow[r, mapsto] \arrow[dr] & \mu_x & \mu_{B_2} \arrow[l, mapsto] \\
& \mu_{B_3} \arrow[u, mapsto] &
\end{tikzcd}$$

$$\begin{tikzcd}
H_n(M|B_1) \arrow[r] & H_n(M|B_3) \\
\mu_{B_1} \arrow[r, mapsto] & \mu_{B_3}
\end{tikzcd}$$

$$\begin{tikzcd}
H_n(M|B_2) \arrow[r] & H_n(M|B_3) \\
\mu_{B_2} \arrow[r, mapsto] & \mu_{B_3}.
\end{tikzcd}$$

Hence, $\widetilde{M}$ is a topological space with basis $\{U(B)\}$

One must check the followings!

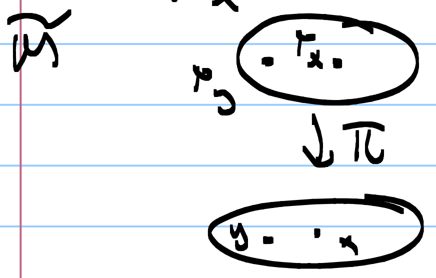
1) $\widetilde{M}$ is a topological manifold of dimension n
2) $\pi : \widetilde{M} \to M$ is a covering projection.

<u>Exercise</u>: Prove above statements.

Note that $\widetilde{M}$ is orientable because each point $\mu_x \in \widetilde{M}$ has a canonical local orientation

$\tilde{\mu}_x \in H_n(\widetilde{M}|\mu_x)$ which corresponds to μ_x under the isomorphisms

$$H_n(\widetilde{M}|\mu_x) \simeq H_n(U(\mu_B)|\mu_x) \overset{\pi_*}{\simeq} H_n(B|x)$$
$$\tilde{\mu}_x \longmapsto \mu_x$$



$\underline{\underline{U(\mu_B)}} = \{\mu_x \mid x \in B, \mu_B \mapsto \mu_x\}$

<u>Proposition</u>: If M is connected, then M is orientable if and only if $\widetilde{M}$ has two components. In particular, M is orientable if it is simply connected, or more generally $\pi_1(M)$ has no subgroup of index two.

<u>Proof</u>: Let's first prove the second statement.

If $\pi_1(M) = (e)$ or has no subgroup of index two then M has <u>no</u> connected two cover.

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Hence, the orientation double cover $\tilde{M} \to M$ cannot be connected. Hence, by the first statement M must be orientable.

First assume that $\tilde{M}$ has two components.

$$\begin{tikzcd}\tilde{M} \arrow[d, "2:1"] \\ M\end{tikzcd}$$

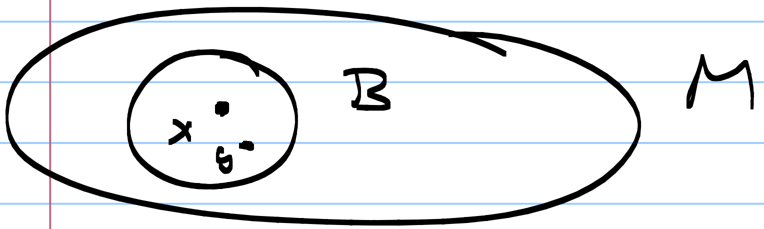
Then each component of $\tilde{M}$ is a 1-fold covering of M and thus both components of $\tilde{M}$ are homeomorphic to M. Since $\tilde{M}$ is orientable so are the components and thus M is orientable.

Now assume that M is orientable.
<u>must show:</u> $\tilde{M}$ is not connected.

$$\begin{tikzcd}\tilde{M} \arrow[d, "p"'] & \{\mu_x, -\mu_x\} \arrow[d] \\ M \arrow[r, phantom, "\ni"] & x\end{tikzcd}$$

$\underline{\underline{\mu_x}} \in H_n(M|x)$

Let $A = \{\mu_x \mid x \in M,\ \mu_x$ is the chosen local orientation of M at $x\}$.
A is open in $\tilde{M}$ because of local consistency of the orientation on M.



$$\begin{tikzcd}\mu_B \arrow[d, mapsto] \arrow[r, phantom, "\in"] & H_n(M|B) \arrow[d] \\ \mu_x & H_n(M|x)\end{tikzcd}$$

$\mu_x \in A$ then $\{\mu_y \mid y \in B,\ \mu_B \mapsto \mu_y\} \subseteq \tilde{M}$

$\Rightarrow A$ is open.

Let $B = \{\mu_x \in \tilde{M} \mid -\mu_x$ is the chosen local orientation$\}$

$\tilde{M} = A \cup B$, $A \cap B = \emptyset$. Similarly, B is open in $\tilde{M}$. Hence, $\tilde{M}$ cannot be connected.
This finishes the proof. ■

Remark. The orientation double cover $\tilde{M} \to M$ can be embedded into a larger covering space $M_{\mathbb{Z}} \to M$, where $M_{\mathbb{Z}}$ consists of all elements $\alpha_x \in H_n(M|x)$ as x ranges over M.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, phantom, "\wr"] & M_{\mathbb{Z}} \arrow[d] \arrow[r, phantom, "\supseteq"] & \tilde{M} \arrow[d] & \mathbb{Z}_2 = \{\pm\mu_x\} \arrow[l] \\
H_n(M|x) & M & M &
\end{tikzcd}$$

A continuous map $M \to M_{\mathbb{Z}}$ of the form $x \mapsto \alpha_x$ is called a section of the covering space.

$$\begin{tikzcd}
M_{\mathbb{Z}} \arrow[d, "p"] \\
M \arrow[u, bend left, "\alpha"]
\end{tikzcd} \qquad s(x) = \alpha_x \qquad (p \circ \alpha)(x) = x \ \forall x \in M.$$

Indeed, the same construction can be done for any commutative ring R with unity.

$M_R = \{\alpha_x \mid \alpha_x \in H_n(M|x;R) \simeq H_n(M|x) \otimes_{\mathbb{Z}} R\}$.

$\alpha_x = \mu_x \otimes r$

If $r \in R$ is a fixed element then the subset $M_r = \{\mu_x \otimes r \mid \mu_x \in \underset{\overset{\wr}{\mathbb{Z}}}{H_n(M|x)}$ generator$\}$ is a subcovering of $M_R \to M$.

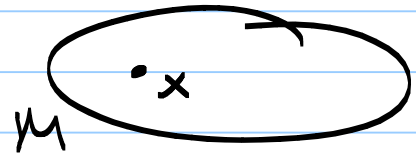
Note that $M_r \simeq M_{-r}$ and $r_1 \neq \pm r_2$ then M_{r_1} and M_{r_2} are disjoint subcovers of M_R.

Example: $R = \mathbb{Z}_2$ then $M_R \simeq M$

$$\mu_x \in H_n(M|x) \otimes \mathbb{Z}_2 \, , \quad \mu_x = -\mu_x$$

Since M_R is R-orientable, M is $\mathbb{Z}_2$-orientable

In other words, any manifold is $\mathbb{Z}_2$-orientable.



$$\mu_x \in H_n(M|x;\mathbb{Z}) = H_n(M|x) \otimes \mathbb{Z}_2 \simeq \mathbb{Z}_2$$

μ_x is the only nonzero element in $H_n(M|x;\mathbb{Z}_2)$.

Theorem: Let M be a closed connected n-manifold. Then:

a) If M is R-orientable, the map $H_n(M;R) \longrightarrow H_n(M|x;R) \simeq R$ is an isomorphism for each $x \in M$.

b) If M is not R-orientable, the map $H_n(M;R) \longrightarrow H_n(M|x;R) \simeq R$ is injective with image $\{r \in R \mid 2r = 0\}$ for all $x \in M$.

c) $H_i(M;R) = 0$ for $i > n$.

Remark: If M is orientable ($\mathbb{Z}$-orientable) then $H_n(M;\mathbb{Z}) \simeq \mathbb{Z}$ and if M is not orientable then $H_n(M;\mathbb{Z}) = 0$.

On the other hand, $H_n(M;\mathbb{Z}_2) \simeq \mathbb{Z}_2$ since any manifold is $\mathbb{Z}_2$-orientable.

A class $\mu \in H_n(M;R)$ is called a fundamental class if the image of μ under the map

$$\begin{tikzcd} H_n(M;R) \arrow[r] & H_n(M|x;R) \\ \mu \arrow[r, mapsto] & \mu_x \end{tikzcd}$$

is a generator for every $x \in M$.

Note that M has a fundamental class if M is closed and orientable.

<u>Claim</u>: If M has a fundamental class then M is closed and orientable.

<u>Proof</u>: Clearly M is orientable.

$$\begin{tikzcd} H_n(M;R) \arrow[r] & H_n(M|x;R) \\ \mu \arrow[r, mapsto] & \mu_x \end{tikzcd}$$

For local consistency condition choose a ball B with $x \in B \subseteq \mathbb{R}^n \subseteq M$ (finite radius) then

$$\begin{tikzcd} H_n(M;R) \arrow[r, "\simeq"] & H_n(M|B) \arrow[r, "\simeq"] & H_n(M|x;R) \\ \mu \arrow[r, mapsto] & \mu_B \arrow[r, mapsto] & \mu_x \end{tikzcd}$$

To see that M must be closed (i.e. compact) note that if M is <u>not</u> compact then there is a point $x \in M \setminus \mathrm{supp}(\mu)$

$\mathrm{supp}(\mu) \subseteq M\setminus\{x\}$

$$\begin{tikzcd} H_n(M) \arrow[r, "\simeq"] & H_n(M|x) \arrow[d, phantom, "\shortparallel"] \\ \mu \arrow[r, mapsto] & 0 \quad H_n(M, \underline{\underline{M\setminus x}}) \end{tikzcd}$$

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This would give a contradiction. Hence, M must be compact.

The above theorem is a consequence of the following

Lemma: Let M be a manifold of dimension n and let $A \subseteq M$ be a compact subset. Then:

a) If $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique class $\alpha_A \in H_n(M|A; R)$, whose image in $H_n(M|x; R)$ is α_x, for all $x \in A$.

b) $H_i(M|A; R) = 0$, for all $i > n$.

Proof of theorem: Choose $A = M$. The

Part (c) of the theorem follows from Part (b) of the lemma. If $i > n$

$$0 = H_i(M|A; R) = H_i(M, M\setminus A; R) = H_i(M, \emptyset; R) = H_i(M; R).$$

For Part (c) and (b) of the theorem, let $\Gamma_R(M)$ be the set of all sections of $M_R \to M$. The summation and scalar multiplications turns $\Gamma_R(M)$ into an R-module:

$$\alpha : M \to M_R, \quad x \mapsto \alpha_x \in H_n(M|x; R) \simeq R$$
$$\beta : M \to M_R, \quad x \mapsto \beta_x \in H_n(M|x; R) \simeq R$$

$$(r\alpha + \beta)(x) = r\alpha_x + \beta_x.$$

Also there is a homomorphism

$$H_n(M; R) \longrightarrow \Gamma_R(M)$$
$$\alpha \longmapsto (x \mapsto \alpha_x \in H_n(M|x; R)),$$

where α_x is the image of α under the map
$H_n(M;R) \longrightarrow H_n(M|x;R)$.
$\alpha \longmapsto \alpha_x$

The lemma implies the map $H_n(M;R) \to \Gamma_R(M)$ is an isomorphism.

Also any section is determined by its value at a single point: If $\alpha_x = \beta_x$ at some $x \in M$ then $\alpha = \beta$.

$$\begin{tikzcd} & M_R \arrow[d] \\ M \arrow[ur, "{\alpha, \beta}"] \arrow[r, "id"'] & M \end{tikzcd}$$

$$H_n(M;R) \xrightarrow{\simeq} \Gamma_R(M) \longrightarrow H_n(M/x_0;R) \simeq R$$
$$\alpha \longmapsto (x \mapsto \alpha_x) \longmapsto \alpha_{x_0}.$$

If M is orientable then $M_R \to M$ has a section so that α_x is a unit for all $x \in M$. Therefore, if β is any other element in $H_n(M;R)$ then $\beta_{x_0} = r\alpha_{x_0}$, for some $r \in R$. Then $\beta = r\alpha$. Hence, $H_n(M;R) \simeq \langle \alpha \rangle \simeq R$.

If M is not orientable (6) follows from previous arguments.

Proof of the lemma: We'll omit R from the notation for simplicity. Proof consists of 4 steps.

1) If the lemma holds for compact sets A, B and $A \cap B$, then it holds also for $A \cup B$.

We'll use Meyer-Vietoris sequence:

$\lceil X \supseteq Y = A \cup B$

$C_n(X,A) = C_n(X)/C_n(A)$, $C_n(X,B) = C_n(X)/C_n(B)$

$C_n(X, A\cap B) = C_n(X)/C_n(A\cap B)$

$C_n(X, A\cup B = C_n(X)/C_n(A\cup B)$

So we have a short exact sequence:

$$0 \to C_n(X,A\cap B) \xrightarrow{\Phi} C_n(X,A)\oplus C_n(X,B) \xrightarrow{\Psi} C_n(X,A+B) \to 0$$

$$\alpha \longmapsto (\alpha, -\alpha)$$

$$(\alpha,\beta) \longmapsto \alpha+\beta$$

This gives a long exact ~~sequence~~

$$\begin{tikzcd} & & & H_n(X,A\cup B) \arrow[d, phantom, "\parallel"] & \\ \to H_n(X,A\cap B) \arrow[r] & H_n(X,A)\oplus H_n(X,B) \arrow[r] & & H_n(X,A+B) \arrow[r] & H_{n-1}(X,A\cap B) \end{tikzcd}$$

$\rfloor$

We apply the above M-V sequence to the pairs $(M, M\setminus A)$, $(M, M\setminus B)$, $(M, M\setminus(A\cup B))$ and $(M, M\setminus(A\cap B))$ and obtain the long exact sequence

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, phantom, "\parallel"] & H_n(M|A\cup B) \arrow[r, "\Phi"] & H_n(M|A)\oplus H_n(M|B) \arrow[r, "\Psi"] & H_n(M|A\cap B) \arrow[r] & \cdots \\ H_{n+1}(M|A\cap B) & & & & \end{tikzcd}$$

If $i > n$, then

$$\underset{\overset{\parallel}{0}}{H_{i+1}(M|A\cap B)} \longrightarrow \underset{\overset{\cong}{0}}{H_i(M|A\cup B)} \longrightarrow \underset{\overset{\parallel}{0}}{H_i(M|A)} \oplus \underset{\overset{\parallel}{0}}{H_i(M|B)}$$

$\Rightarrow$

This proves part (b) of the lemma.

For part (a) we must prove: If $x \longmapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique class $\alpha_{A\cup B}$ in $H_n(M|A\cup B; R)$, whose image in $H_n(M|x)$ is α_x, for all $x \in A\cup B$.

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By assumption there are uniquely determined classes α_A, α_B and $\alpha_{A\cap B}$ so that they all restrict to α_x at all $x\in A$, $x\in B$, $x\in A\cap B$, respectively.

$$\begin{tikzcd}
H_n(M|A) \arrow[rr] \arrow[dr] & & H_n(M|x) \\
& H_n(M/A\cap B) \arrow[ur] \arrow[dr] & \\
H_n(M/B) \arrow[ur] \arrow[rr] & & H_n(M/x)
\end{tikzcd}$$

$\alpha_A \longmapsto \alpha_x$

$\alpha_A \mapsto \alpha_A|_{A\cap B}$

$\alpha_B \mapsto \alpha_B|_{A\cap B}$

$\alpha_B \longmapsto \alpha_x$

$\Rightarrow \alpha_A|_{A\cap B} = \alpha_B|_{A\cap B}$

Hence, $\Phi(\alpha_A, -\alpha_B) = \alpha_A|_{A\cap B} - \alpha_B|_{A\cap B} = 0.$

So by exactness of the M-V sequence there is a class, say $\alpha_{A\cup B}$ in $H_n(M|A\cup B)$, so that

$$\Phi(\alpha_{A\cup B}) = (\alpha_{A\cup B}|_A, -\alpha_{A\cup B}|_B) = (\alpha_A, -\alpha_B)$$

$$\alpha_{A\cup B}|_A = \alpha_A, \quad \alpha_{A\cup B}|_B = \alpha_B.$$

So, $\alpha_{A\cup B}|_x = \alpha_x$ for all $x\in A\cup B$.

Uniqueness of $\alpha_{A\cup B}$ follows from uniqueness of α_A, α_B and the injectivity of the map

$$H_n(M|A\cup B) \xrightarrow{\Phi} H_n(M|A) \oplus H_n(M|B).$$

(2) This step reduces the theorem to the case $M=\mathbb{R}^n$.

Since $A\subseteq M$ is compact and M is a union of open subsets which are all homeomorphic to $\mathbb{R}^n$ we can find finitely many compact sets $A_1, \ldots, A_m$ each contained in some

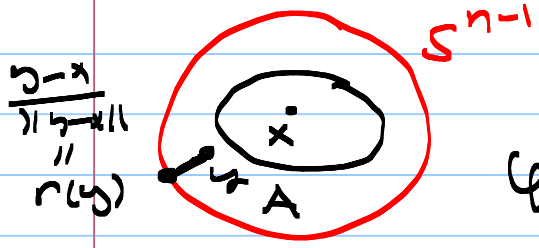
$\mathbb{R}^n \subseteq M$ so that $A = A_1 \cup A_2 \cup \cdots \cup A_m$.
Assume that the result holds for $M = \mathbb{R}^n$.
We apply Step 1 to $A_1 \cup A_2 \cdots \cup A_{k-1}$ and A_k.
Then $(A_1 \cup A_2 \cup \cdots \cup A_{k-1}) \cap A_k = (A_1 \cap A_k) \cup \cdots \cup (A_{k-1} \cap A_k)$
which lies in $A_k \subseteq \mathbb{R}^n$.
Hence, by induction the result holds for all k and thus for $k = m$, A.

3) Now let's prove the lemma for $M = \mathbb{R}^n$.
First assume that the compact set A is a union of convex compact subsets, say $A = A_1 \cup A_2 \cup \cdots \cup A_m$. So if the result holds for $(\mathbb{R}^n, A_i)$ then again by induction similar to above in Step 2 the result holds for A.

Need to show: If result holds for $A_1 \cup \cdots \cup A_{k-1}$ and A_k then it holds for $(A_1 \cup \cdots \cup A_{k-1}) \cap A_k$.

However, $(A_1 \cup \cdots \cup A_{k-1}) \cap A_k = (A_1 \cap A_k) \cup \cdots \cup (A_{k-1} \cap A_k)$, which is a union of $k-1$ compact convex subsets and thus the result holds by induction hypothesis.

Now assume $A \subseteq \mathbb{R}^n$ is a compact convex subset. Then both $\mathbb{R}^n \setminus A$ and $\mathbb{R}^n \setminus \{x\}$ deformation retracts onto a sphere centered at x.



$$\varphi : \mathbb{R}^n \setminus \{x\} \longrightarrow \mathbb{R}^n \setminus \{x\}$$
$$y \longmapsto \underline{\frac{y-x}{\|y-x\|}(1-t) + ty}$$
$$\varphi : \mathbb{R}^n \setminus A \longrightarrow \mathbb{R}^n \setminus A$$
$$y \longmapsto \frac{y-x}{\|y-x\|}(1-t) + ty$$

$$\begin{tikzcd} \mathbb{R}^n\setminus\{x\} \arrow[r, "h.e.", "\varphi"'] & S^{n-1} \\ \mathbb{R}^n\setminus A \arrow[u, phantom, "\cup"] \arrow[r, "\varphi", "h.e."'] & S^{n-1} \end{tikzcd}$$

$$\begin{tikzcd} & S^{n-1} \\ & \cong \\ H_n(\mathbb{R}^n|x)=H_n|\mathbb{R}^n & \mathbb{R}^n\setminus\{x\}) \\ H_n(\mathbb{R}^n|A)=H_n(\mathbb{R}^n & \mathbb{R}^n\setminus A) \arrow[u, "i_*"] \\ & \cong \\ & S^{n-1} \end{tikzcd}$$

Here, i_* is an isomorphism.

So we've proved the result for $M=\mathbb{R}^n$ and A a union of finitely many compact convex subsets.

4) Now assume that $A\subseteq\mathbb{R}^n$ is an arbitrary compact subset. Let $\alpha\in H_n(\mathbb{R}^n|A)=H_n(\mathbb{R}^n,\mathbb{R}^n\setminus A)$ be represented by a relative cycle z and let $C\subseteq\mathbb{R}^n\setminus A$ be the union of the images of the singular simplices in ∂z.

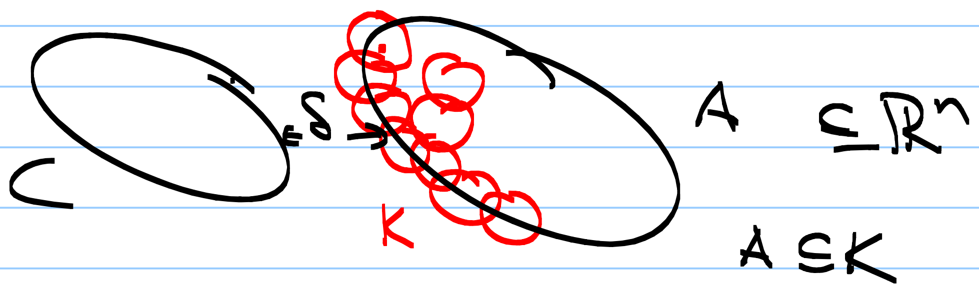
$\alpha=[z]$, $\quad z\in C_n(\mathbb{R}^n,\mathbb{R}^n\setminus A)$

$\partial z\in C_{n-1}(\mathbb{R}^n\setminus A)$, $\quad \partial z=\sum \sigma_i$

$\sigma_i:\Delta^{n-1}\to X$, $\quad \bigcup_i \operatorname{Im}\sigma_i = C$.

Since C is compact it has a positive distance $\delta>0$ from A.

$C\subseteq\mathbb{R}^n\setminus A$, $\quad C\cap A=\emptyset$



Let's cover A by finitely many closed balls of radius less than δ centered at points of A. Let K be the union of these balls, so that $K \cap C = \emptyset$. Then the relative cycle z defines an element

$$\alpha_K \in H_i(\mathbb{R}^n | K) = H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K)$$

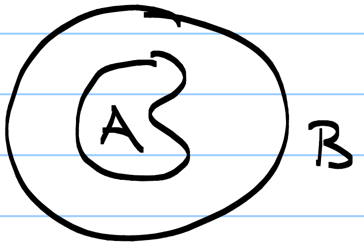
mapping to the class $\alpha \in H_i(\mathbb{R}^n | A)$.

$$\begin{tikzcd} H_i(\mathbb{R}^n | K) \arrow[r] & H_i(\mathbb{R}^n | A) \\ \alpha_K = [z] \arrow[r, mapsto] & {[z]} = \alpha . \end{tikzcd}$$

If $i > n$ then by (3) we have $H_i(\mathbb{R}^n | K) = 0$ because K is a union of closed balls, which are convex compact subsets. Hence, $\alpha_K = 0$ which implies $\alpha = 0$. Hence, $H_i(\mathbb{R}^n | A) = 0$.

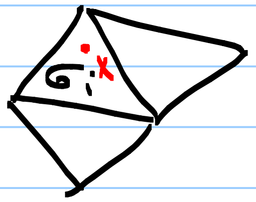
If $i = n$ and α_x is zero in $H_n(\mathbb{R}^n | x)$ for all $x \in A$, then in fact this holds for all $x \in K$, when α_x is the image of α_K. This is so since K is a union of balls B meeting A and $H_n(\mathbb{R}^n | B) \to H_n(\mathbb{R}^n | x)$ is an isomorphism for all $x \in B$.

Finally, since $\alpha_x = 0$ for all $x \in K$, then by (3) $\alpha_K = 0$ and thus $\alpha = 0$. This proves the uniqueness part of (a). For the existence part we let α_A to be image of the element α_B associated to any ball $B \supseteq A$.



Video 34

Remark: If M is a closed having a Δ-complex structure then we can construct the fundamental class more explicitly.



$\alpha = \sum k_i \sigma_i \qquad k_i \in \mathbb{Z}$

$\sigma_i : \Delta^n \to M$

$\alpha \in H_n(M|M) = H_n(M, \overset{\phi}{M \setminus M}) \cong H_n(M)$ so that

$\alpha \longmapsto \underset{\text{(a generator)}}{\alpha_x} \in \underset{\mathbb{Z}}{\underset{\wr}{H_n(M|x)}} = H_n(M, M\setminus x)$, for all $x \in M$

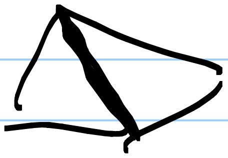
$\alpha \longmapsto \pm k_i \in H_n(M|x)$

So we must have $k_i = \pm 1$.

$\alpha = \sum k_i \sigma_i$, $k_i \in \{\pm 1\}$ and $\partial\alpha = 0$.

Thus M is orientable if and only if there is a choice of $k_i = \pm 1$ so that $\partial\alpha = \sum k_i \partial\sigma_i = 0$.

In case $R = \mathbb{Z}_2$ then $\alpha = \sum \sigma_i$ is a fundamental class.



$\partial\alpha = \sum \partial\sigma_i = 0$.

$0 \neq \alpha \in H_n(M; \mathbb{Z}_2) \simeq \mathbb{Z}_2$.

Proposition: If M is a connected noncompact n-manifold, then $H_i(M; R) = 0$, for $i \geq n$.

Sketch of proof Let z be a cycle representing an element in $H_i(M;R)$. Thus z has compact image there is an open set $U \subseteq M$ containing the image of z and having compact closure $\overline{U} \subset M$. Let $V = M \setminus \overline{U}$. Consider the exact sequence of the triple $(M, U \cup V, V)$

$$\begin{tikzcd}
\underline{\underline{H_{i+1}(M, U \cup V)}} \arrow[r] & H_i(U \cup V, V) \arrow[r] & H_i(M, V) \arrow[r] & {} \\
& H_i(U) \arrow[r] \arrow[u, "(\text{Excision}) \simeq"] & H_i(M) \arrow[u] & =
\end{tikzcd}$$

Case 1 $i > n$. By the above lemma

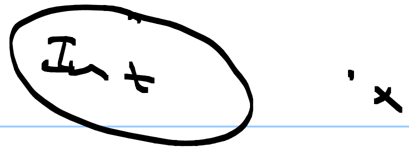
$H_{i+1}(M, \underline{\underline{U \cup V}}) = 0$ and $H_i(M, \underline{\underline{V}}) = 0$ because $U \cup V$ and V are the complements of compact sets in M. Hence, $H_i(U \cup V, V) = 0$ and thus $H_i(U) = 0$. Thus $[z] = 0$ in $H_i(U)$ and hence in $H_i(M)$.

$\Rightarrow H_i(M) = 0$ since z was arbitrary.

Case 2 $i = n$. Then the class $[z] \in H_n(M;R)$ defines a section $x \mapsto [z]_x$ by the map

$$H_n(M) \xrightarrow{\cong} H_n(M|x).$$
$$[z] \doteq \alpha \longmapsto \alpha_x = [z]_x$$

We know that the above map is an isomorphism for each x. Thus it is enough to show that $\alpha_x = [z]_x$ is zero for some $x \in M$. Since z has compact image choose some $x \in M \setminus \operatorname{Im} z$. Then



$[z] \longmapsto [z]_x \in H_n(M|x) = H_n(M, M\setminus\{x\})$

is zero since $H_n(M, M\setminus\{x\})$ is the nth homology of the complex $C_i(M)/C_i(M\setminus\{x\})$ and $z \in C_n(M\setminus\{x\})$.

This finishes the proof. ■

Video 35

The Duality Theorem:

Cap Product : $\frown C_k(X;R)\times C^\ell(X;R)\to C_{k-\ell}(X;R)$
$(k\geqslant \ell)$

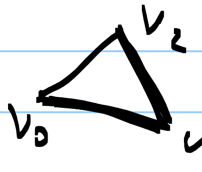
$$\sigma\frown\varphi=\varphi(\sigma|_{[v_0,\dots,v_\ell]})\cdot\sigma|_{[v_\ell,\dots,v_k]},$$

where $\sigma:\Delta^k\longrightarrow X$ and $\varphi\in C^\ell(X;R)$.

Ex $k=2,\quad \ell=1$

$$\sigma\frown\varphi=\varphi(\sigma|_{[v_0,v_1]})\cdot\sigma|_{[v_1,v_2]}$$

$\sigma:[v_0,v_1,v_2]\longrightarrow X$



Lemma: $\partial(\sigma\frown\varphi)=(-1)^\ell(\partial\sigma\frown\varphi-\sigma\frown\delta\varphi)$

Proof:
$$\partial\sigma\frown\varphi=\underbrace{\sum_{i=0}^{\ell}(-1)^i\varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{\ell+1}]})\cdot\sigma|_{[v_{\ell+1},\dots,v_k]}}_{①}$$
$$+\underbrace{\sum_{i=\ell+1}^{k}(-1)^i\varphi(\sigma|_{[v_0,\dots,v_\ell]})\sigma|_{[v_\ell,\dots,\hat{v}_i,\dots,v_k]}}_{②}$$

$$\sigma\frown\delta\varphi=\delta\varphi(\sigma|_{[v_0,\dots,v_{\ell+1}]})\cdot\sigma|_{[v_{\ell+1},\dots,v_k]}$$
$$=\varphi(\partial\sigma|_{[v_0,\dots,v_{\ell+1}]})\cdot\sigma|_{[v_{\ell+1},\dots,v_k]}$$
$$=\underbrace{\sum_{i=0}^{\ell+1}(-1)^i\varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{\ell+1}]})\sigma|_{[v_{\ell+1},\dots,v_k]}}_{(1)}$$

Finally,

$$\partial(\sigma \frown \varphi) = \partial(\varphi(\sigma|_{[v_0,\dots,v_\ell]}) \cdot \sigma|_{[\overset{u_0}{v_\ell},\dots,\overset{u_{k-\ell}}{v_k}]})$$

$$= \varphi(\sigma|_{[v_0,\dots,v_\ell]}) \sum_{i=0}^{k-\ell} (-1)^i \sigma|_{[u_0,\dots,\hat{u}_i,\dots,u_{k-\ell}]}$$

$$= \varphi(\sigma|_{[v_0,\dots,v_\ell]}) \cdot \underbrace{\sum_{i=\ell}^{k} (-1)^{i-\ell} \sigma|_{[v_\ell,\dots,\hat{v}_i,\dots,v_k]}}_{(-1)^\ell \text{②}}$$

So by the lemma $\partial(\sigma \frown \varphi) = \pm(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$. Hence, if σ is a cycle and φ is a cocycle then $\partial(\sigma \frown \varphi) = 0$ so that $\sigma \frown \varphi$ is a $k-\ell$ cycle. Thus the cap product induces an operation on (co)homology level:

$$H_k(X;R) \times H^\ell(X;R) \longrightarrow H_{k-\ell}(X;R),$$ which

is linear on both arguments.

Similarly, for pairs of spaces (X,A) we have

$$H_k(X,A;R) \times H^\ell(X;R) \longrightarrow H_{k-\ell}(X,A;R)$$

$$H_k(X,A;R) \times H^\ell(X,A;R) \longrightarrow H_{k-\ell}(X;R).$$

For the second line note that the cap product

$$C_k(X;R) \times C^\ell(X;R) \xrightarrow{\frown} C_{k-\ell}(X;R)$$ restricts to zero on the submodule

$$C_k(A;R) \times C^\ell(X,A;R) \xrightarrow{\frown} 0$$

and thus induces a well defined product

$$C_k(X;R)/C_k(A;R)$$

$$\shortparallel$$

$$C_k(X,A;R) \times C^l(X,A;R) \longrightarrow C_{k-l}(X;R)$$

$$\Rightarrow H_k(X,A;R) \times H^l(X,A;R) \xrightarrow{\frown} H_{k-l}(X;R)$$

Indeed, more generally we have

$$H_k(X, A\cup B;R) \times H^l(X,A;R) \xrightarrow{\frown} H_{k-l}(X,B;R),$$

when A and B are open subsets of X.
Here we use chain groups
$C_n(X, A+B;R) = C_n(X;R)/C_n(A+B;R)$.

Naturality of Cap product:

$f: X \to Y$, $f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$

$$\begin{tikzcd}
H_k(X) \arrow[d, "f_*"'] \arrow[d, bend right=60] & \times & H^l(X) \arrow[r, "\frown"] & H_{k-l}(X) \arrow[d, "f_*"] \\
H_k(Y) & \times & H^l(Y) \arrow[u, "f^*"'] \arrow[r, "\frown"] & H_{k-l}(Y)
\end{tikzcd}$$

This follows from the definitions, namely

$$f\sigma \frown \varphi = \varphi(f\sigma|_{[v_0,\dots,v_l]}) \cdot f\sigma|_{[v_l,\dots,v_k]}$$

$$\sigma \frown f^*\varphi = \underbrace{f^*\varphi(\sigma}|_{[v_0,\dots,v_l]})\, \sigma|_{[v_l,\dots,v_k]}$$

$$\Rightarrow f(\sigma \frown f^*\varphi) = f^*\varphi(\sigma|_{[v_0,\dots,v_l]}) \cdot f\sigma|_{[v_l,\dots,v_k]}$$

$$\shortparallel$$

$$\varphi(f\sigma|_{[v_0,\dots,v_l]}) \cdot f\sigma|_{[v_l,\dots,v_k]}$$

Video 36

Theorem (Poincaré Duality)
If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map

$$D: H^k(M;R) \longrightarrow H_{n-k}(M;R),\ D(\alpha) = [M] \frown \alpha,$$

is an isomorphism for all k.

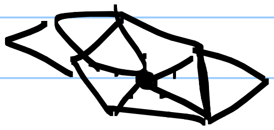
(Here $[M]$ is a class in $H_n(M;R)$ so that its restriction under the map $H_n(M;R) \to H_n(M|x;R) \cong R$ is a generator for each $x \in M$.)

Remark: Note that any n-manifold is $R = \mathbb{Z}_2$-orientable.

To state and prove Poincaré Duality for non-compact manifolds we need so called "Cohomology with compact supports".

To define this consider the subgroup $\Delta^i_c(X;G)$ of the simplicial cochain group $\Delta^i(X;G)$ consisting of cochains that are compactly supported in the sense that they take nonzero values on only finitely many simplices.

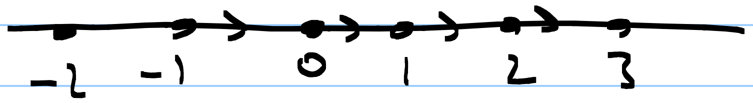
X: Delta complex



Note that if $\varphi \in \Delta^i_c(X;G)$ then $\delta\varphi \in \Delta^{i+1}_c(X;G)$

Thus we may define $H^i_c(X;G) = \dfrac{\ker(\delta: \Delta^i_c \to \Delta^{i+1}_c)}{\mathrm{Im}(\delta: \Delta^{i-1}_c \to \Delta^i_c)}$

Example: $\mathbb{R}$:



$\Delta_0(\mathbb{R}) = \{ \sum c_n \tau_n \mid c_n \in \mathbb{Z},\ c_n \neq 0$ for finitely many $n \}$

$\tau_n(v_0) = n$

, $\Delta_1(\mathbb{R}) = \{ \sum c_n \sigma_n \mid c_n \in \mathbb{Z} \}$

$c_n \neq 0$ for finitely many n

$\sigma_n : [v_0, v_1] \to [n, n+1]$

$$\Delta^0(\mathbb{R}) = \mathrm{Hom}(\Delta_0(\mathbb{R}), \mathbb{Z})$$
$$\cong \{ f : \Delta_0(\mathbb{R}) \to \mathbb{Z} \}$$
$$\cong \mathbb{Z}^{\mathbb{Z}} = \text{the set of integer sequences}$$

$(a_n)_{n \in \mathbb{Z}} \mapsto f(\sigma_n) = a_n$

$f \in \Delta^0_c(\mathbb{R})$, $f(\sigma_n) = 0$ for all but finitely many n.

So if $f \in \Delta^0_c(\mathbb{R})$ is a coboundary, i.e., $\delta f = 0$, then $\delta f(\sigma_n) = 0$, for all n.

$$0 = \delta f(\sigma_n) = f(\partial \sigma_n) = f(\tau_{n+1} - \tau_n)$$
$$= f(\tau_{n+1}) - f(\tau_n)$$

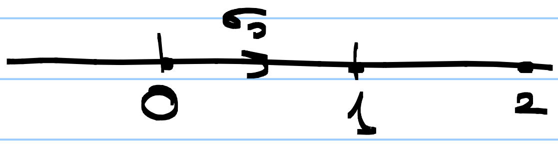
$\Rightarrow f(\tau_{n+1}) = f(\tau_n)$ for all n.

$\Rightarrow f$ is constant.

Hence, if f is compactly supported 0-cocycle then $f = 0$. In particular, $H^0_c(\mathbb{R}) = 0$, whereas $H^0(\mathbb{R}) \cong \mathbb{Z}$.

Similarly, $H^1(\mathbb{R}; G) = 0$. Now let $\varphi \in \Delta^1_c(\mathbb{R})$,

such that $\varphi(\sigma_0) = 1$ and $\varphi(\sigma_i) = 0$ if $i \neq 1$.



Since $\mathbb{R}$ has dimension one as a Δ-complex $\Delta^2(\mathbb{R}) = 0$ and thus any 1-cochain is a cocycle. Note that if $\phi \in \Delta_c^0(\mathbb{R})$ then

$$\delta\phi(\sigma) = \phi(\partial\sigma)$$
$$= \phi(\sigma(v_1) - \sigma(v_0))$$
$$= \phi(\sigma(v_1)) - \phi(\sigma(v_0)).$$

Since ϕ is compactly supported there is some $n_0 \in \mathbb{N}$ so that $\phi(n)=0$, for all $|n| \geq n_0$.

Thus, $\delta\phi(\sigma_{-n_0} + \dots + \sigma_0 + \dots + \sigma_{n_0}) = \delta\phi([-n_0, n_0+1])$

$$= \phi(\delta([-n_0, n_0+1]))$$
$$= \phi(n_0+1) - \delta(-n_0)$$
$$= 0 - 0$$
$$= 0.$$

However, $\varphi(\sigma_{-n_0} + \dots + \sigma_0 + \dots + \sigma_{n_0}) = \varphi(\sigma_0) = 1$ and thus $\varphi \neq \delta\phi$, for any $\phi \in \Delta_c^0(\mathbb{R})$.

Moreover, if $\psi \in \Delta_c^1(\mathbb{R})$ then there is some $n_0 \in \mathbb{N}$ so that if $n \geq n_0$ then $\psi([-n,n]) = \psi([-n_0,n_0])$. Let $k = \psi([-n_0,n_0])$ and consider the cocycle $\psi - k\varphi$.

Let $(\psi - k\varphi)(\overset{\sigma_i}{[i,i+1]}) = c_i$, $i = -n_0, \dots, n_0-1$.

Since $(\psi - k\varphi)([-n_0,n_0]) = k - k\cdot 1 = 0$, we see that

$$c_{-n_0} + \dots + c_0 + \dots + c_{n_0-1} = 0.$$

Let $\theta \in \Delta_c^0(\mathbb{R})$ be defined by $\theta(-n_0+1) = c_{-n_0}$, $\dots$, $\theta(-n_0+k) = c_{-n_0} + \dots + c_{-n_0+k-1}$ and $\theta(n) = 0$, for all other integers.

Then $\underline{\underline{\delta\theta}}([i,i+1]) = \theta(i+1) - \theta(i)$

$$= c_i$$
$$= (\underline{\psi - k\varphi})([i,i+1]), \ \forall i.$$

Hence, $\psi - k\varphi = \delta\theta$ so that $[\psi - k\varphi] = 0$ in $H_c^1(\mathbb{R};\mathbb{Z}) = 0$ or $[\psi] = k[\varphi]$.

Thus $H^1_c(\mathbb{R};\mathbb{Z}) \simeq \mathbb{Z}$.

Finally, to see that $H^1_c(\mathbb{R};G)\simeq G$ consider G as a $\mathbb{Z}$-module. Then for any fixed $g\in G$ replace any integer value, say n, of a cocycle by $n\cdot g$. More explicitly, for example let $\varphi\in\Delta^1_c(\mathbb{R})$ be the cocycle defined by $\varphi(\sigma_0)=1\cdot g=g$ and $\varphi(\sigma_n)=0$, for all $n\neq 0$.

As a conclusion, we see that $H^1_c(\mathbb{R};G)\simeq G$.
(See also my Math 709 lecture notes and Video 26 after 11:50, for the de Rham version!)

Remark: V finite dim'l k-vector space.

Then $V^*=\mathrm{Hom}(V;k)\simeq V$
If $\mathcal{B}$ is a basis for V say, $\mathcal{B}=\{v_1,\dots,v_n\}$
then $\mathcal{B}^*=\{v_1^*,\dots,v_n^*\}$

$$v_i^*: V\to k,\quad v_i^*(v_j)=\delta_{ij}.$$

In particular, $\dim_k V=\dim_k V^*$.

However, this is no longer true in case of infinite dimensional vector spaces.

Example: $V=\mathbb{R}[x]$ is infinite dimensional vector space with basis $\mathcal{B}=\{1,x,x^2,\dots,x^n,\dots\}$.
$V^*=\{f:V\to\mathbb{R} \mid f \text{ is a homomorphism}\}$.

$\dim V^* > \dim V$.

$$V=\bigoplus_{n=0}^{\infty}\mathbb{R}\langle x^n\rangle,\quad V^*=\prod_{n=0}^{\infty}\overbrace{(\mathbb{R}\langle x^n\rangle)^*}^{\overset{\mathbb{R}}{\wr}}$$

$\mathbb{R}$
is

V^* is bigger that $\bigoplus_{n=0}^{\infty} (\mathbb{R}\langle x^n\rangle)^*$.

Note that any $\psi \in \bigoplus_{n=0}^{\infty} (\mathbb{R}\langle x^n\rangle)^*$ then

$\varphi(x^n) = 0$ for all but finitely many n.

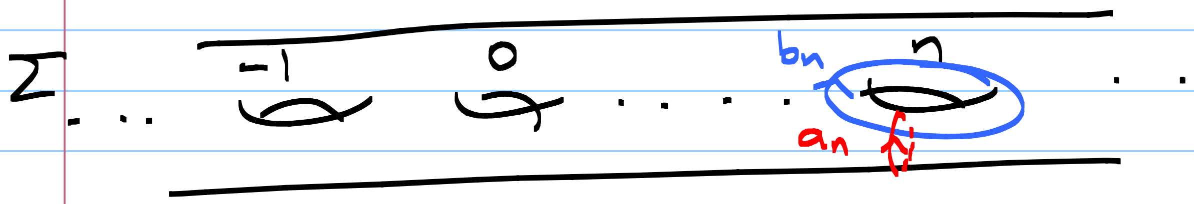
However, in V^* there is an element say

$\phi : \bigoplus_{n=0}^{\infty} \mathbb{R}\langle x^n\rangle \to \mathbb{R}, \quad \phi(x^n) = n.$

Hence, $V^* = (\mathbb{R}[x])^* \neq \bigoplus_{n=0}^{\infty} (\mathbb{R}[x^n])^*$

Remark: V^* does not have a countable basis.

Ex



$H_1(\Sigma; \mathbb{Z}) \simeq \langle a_n, b_n \mid n \in \mathbb{Z} \rangle$ countable gen

$H^1(\Sigma; \mathbb{Z})$ is uncountable.

$H^1_c(\Sigma; \mathbb{Z}) \simeq \langle a_n^*, b_n^* \mid n \in \mathbb{Z} \rangle \simeq H_1(\Sigma; \mathbb{Z})$

$a_n^*(a_m) = \delta_{nm}, \quad a_n^*(b_m) = 0.$
$b_n^*(a_m) = 0, \quad b_n^*(b_m) = \delta_{nm}$

Video 37

Compactly supported Singular Cohomology

X topological space, G abelian group.

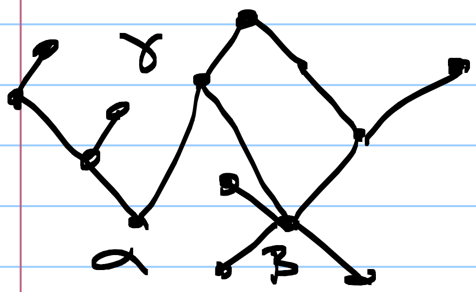
Define $C^i_c(X;G)$ as the union of the subgroups $C^i(X, X\setminus K; G)$ as K ranges over compact subsets of X.

$C^i(X,X\setminus K;G) = \{\varphi \in C^i(X;G) \mid \varphi(\sum a_k\sigma_k) = 0$ if each $\operatorname{Im}\sigma_k \subseteq X\setminus K\}$.

Note that if $K \subset L$, then $X\setminus L \hookrightarrow X\setminus K$ and we get an inclusion map
$C^i(X, X\setminus K; G) \hookrightarrow C^i(X, X\setminus L; G)$.

$\Rightarrow H^i(X, X\setminus K; G)$ and we may define $H^i_c(X;G)$ as a "kind of limit" of $H^i(X, X\setminus K; G)$ as K varying over compact subsets of X.

Directed Limit: Let I be a partially ordered set. Assume that for any $\alpha, \beta \in I$ there is some $\gamma \in I$ so that $\gamma \geq \alpha$, $\gamma \geq \beta$. In this case, we'll call I a directed set.



Let G_α be a family of groups indexed by elements of a directed set I. Suppose that

1) For each $\beta \geq \alpha$ elements of I there is a homomorphism $f_{\alpha\beta} : G_\alpha \to G_\beta$ so that

$f_{\alpha\alpha} = id_{G_\alpha}$, and

iii) If $\alpha \leq \beta \leq \gamma$ then $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$

Now define the direct limit $\varinjlim G_\alpha$ of the directed set of groups as the quotient group of the disjoint union

$\coprod_{\alpha\in I} G_\alpha$ by the relation $\sim$ defined as

$g_1, g_2 \in \coprod_{\alpha\in I} G_\alpha$, $g_1 \sim g_2$ iff and only if

$g_1 \in G_\alpha$, $g_2 \in G_\beta$ and there is some $\gamma \in I$
$\gamma \geq \alpha$, $\gamma \geq \beta$ and $f_{\alpha\gamma}(g_1) = f_{\beta\gamma}(g_2)$.

$$\begin{tikzcd}
 & G_\gamma & \\
G_\alpha \arrow[ur, "f_{\alpha\gamma}"] & & G_\beta \arrow[ul, "f_{\beta\gamma}"'] \\
g_1 \arrow[u, phantom, "\in" rotate=90] & & g_2 \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

Then $\varinjlim G_\alpha$ is a group whose product is defined as

$[g_1], [g_2] \in \varinjlim G_\alpha$, $g_1 \in G_\alpha$, $g_2 \in G_\beta$

$\exists \gamma \in I$ s.t. $\gamma \geq \alpha$, $\gamma \geq \beta$, $f_{\alpha\gamma}(g_1), f_{\beta\gamma}(g_2) \in G_\gamma$

$[g_1] \cdot [g_2] \doteq [f_{\alpha\gamma}(g_1) \cdot f_{\beta\gamma}(g_2)]$.

Exercise: Show that this really defines a group.

Example: $I = (\mathbb{N}, I)$ $n, m \in \mathbb{N}$, $G_n \doteq S_n$

$S_n = \mathrm{Sym}\{1, 2, \dots, n\}$.

$m \geq n$, $f_{nm} : S_n \longrightarrow S_m$

$\sigma \longmapsto \sigma$

$\varinjlim S_n$ = locally finite permutations on the set of positive integers $\mathbb{N}$.

Example: $I = (\mathbb{N}, >)$, $m, n \in \mathbb{N}$

$m \geq n$ if and only $n | m$.

$2 \not\geq 3$, $3 \not\geq 2$, $4 \geq 2$, $15 \geq 5$

$$G_n = \left\{ \frac{k}{n} \in \mathbb{Q} \mid k \in \mathbb{Z} \right\} \simeq \mathbb{Z}$$

$$1/n \longleftrightarrow 1$$

$m \geq n$ ($\Leftrightarrow n | m \Leftrightarrow m = ln$, $l \in \mathbb{N}$)

$$f_{nm} : G_n \longrightarrow G_m , \; f_{nm}\left(\frac{k}{n}\right) = \frac{kl}{nl} = \frac{kl}{m}$$

$$\begin{tikzcd} G_n \arrow[r] \arrow[d, phantom, "\wr"] & G_m \arrow[d, phantom, "\wr"] \\ \mathbb{Z} \arrow[r, "\times l"] & \mathbb{Z} \\ 1 \arrow[r, mapsto] & \frac{m}{n} = l \end{tikzcd}$$

$\varinjlim G_n = (\mathbb{Q}, +)$

$$\begin{tikzcd} 3/7 \arrow[d, mapsto] & \in G_7 \arrow[d] & , & \frac{4}{5} \arrow[d, mapsto] & \in G_5 \arrow[d] \\ 15/35 & G_{35} & & 28/35 & G_{35} \end{tikzcd}$$

$35 \geq 7$, $35 \geq 5$

$$\frac{3}{7} + \frac{4}{5} = \frac{15}{35} + \frac{28}{35} = 43/35 .$$

Assume that X is a topological space which is expressed as the union of a collection of subspaces X_α forming a directed set with respect to inclusion relation. Then the groups $H_i(X_\alpha; G)$ for fixed i and G, form a directed system using the homomorphisms

$$X_\alpha, X_\beta \subseteq X, \qquad X_\alpha \subseteq X_\beta \quad (\Leftrightarrow X_\beta \geq X_\alpha)$$

$f_{\alpha\beta}: H_i(X_\alpha; G) \longrightarrow H_i(X_\beta; G)$ induced by the inclusion $X_\alpha \hookrightarrow X_\beta$.

The natural maps $H_i(X_\alpha, G) \longrightarrow H_i(X; G)$ induce a homomorphism

$$\varinjlim_{\alpha} H_i(X_\alpha; G) \longrightarrow H_i(X; G)$$

Proposition If a space X is the union of a directed set of subspaces X_α with the property that each compact set in X is contained in some X_α, then the natural map

$$\varinjlim H_i(X_\alpha; G) \longrightarrow H_i(X; G)$$

is an isomorphism for all i and G.

Proof. Let $\sum c_\sigma \sigma$ be a singular chain in X representing a class in $H_i(X; G)$. Since the image of each σ is compact and this is a finite sum the image of $\sum c_\sigma \sigma$ is compact and thus there is some $X_\alpha \subseteq X$ so that the image of $\sum c_\sigma \sigma$ lies in X_α.

Here $[\sum c_\sigma \sigma]$ defines a class in $H_i(X_\alpha; G)$.

So, the homomorphism $H_i(X_\alpha, G) \to H_i(X; G)$ maps $[\sum c_\sigma \sigma]$ maps to itself.

Hence, $\varinjlim H_i(X_\alpha; G) \to H_i(X; G)$ is onto.

For injectivity let $[\sum c_\sigma \sigma] \in H_i(X_\alpha; G)$ for some α and assume that

$[\sum c_\sigma \sigma] = 0$ in $H_i(X; G)$. So there is some $\tau \in C_{i+1}(X; G)$ so that

$$\partial \tau = \sum c_\sigma \sigma.$$

Similarly, $\tau \in C_{i+1}(X_\beta; G)$ for some β. Choose γ so that $X_\alpha \subseteq X_\gamma$ and $X_\beta \subseteq X_\gamma$. Then

$$\partial \tau = \sum c_\sigma \sigma \text{ in } X_\gamma.$$

$\Rightarrow [\sum c_\sigma \sigma] = 0$ in $H_i(X_\gamma; G)$ and thus it is trivial in $\varinjlim H_i(X_\alpha, G)$.

This finishes the proof. ∎

What about cohomology?

For a space X the collection of compacts subsets $K \subseteq X$ form a directed set under inclusion. To each compact set $K \subseteq X$ we associate a group $H^i(X, X \setminus K; G)$, with fixed i and G and to each inclusion $K \subseteq L$ of compact subsets we associate a homomorphism

$$H^i(X, X \setminus K; G) \to H^i(X, X \setminus L; G).$$

The resulting limit group $\varinjlim H^i(X, X \setminus K; G)$

is equal $H^i_c(X;G)$, when $H^i_c(X;G)$ is defined as following:

Let $C^i_c(X;G)$ be the subgroup of $C^i(X;G)$ consisting of cochains $\varphi : C_i(X) \to G$, for which there is some compact set $K = K_\varphi \subseteq X$ s.t. φ is zero on all chains in $X \setminus K$.

$$\delta : C^i_c(X;G) \longrightarrow C^{i+1}_c(X;G).$$

Since, $\delta^2 = 0$ we obtain so called compactly supported cohomology

$$H^i_c(X;G).$$

<u>Exercise</u>: $H^*_c(\mathbb{R}^n;G) = \begin{cases} 0 & \text{if } i \neq n \\ G & \text{if } i = n \end{cases}$

<u>Duality For Non-Compact Manifolds</u>

Let M be an R-orientable n-manifold. We define $D_M : H^k_c(M;R) \longrightarrow H_{n-k}(M;R)$ as follows

For compact subsets $K \subseteq L \subseteq M$ (M may <u>not</u> be compact)

$$\begin{tikzcd}
H_n(M, M\setminus L; R) \times H^k(M, M\setminus L; R) \arrow[dr, "\frown"] \arrow[d, "i_*"'] & \\
H_n(M, M\setminus K; R) \times H^k(M, M\setminus K; R) \arrow[r, "\frown"] \arrow[u, "i^*"', shift left=8em] & H_{n-k}(M;R)
\end{tikzcd}$$

Assume M is oriented (i.e. we have chosen a generator μ_x for each $H_n(M|x;R)$.)

By a previous lemma there are unique classes $\mu_K \in H_n(M|K;R)$ and $\mu_L \in H_n(M|L;R)$ which restricts to μ_x in $H_n(M|x;R)$

$\mu_K \longmapsto \mu_x$, $\mu_L \longmapsto \mu_x$

$$\begin{tikzcd}
\mu_L \arrow[ddrr, mapsto] & H_n(M|L;R) \arrow[rr, "i_*"] \arrow[dr, "i_*"'] & & H_n(M|K;R) \arrow[dl, "i_*"] & \mu_K \arrow[ddll, mapsto] \\
 & & H_n(M|x;R) & & \\
 & & \mu_x & &
\end{tikzcd}$$

By uniqueness we see that μ_L maps μ_K:

$i_*(\mu_L) = \mu_K$.

By the naturality of cap product we have:
For any $x \in H^k(M|K;R)$

$$i_*(\mu_L) \cap x = \mu_L \cap i^*(x)$$ and thus

$$\mu_K \cap x = \mu_L \cap i^*(x)$$

Therefore, letting K vary over compact sets in M, the homomorphism $H^k(M|K;R) \longrightarrow H_{n-k}(M;R)$, $x \longmapsto \mu_K \cap x$ induces a homomorphism in the limit

$$\begin{tikzcd}
D_M : H^k_c(M;R) \arrow[r] \arrow[d, equal] & H_{n-k}(M;R) \\
\varinjlim_K H^k(M|K;R) &
\end{tikzcd}$$

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Theorem. The duality map $D_M : H^k_c(M;R) \to H_{n-k}(M;R)$ is an isomorphism for all k, whenever M^n is an oriented n-manifold.

Remark. If M is compact then $H^k_c(M;R) = H^k(M;R)$ so that the above theorem gives the version for compact manifolds.

Lemma. If M is a union of two open sets U, V, then there is a diagram of Mayer–Vietoris sequences, commutative up to sign:

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & H^k_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^k_c(U)\oplus H^k_c(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H^k_c(M) \arrow[r] \arrow[d, "D_M"] & H^{k+1}_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\cdots \arrow[r, dashed] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(M)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Remark that in the top row arrows are in the opposite direction to the usual one in cohomology:

$$i : U \hookrightarrow M, \quad i^* : H^k(M) \to H^k(U)$$

In compactly supported cohomology we have natural inclusions

$$C^k_c(U) \subseteq C^k_c(M)$$

$$\Rightarrow i^* : H^k_c(U) \to H^k_c(M)$$

$$0 \to C^k_c(U\cap V) \to C^k_c(U)\oplus C^k_c(V) \to C^k_c(U\cup V) \to 0$$

$$\varphi \longmapsto (\varphi_1 - \varphi)$$

Exact

$$(\varphi, \phi) \longmapsto \varphi + \phi$$

$$\Rightarrow \cdots \to H^k_c(U\cap V) \to H^k_c(U)\oplus H^k_c(V) \to H^k_c(U\cup V) \to H^{k+1}_c(U\cap V)$$

$H^k_c(U) = \lim_{K} H^k_c(U, U\setminus K)$

For compact sets $K \subseteq U$, $L \subseteq V$ use the Mayer–Vietoris sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(M, K\cap L) \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\leftarrow \text{Excision} \rightarrow"] & H^k(M|K)\oplus H^k(M|L) \arrow[r] \arrow[d, "\simeq"] & H^k(M|K\cup L) \arrow[r] \arrow[dd, "\mu_{K\cap L}\frown"] & \cdots \\
 & H^k(U\cap V|K\cap L) \arrow[d, "\mu_{K\cap L}\frown"] & H^k(U|K)\oplus H^k(V|L) \arrow[d, "\mu_K\frown \,\oplus\, -\mu_L\frown"] & & \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & \cdots
\end{tikzcd}$$

Assuming the commutativity of the diagram, passing to the limit over compact sets $K \subseteq U$ and $L \subseteq V$ we obtain diagram for compactly supported cohomology as stated in the Lemma

I leave as an exercise the commutativity of the squares in the diagram (up to sign)

■

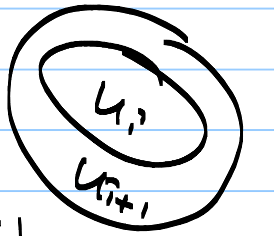
Proof of the Poincaré Duality

There are one finite and one infinite induction steps:

A) If M is the union of open sets U and V and if D_U, D_V and $D_{U\cap V}$ are isomorphisms then so is $D_{U\cup V}$. This follows from the above lemma and the 5-lemma.

B) Let M be the union of open subsets $U_1 \subseteq U_2 \subseteq \cdots$ and assume that each $D_{U_i} : H^k_c(U_i) \to H_{n-k}(U_i)$ is an isomorphism.

The D_M is also an isomorphism.



By excision we have
$$H_c^k(U_i) = \lim_{K\subseteq U_i} H^k(U_i|K) = \lim_{K\subseteq U_i} H^k(M|K)$$

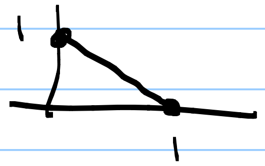
There are also maps $H^k_c(U_i)\longrightarrow H^k_c(U_{i+1})$.
Also note that $H_c^k(M)\cong \lim_i H^k_c(U_i)$, because

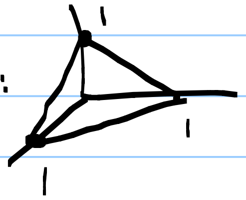
$$H^k_c(M)=\lim_{K\subseteq M}H^k(M|K), \quad H^k_c(U_i)=\lim_{K\subseteq U_i}H^k(U_i|K)$$

and each compact subset of M lies in some U_i since $M=\bigcup_i U_i$.

Now using these two facts we proceed in steps:

1) The case $M=\mathbb{R}^n$. Regard $M=\mathbb{R}^n$ as the interior of the standard n-simplex $\Delta^n=\{(x_0,\dots,x_n)\in\mathbb{R}^{n+1} \mid x_i\geq 0, \sum x_i=1\}$.

Δ^1:


Δ^2:


Then the duality map D_M can be identified with the map

$$\begin{array}{l} H^k(\Delta^n,\partial\Delta^n) \longrightarrow H_{n-k}(\Delta^n)\\ \quad\cong \\ H^k(\mathrm{Int}(\Delta^n),\emptyset)\\ \quad\cong\\ H^k(\mathbb{R}^n)\end{array}$$

given by the cap product with a unit times a generator $[\Delta^n]\in H_n(\Delta^n,\partial\Delta^n)$.

Here the generator $[\Delta^n]$ can be defined as the image of the identity map $\underset{\mathrm{id}}{\sigma}:\Delta^n\longrightarrow\Delta^n$, which is a relative cycle.

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Note that both groups are zero unless $k=n$.

$H^n(\Delta^n,\partial\Delta^n)\xrightarrow{\cap[\Delta^n]}H_0(\Delta^n)$ is an isomorphism R since a generator of $H^n(\Delta^n,\partial\Delta^n)\simeq \mathrm{Hom}(H_n(\Delta^n,\partial\Delta^n),R)$ is represented by a cocycle φ taking the value 1 on Δ^n, and thus by the definition of cap product $\underset{\overset{\shortparallel}{[v_0,\dots,v_n]}}{\Delta^n}\cap\varphi=\varphi([v_0,\dots,v_n])\cdot[v_n]$ represents a generator of $H_0(\Delta^n)$.

2) Now take M an arbitrary open set in $\mathbb{R}^n$. Note that M can be written as the countable union of nonempty convex open sets U_i (for example take U_i's as balls in $\mathbb{R}^n$).

$M=\bigcup_i U_i \qquad U_i\subseteq M$ convex subsets

Let $V_i=\underline{U_1}\cup\underline{U_2}\cup\dots\cup\underline{U_{i-1}}$

Both V_i and $U_i\cap V_i=(U_1\cap U_i)\cup(U_2\cap U_i)\cap\dots\cap(U_{i-1}\cap U_i)$ are unions of $i-1$ convex open subsets. So by induction on the number of such sets in a cover we may assume that D_{V_i} and $D_{U_i\cap V_i}$ are isomorphisms. By (1) D_{U_i} is an isomorphism since U_i is homeomorphic to $\mathbb{R}^n$. Now if we assume duality map is an isomorphism for the open sets which can be written as the union of $i-1$ convex open sets then D_{U_i}, D_{V_i} and $D_{U_i\cap V_i}$ are all isomorphisms. Finally, by (A) the map $D_{U_i\cup V_i}=D_{V_{i+1}}$ is an isomorphism.

Thus, D_{V_i} is an isomorphism for all i. However, $\cup V_i=\cup U_i=M$ and thus by (B) the map $D_M=D_{\cup U_i}$ is an isomorphism.

3) Assume M is a finite or countable union of open sets U_i homeomorphic to $\mathbb{R}^n$.
Now by (2) D_{U_i} is an isomorphism.

$M = \cup U_i$, let $V_i = U_1 \cup \dots \cup U_{i-1}$,
$V_i \cap U_i = (U_1 \cap U_i) \cup \dots \cup (U_{i-1} \cap U_i)$

$\Rightarrow V_i \cup (U_i) = V_{i+1}.$, $V_1 \subseteq V_2 \subseteq \dots \subseteq V_n \subseteq \longrightarrow M$

Almost the same arguments in part (2) prove (3).

Note that if M is a closed manifold then M is a finite union of open sets which are homeomorphic to $\mathbb{R}^n$.

To finish the proof for non-compact manifolds we use Zorn's lemma. Consider the collection of open sets $U \subseteq M$, for which the duality maps D_U are isomorphisms. This collection is partially ordered by inclusion: $U, V \subseteq M$
D_U, D_V isom. $U \subseteq V$.
Note that the union of every totally ordered subcollection is again in the collection by the argument in $\underline{\underline{(B)}}$: $\{U_\lambda\}_{\lambda \in \Lambda}$

$\lambda_1, \lambda_2 \in \Lambda$, $U_{\lambda_1} \subseteq U_{\lambda_2}$ or $U_{\lambda_2} \subseteq U_{\lambda_1}$

$\underset{\lambda \in \Lambda}{\cup} U_\lambda = W$ $(\tilde{B})$: If D_{U_λ} is an isomorphism for each $\lambda \in \Lambda$, then D_W is an isomorphism.

Now by Zorn's lemma there is a maximal element in the collection say $U \subseteq M$.

If $U \neq M$ then choose a point $x \in M \setminus U$ and open set $x \in V$ homeomorphic to $\mathbb{R}^n$. Then $U \cap V$ is an open subset of $\mathbb{R}^n$ and thus D_U, D_V and $D_{U \cap V}$ are all isomorphisms. So by (A) $D_{U \cup V}$ is an isomorphism. Since $x \notin U$, $U \cup V$ is bigger than the maximal element U of the collection, which is a contradiction. Hence, we must have $U = M$.

This finishes the proof of Poincaré Duality. ■

Corollary: A closed manifold M of odd dimension has zero Euler characteristic.

Proof: First assume M is orientable. Then
$$D_M : H^k(M;\mathbb{Z}) \longrightarrow H_{n-k}(M;\mathbb{Z})$$
is an isomorphism.

Let $b_k = \operatorname{rank} H_k(M;\mathbb{Z})$. Then $\chi(M) = \sum_{k=0}^{n} (-1)^k b_k$.

Since D_M is an isomorphism
$$\begin{aligned} b_{n-k} = \operatorname{rank} H_{n-k}(M;\mathbb{Z}) &= \operatorname{rank} H^k(M;\mathbb{Z}) \\ &= \operatorname{rank}(\operatorname{Fr} H_k(M;\mathbb{Z}) \oplus \operatorname{Tor} H_{k-1}(M;\mathbb{Z})) \\ &= \operatorname{rank}(\operatorname{Fr} H_k(M;\mathbb{Z})) \\ &= \operatorname{rank}(H_k(M;\mathbb{Z})) \\ &= b_k \end{aligned}$$

$\underline{n = 2m+1}$

Finally, $\chi(M) = \sum_{k=0}^{2m+1} (-1)^k b_k$
$$= \underbrace{b_0 - \underbrace{b_1 + b_2 + - \cdots + \overbrace{(-1)^m b_m + (-1)^{m+1} b_{m+1}}^{=0} + - \cdots}_{=0} - b_n}_{=0}$$

$= 0.$

If M is not orientable then we work with $\mathbb{Z}_2$-coefficients.

Exercise: Finish the proof. ∎

Connection with Cup Product

Lemma: For any $\alpha \in C_{k+l}(X;R)$, $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ we have

$$\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)$$

Proof: Both sides are linear w.r.t. all components, in particular w.r.t. α and thus it is enough to prove this for a singular simplex $\sigma : \Delta^{k+l} \to X$.

Now,

$$\psi(\sigma \frown \varphi) = \psi(\varphi(\sigma|_{[v_0,\dots,v_k]})\sigma|_{[v_k,\dots,v_{k+l}]})$$

$$= \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_k,\dots,v_{k+l}]})$$

$$= (\varphi \smile \psi)(\sigma)$$

∎

Using the above observation we prove the following formulation of Poincaré duality:

Let M be a closed R-oriented n-manifold and consider the cup product pairing:

$$H^k(M;R) \times H^{n-k}(M;R) \longrightarrow R$$
$$([\varphi],[\psi]) \longmapsto (\varphi \smile \psi)[M]$$

Note that this pairing is bilinear. Recall that such a pairing is called nonsingular (non-degenerate) if for any $[\varphi]\neq 0$ there is some $[\psi]$ s.t. that

$$([\varphi]\cup[\psi] \mid ([M]))\neq 0.$$

Proposition: Assume the above set up. Then the cup product pairing is nonsingular when R is a field or $R=\mathbb{Z}$ and torsion in $H^*(M;\mathbb{Z})$ is factored out, (i.e., replace $H^*(M;\mathbb{Z})$ by $H^*(M;\mathbb{Z})/\mathrm{Tor}(H^*(M;\mathbb{Z}))$.)

Proof: $R=\mathbb{Z}$. Take any $[\varphi]\in H^k(M;\mathbb{Z})/\mathrm{Tor}$ class, $[\varphi]\neq 0$ in $H^k(M;\mathbb{Z})/\mathrm{Tor}$.

M closed and $\mathbb{Z}$-orientable implies that $D_M([\varphi])=[\varphi\cap M]$ is not torsion in $H_{n-k}(M;\mathbb{Z})$. Now recall the U.C.T. that

$$H^{n-k}\underline{\underline{(M;\mathbb{Z})}}=\mathrm{Hom}(\widetilde{H_{n-k}(M};\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Tor}$$

$$\mathrm{Hom}(H_{n-k}(M;\mathbb{Z}),\mathbb{Z})=\mathrm{Hom}(\underbrace{\mathrm{Fr}(H_{n-k}(M;\mathbb{Z})}_{\cong});\mathbb{Z})$$

$$\underline{\mathcal{B}=\{e_1,e_2,\dots,e_n\}} \leftarrow \boxed{\mathbb{Z}\oplus\dots\oplus\mathbb{Z}}$$

$$\underline{\underline{\boxed{[\varphi\cap M]}}}=l\cdot\underline{\underline{e_1}} \qquad l\in\mathbb{Z}$$

$$\underline{\underline{\psi}}:\mathbb{Z}\oplus\dots\oplus\mathbb{Z}\to\mathbb{Z}$$
$$e_1\longmapsto 1$$
$$e_i\longmapsto 0,\ i\neq 1$$

So there is a cohomology class $[\psi] \in H^{n-k}(M;\mathbb{Z})$ so that

$$[\psi]([\varphi] \cap [M]) \neq 0.$$

Finally, by the lemma $([\psi] \cup [\varphi])([M]) \neq 0$ and hence the proof finishes for $R = \mathbb{Z}$ case.

For the $R = \mathbb{F}$ field case just note that

$$H^k(M;\mathbb{F}) \simeq \mathrm{Hom}(H_k(M;R);R)$$

so that the same proof works. ∎

Remark: $D_M : H^k(M;\mathbb{Z}) \longrightarrow H_{n-k}(M;\mathbb{Z})$ is an isomorphism. Thus for any $[\varphi] \in H^k(M;\mathbb{Z})$ is a primitive element if and only if $D_M([\varphi]) = [\varphi] \cap [M]$ is primitive in $H_{n-k}(M;\mathbb{Z})$. Hence, it follows from the above proof that if $[\varphi]$ is primitive then there is another primitive element $[\psi] \in H^{n-k}(M;\mathbb{Z})$ so that

$$([\varphi] \cup [\psi])([M]) = 1.$$

In other words, we have the following

Corollary: If M is a closed connected orientable n-manifold, then for each element $\alpha \in H^k(M;\mathbb{Z})$ of infinite order that is *not* a proper multiple of another element, then exists an element $\beta \in H^{n-k}(M;\mathbb{Z})$ such that $[\alpha \cup \beta][M] = 1$ and thus $[\alpha \cup \beta]$ is a generator of $H^n(M;\mathbb{Z})$.

In field coefficients theorem holds for any nonzero α.

Example: Cohomology Ring of $\mathbb{CP}^n$.

We've already seen that $H^k(\mathbb{CP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0,2,\dots,2n \\ 0, & \text{otherwise} \end{cases}$

Let $\alpha \in H^2(\mathbb{CP}^n; \mathbb{Z})$ be a generator.

Claim: $\alpha^k \in H^{2k}(\mathbb{CP}^n; \mathbb{Z})$ is a generator for all $k = 0, 1, \dots, n$. In particular,
$H^*(\mathbb{CP}^n; \mathbb{Z}) \simeq \mathbb{Z}[\alpha]/(\alpha^{n+1})$.

Proof: Proof by Induction of n.

$n=1$, $\alpha^1 \in H^2(\mathbb{CP}^1; \mathbb{Z})$ is a generator.
Assume the result for $n-1$. So we have $\alpha^{2k} \in H^{2k}(\mathbb{CP}^{n-1}; \mathbb{Z})$ is a generator, for any $k = 0, \dots, n-1$.

Consider the embedding $\mathbb{CP}^{n-1} \overset{i}{\hookrightarrow} \mathbb{CP}^n$
$[z_0 : \dots : z_{n-1}] \longmapsto [z_0 : \dots : z_{n-1} : 0]$.

Note that $i_* : H_k(\mathbb{CP}^{n-1}; \mathbb{Z}) \to H_k(\mathbb{CP}^n; \mathbb{Z})$ is an isomorphism for any $k \leq 2n-1$:

$\mathbb{CP}^n = \mathbb{CP}^{n-1} \cup e^{2n}$ and thus by cellular homology i_* is an isomorphism for $k \leq 2n-1$.

Similarly, $i^* : H^k(\mathbb{CP}^n; \mathbb{Z}) \to H^k(\mathbb{CP}^{n-1}; \mathbb{Z})$ is an isomorphism for all $k \leq 2n-1$.
$(\underline{\underline{i}}^*[\varphi])([\sigma]) = [\varphi](\underline{\underline{i}}_*([\sigma]))$

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Therefore, since α^k is a generator for $H^{2k}(\mathbb{CP}^{n-1};\mathbb{Z})$ for all $k=0,\dots,n-1$, α^k is a generator for $H^{2k}(\mathbb{CP}^n;\mathbb{Z})$.

So to finish the proof we must show that α^n is a generator for $H^{2n}(\mathbb{CP}^n;\mathbb{Z})$.

We know that $\alpha \in H^2(\mathbb{CP}^n;\mathbb{Z})$ and $\alpha^{n-1} \in H^{2n-2}(\mathbb{CP}^n;\mathbb{Z})$ are both generators. Since the pairing

$$H^2(\mathbb{CP}^n;\mathbb{Z}) \times H^{2n-2}(\mathbb{CP}^n;\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$([\varphi],[\psi]) \longmapsto [\varphi \smile \psi]([\mathbb{CP}^n])$$

is nondegenerate the product $\alpha \smile \alpha^{n-1} = \alpha^n$ is not trivial.

Indeed, since $\alpha \in H^2(\mathbb{CP}^n;\mathbb{Z})$ is primitive there is some $\beta \in H^{2n-2}(\mathbb{CP}^n;\mathbb{Z})$ so that $(\alpha \smile \beta)[\mathbb{CP}^n] = 1$. Since α^{n-1} is also a generator $\beta = l\alpha^{n-1}$ for some integer l.

Then $1 = (\alpha \smile \beta)[\mathbb{CP}^n] = l\,\underbrace{\alpha^n([\mathbb{CP}^n])}_{\in \mathbb{Z}}$

$\Rightarrow l = \pm 1$ and α^n is a generator for $H^{2n}(\mathbb{CP}^n;\mathbb{Z})$. ∎

Exercise: State and prove the analogous result for $H^*(\mathbb{RP}^n;\mathbb{Z}_2)$.

Example: $M = S^2 \times S^4$. Since both S^2 and S^4 are closed and orientable so is M.
We know that

$$H^n(M;\mathbb{Z}) = \bigoplus_{i+j=n} H^i(S^2;\mathbb{Z}) \otimes H^j(S^4;\mathbb{Z})$$

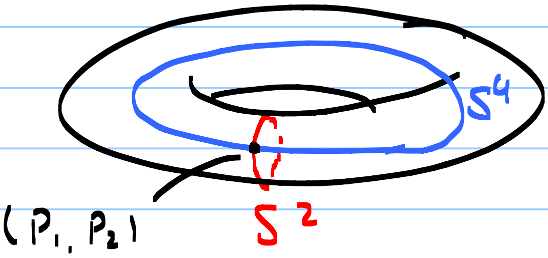
$$H^n(S^2 \times S^4; \mathbb{Z}) = \begin{cases} \mathbb{Z} & , n = 0, 2, 4, 6 \\ 0 & , \text{otherwise.} \end{cases}$$

From the above formula we have of

$i: S^2 \longrightarrow S^2 \times S^4$ $\quad$ $(i: S^4 \longrightarrow S^2 \times S^4)$

$\quad x \longmapsto (x, p)$ $\qquad\qquad\qquad$ $x \longmapsto (p, x)$

(p is a fixed point in S^4 or S^2, respectively)



$H^2(S^2 \times S^4; \mathbb{Z}) \times H^4(S^2 \times S^4; \mathbb{Z}) \longrightarrow \mathbb{Z}$ is nondegenerate and unimodular.

Hence, $\alpha \cup \beta$ is a generator for $H^6(S^2 \times S^4; \mathbb{Z})$ provided that $\alpha \in H^2(S^2 \times S^4; \mathbb{Z})$ and $\beta \in H^4(S^2 \times S^4; \mathbb{Z})$ are generators.

Thus, $H^*(S^2 \times S^4; \mathbb{Z}) = \mathbb{Z}[\alpha, \beta] \Big/ (\alpha^2, \beta^2)$ $\quad$ $\deg \alpha = 2$, $\deg \beta = 4$

Although $H^k(S^2 \times S^4; \mathbb{Z}) \simeq H^k(\mathbb{CP}^3; \mathbb{Z})$ for all k, the cohomology rings are non isomorphic and thus $\mathbb{CP}^3$ and $S^2 \times S^4$ are <u>not</u> even homotopy equivalent.

An application: If M is a $4k+2$ dimensional closed orientable manifold then $H^{2k+1}(M;\mathbb{Z})$ has even rank.

Proof: Since M is closed and orientable the pairing $H^{2k+1}(M;\mathbb{Z})\times H^{2k+1}(M;\mathbb{Z})\longrightarrow \mathbb{Z}$

$$([\varphi],[\psi]) \longmapsto (\varphi\cup\psi)[M]$$

is non degenerate. Indeed the same holds for $\mathbb{Q}$ coefficients also.

$H^{2k+1}(M;\mathbb{Q})\times H^{2k+1}(M;\mathbb{Q})\longrightarrow \mathbb{Q}$ is non degenerate

Also note that if $\alpha,\beta\in H^{2k+1}(M;\mathbb{Q})$ then $\alpha\cup\beta = -\beta\cup\alpha$.

Thus the above pairing is symplectic.

Algebraic fact: $\mathbb{F}$ - field, V, $\mathbb{F}$-vector space, $n = \dim_{\mathbb{F}} V$.

$\omega: V\times V\longrightarrow \mathbb{F}$ bilinear map. ω is called symplectic if ω is non-degenerate (i.e., if $u\in V$, $u\neq 0$, there is some $v\in V$ so that $\omega(u,v)\neq 0$) and $\omega(u,v) = -\omega(v,u)$, for all $u,v\in V$.

Fact: The vector space V has an $\mathbb{F}$-basis $\mathcal{B}=\{e_1,f_1,\dots,e_m,f_m\}$ so that

$\omega(e_i,e_j)=0=\omega(f_i,f_j)$, $\omega(e_i,f_j)=\delta_{ij}$ for all $i,j=1,\dots,m$.

In particular, $n=2m$ is an even integer.

Proof Exercise.

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Now the algebraic fact implies that the cohomology vector space $V = H^{2k+1}(M;\mathbb{Q})$ has even dimension.

$$H^{2k+1}(M;\mathbb{Q}) \cong \text{Hom}(H_{2k+1}(M;\mathbb{Z}),\mathbb{Q}) \oplus \overbrace{\text{Tor} -}^{=0}$$

$$\cong \text{Hom}(\text{Fr}\, H_{2k+1}(M;\mathbb{Z});\mathbb{Q})$$

$$\cong H_{2k+1}(M;\mathbb{Z}) \otimes \mathbb{Q}$$

$\Rightarrow \text{rank}\, H_{2k+1}(M;\mathbb{Z}) = \dim_{\mathbb{Q}} H^{2k+1}(M;\mathbb{Q})$, which is even.

Remark: $M = \mathbb{CP}^2$, $\dim M = 4 = 4k$

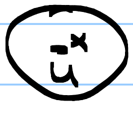
$H^2(\mathbb{CP}^2;\mathbb{Z}) \simeq \mathbb{Z}$ so that $\text{rank}\, H^2(\mathbb{CP}^2) = 1$ is not even.

Other Forms of Duality:

Definition: A n-manifold with boundary is a Hausdorff space in which each point has an open neighborhood homeomorphic either to $\mathbb{R}^n$ or the half-space $\mathbb{R}^n_+ = \{(x_1,\ldots,x_n)\in\mathbb{R}^n \mid x_n \geq 0\}$.

If a point x of the manifold corresponds to a point $(x_1,\ldots,x_n)\in\mathbb{R}^n_+$ with $x_n = 0$ then

$$H_n(M, M\setminus\{x\};\mathbb{Z}) \simeq H_n(U, U\setminus\{x\}, \mathbb{Z}) \quad \text{(Excision)}$$

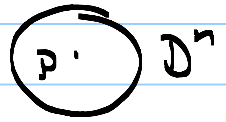


$$\simeq H_n(\mathbb{R}^n_+, \mathbb{R}^n_+\setminus\{(0,\ldots,0)\};\mathbb{Z})$$

$$\simeq H_n(\mathbb{R}^+_n, \overset{\simeq\ \text{h.e.}}{\mathbb{R}^n_+};\mathbb{Z}) = 0.$$

If $x \in M$ corresponds to a point in $\mathbb{R}^n_+$ with $x_n > 0$ then

$$\begin{aligned} H_n(M, M\setminus\{x\}) &\simeq H_n(\mathbb{R}^n_+, \mathbb{R}^n_+\setminus\{(0,\dots,0,1)\}; \mathbb{Z}) \\ &\simeq H_n(D^n, D^n\setminus\{p\}; \mathbb{Z}) \\ &\simeq H_n(D^n, \partial D^n; \mathbb{Z}) \\ &\simeq H_{n-1}(\underset{\substack{\rotatebox{90}{$\simeq$} \\ S^{n-1}}}{\partial D^n}; \mathbb{Z}) \simeq \mathbb{Z} \end{aligned}$$



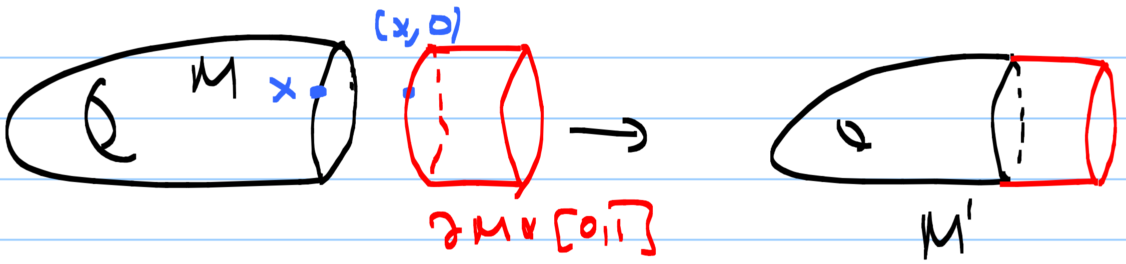
Thus the subset $\partial \underline{M} \doteq \{x \in M \mid x$ corresponds to a point in $\mathbb{R}^n_+$, whose last coordinate is zero$\}$ is well defined. Moreover, by definition ∂M is a $n-1$ dimensional manifold without boundary.

Let M be a manifold with boundary. A collar neighborhood of ∂M in M is an open neighborhood homeomorphic to $\partial M \times [0,1)$ via a homeomorphism taking ∂M to $\partial M \times \{0\}$.

Proposition: If M is a compact manifold with boundary, then ∂M has a collar neighborhood.

Proof: Let M' be the manifold M glued to $\partial M \times [0,1]$ as follows:

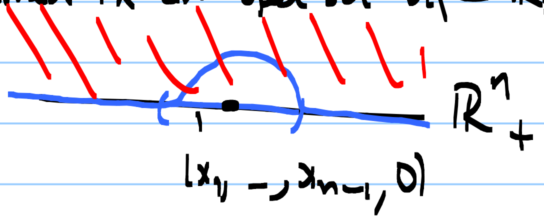
$M' = M \,\dot{\cup}\, \partial M \times [0,1] \big/ x \sim (x,0),\ x \in \partial M$



Clearly, M' is a manifold with boundary homeomorphic to ∂M and M' has a built in collar neighborhood for its boundary.

To finish the proof it is enough to prove that M is homeomorphic to M'.

Since M is compact so is its closed subset ∂M. Choose finitely many continuous functions $\varphi_i : \partial M \longrightarrow [0,1]$ such that the sets $V_i = \varphi_i^{-1}(0,1]$ form an open cover of ∂M and V_i has closure contained in an open set $U_i \cong \mathbb{R}^+_n$



Replacing φ_i by $\varphi_i / \Sigma \varphi_i$, we may assume that $\Sigma \varphi_i = 1$.

For any $k \geq 0$ integer let $\psi_k = \varphi_1 + \cdots + \varphi_k$ and let $M_k \subseteq M'$ be the union of M with the points $(x,t) \in \partial M \times [0,1]$ with $t \leq \psi_k(x)$.

$\psi_0 = 0$, $M_0 = M \subseteq M_1 \subseteq \cdots \subseteq \qquad \subseteq M_N = M'$

$(N = \#$ of φ_i's$)$

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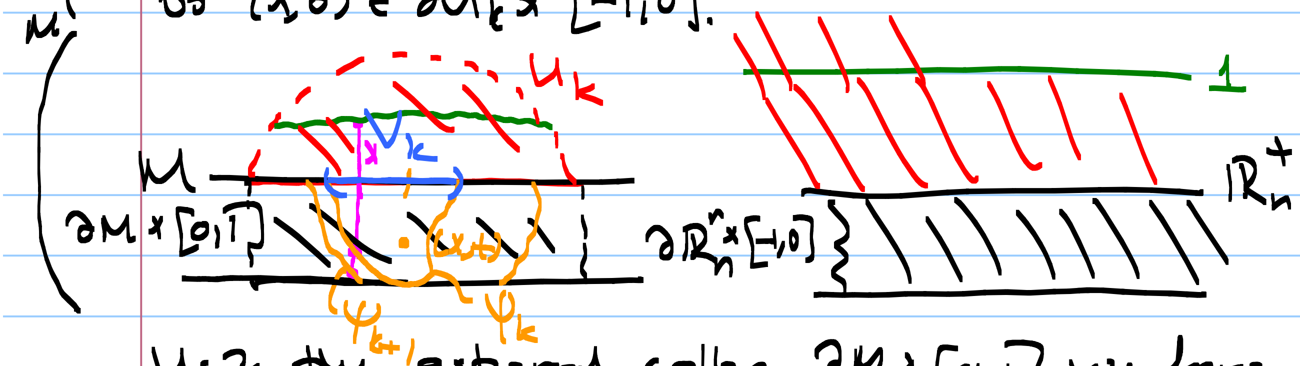
$$\left.\begin{array}{l} \psi_N = \varphi_1 + \dots + \varphi_N = 1 \\ (x,t), \quad t \le 1 = \psi_N(x) \end{array}\right\} \Rightarrow M_N = M'.$$

We'll construct a homeomorphism $h_k : M_{k-1} \to M_k$ for each $k = 1, \dots, N$, as follows:

We know that U_k is homeomorphic to $\mathbb{R}^+_n$. This gives a collar neighborhood $\partial U_k \times [-1,0]$ of ∂U_k in U_k, with $x \in \partial U_k$ corresponding to $(x,0) \in \partial U_k \times [-1,0]$.



Using the external collar $\partial M \times [0,1]$ we have an embedding $\partial U_k \times [-1,1]$ into M'.

Define $h_k : M_{k-1} \to M_k$ to be identity outside of $\partial U_k \times [-1,1]$ and for each $x \in \partial U_k$ let h_k stretch the segment $\{x\} \times [-1, \psi_{k-1}(x)]$ linearly onto $\{x\} \times [-1, \psi_k(x)]$.

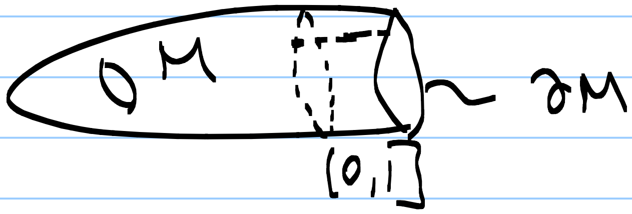
The composition of all the h_k's gives a homeomorphism of $M_0 = M$ to $M_N = M'$.

■

A compact manifold M with boundary is defined to be R-orientable if $M \setminus \partial M$ is R-orientable as a manifold without boundary.

Exercise: If a compact manifold with boundary M is R-orientable then so is its boundary.

Let M be a compact R-orientable manifold with boundary ∂M and let $\partial M \times [0,1)$ is a collar neighborhood of ∂M in M.



Thm $H_i(M,\partial M;R)$ is naturally isomorphic to $H_i(M \setminus \partial M, \partial M \times (0,\epsilon);R)$

Take $A = M$ in $M' = M \sqcup \partial M \times [0,1] / {x \sim (x,0) \atop x \in \partial M}$

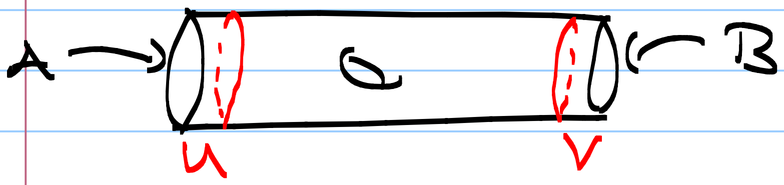
$$H_n(M'|A) = H_n(M', M' \setminus A)$$
$$\simeq H_n(M, \partial M)$$

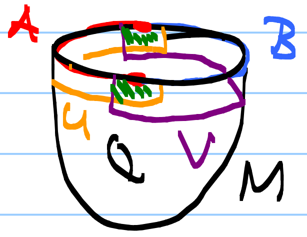
has a fundamental class that restricting to a given orientation at each point of $M \setminus \partial M$. (Lemma 3.27 in Hatcher's Book)

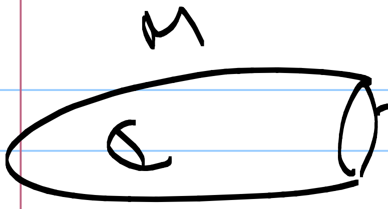
Theorem: Suppose M is a compact R-orientable n-manifold whose boundary ∂M is decomposed as the union of two compact $(n-1)$-dimensional manifolds A and B with common boundary $\partial A = \partial B = A \cap B$. Then cap product with a fundamental class $[M] \in H_n(M, \partial M; R)$ gives isomorphisms

$$D_M : H^k(M,A;R) \longrightarrow H_{n-k}(M;B;R), \text{ for all } k.$$

A, B or $A \cap B$ may be empty, $\partial M = A \cup B$.







$\sim \partial M = A,\ B=\emptyset$
or $B,\ A=\emptyset$.

<u>Proof</u>! $D_M : H^k(M,A;R) \longrightarrow H_{n-k}(M,B;R)$ is
$\alpha \longmapsto [M]\frown\alpha$
defined because by the existence of collar neighborhoods of $A\cap B$ in A and B and ∂M in M there are open neighborhoods U, V in M s.t. $U\cup V$ deformation retracts onto $A\cup B=\partial M$ and $U\cap V$ deformation retracts onto $A\cap B$.

The case $B=\emptyset$:

$$D_M : H^k(M,A;R) \longrightarrow H_{n-k}(M;R).$$

Using the collar neighborhood of $\partial M = A$ we see that

$$H^k(M,\partial M;R) \overset{?}{\simeq} H^k_c(M\setminus\partial M;R)$$

and there are obvious isomorphisms

$$H_{n-k}(M;R) \simeq H_{n-k}(M\setminus\partial M;R).$$

$\simeq$ P.D.

The result then follows from Poincaré duality

$$H^k_c(\underset{M\setminus\partial M}{M};R) \xrightarrow[\simeq]{\frown[M]} H_{n-k}(\underset{M\setminus\partial M}{M};R).$$

The general case reduces to the case $B=\emptyset$ by applying the five lemma to the diagram below:

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$$\begin{tikzcd}
\cdots \arrow[r] & H^k(M,\partial M) \arrow[r] \arrow[d, "{[M]\cap}", "\cong"'] & H^k(M,A) \arrow[r] \arrow[d, "{[M]\cap}", "\cong"'] & H^k(\partial M,A) \arrow[r] \arrow[d, "\cong"] & H^{k+1}(M,\partial M) \arrow[r] \arrow[dd, "{[M]\cap}", "\cong"'] & {} \\
& {} \arrow[r, phantom, "\Rightarrow"] & {} & H^k(B,\partial B) \arrow[d, "\cong", "{[B]\cap}"'] & & \\
\cdots \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k}(M,B) \arrow[r] & H_{n-k-1}(B) \arrow[r] & H_{n-k-1}(M) \arrow[r] & {}
\end{tikzcd}$$

One needs to check the commutativity of the squares in the diagram.

Remark: Proof of $H^k(M,\partial M) \cong H^k_c(M\setminus\partial M)$

$U \subseteq M$ open

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_c^k(U) \arrow[r] \arrow[d] & C_c^k(M) \arrow[r] \arrow[d] & C_c^k(M)/C_c^k(U) \arrow[r] \arrow[d] & 0 \\
& {} & {} & {} &
\end{tikzcd}$$

Long exact sequence in compactly supported cohomology

$$\Rightarrow \cdots \to H^k_c(U) \to H^k_c(M) \to H^k_c(M,U) \to H^{k+1}_c(U) \to \cdots$$

Take $U = M\setminus\partial M$ to get

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(M\setminus\partial M) \arrow[r] & H^k_c(M) \arrow[r] \arrow[d, phantom, "\wr" description] & H^k_c(M, M\setminus\partial M) \arrow[r] \arrow[d, phantom, "\wr\ \text{Fact}" description] & H^{k+1}_c(M\setminus\partial M) \\
& & H^k(M) & H^k_c(\partial M) & \\
& & & H^k(\partial M) &
\end{tikzcd}$$

M is compact $\Rightarrow$

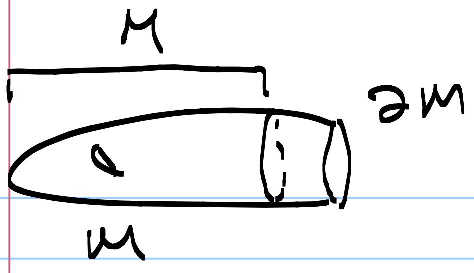
$$\begin{tikzcd}
\Rightarrow H^{k-1}(M) \arrow[r] \arrow[d, "\cong"] & H^{k-1}(\partial M) \arrow[r] \arrow[d, "\cong"] & H^k_c(M\setminus\partial M) \arrow[r] & H^k(M) \arrow[r] \arrow[d, "\cong"] & H^k(\partial M) \arrow[d, "\cong"] \\
H^k(M) \arrow[r] & H^k(\partial M) \arrow[r] & H^{k+1}(M,\partial M) \arrow[r] \arrow[u, "?"] & H^k(M) \arrow[r] & H^k(\partial M)
\end{tikzcd}$$

$$H^k_c(M\setminus\partial M) = \varinjlim_{\substack{K\subseteq M\setminus\partial M\\ \text{compact}}} H^k(M\setminus\partial M, (M\setminus\partial M)\setminus K)$$

$\wr$ h.e. $\quad$ $\wr$ h.e.

$$= \varinjlim_{\substack{K\subseteq M\setminus\partial M\\ \text{compact}}} H^k(M, \underline{\underline{M\setminus K}})$$

$\nwarrow$



$H^k(M,\partial M)$

Finally note that we have natural map $H^k(M,\partial M) \to H^k(M, M\setminus K)$ since $K \subseteq M\setminus \partial M$ $\partial M \subseteq M\setminus K$. Hence, the result follows from 5-lemma. ■

Alexander Duality

Theorem: If K is a compact, locally contractible, nonempty, proper subspace of S^n, then

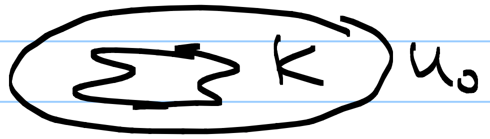
$$\tilde{H}_i(S^n\setminus K;\mathbb{Z}) \simeq \tilde{H}^{n-i-1}(K;\mathbb{Z}),\text{ for all } i.$$

Remark: Recall that we proved this result for K is disc or a sphere previously.

Proof: For $i\neq 0$ we have

$$\begin{aligned} H_i(S^n\setminus K) &\simeq H_c^{n-i}(S^n\setminus K) && \text{(Poincaré Duality)}\\ &\simeq \varinjlim_{\substack{L\subseteq S^n\setminus K\\ \text{compact}}} H^{n-i}(S^n\setminus K, U\setminus K) && \text{(Defn. of c.s.c)}\ (U = S^n\setminus L)\\ &\simeq \varinjlim H^{n-i}(S^n, U) && \text{(Excision of } K)\\ &\simeq \varinjlim \tilde{H}^{n-i-1}(U) && \text{if } i\neq 0\\ & && \text{(long exact seq. of the pair } (S^n,U))\\ &\simeq \tilde{H}^{n-i-1}(K), \end{aligned}$$

where the last isomorphism follows from the fact that K is a retract of some neighborhood U_0 in S^n (Appendix) in which case we can compute the direct limit using neighborhoods



One also needs to check the case $i=0$. (Exercise!)

∎

<u>Corollary</u> If $X \subseteq \mathbb{R}^n$ compact and locally contractible then $H_i(X;\mathbb{Z})$ is zero for $i \geq n$ and torsion free for $i=n-1$ and $n-2$.

<u>An application:</u> M^n compact non-orientable manifold. Assume that $M^n \subseteq \mathbb{R}^{n+1}$.

?

Since M is non-orientable $H_{n-1}(M;\mathbb{Z})$ has a $\mathbb{Z}_2$-component: If M is non-orientable then the orientation double cover $\tilde{M} \to M$ is connected. Hence, $\pi_1(M)$ has an index 2 subgroup. This implies $H_1(M;\mathbb{Z})$ has an index two subgroup, $H \leq H_1(M;\mathbb{Z})$.

$H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{Z})/_H \simeq \mathbb{Z}_2$ onto

$\Rightarrow$ $H^1(M;\mathbb{Z}_2) = \mathrm{Hom}(H_1(M;\mathbb{Z});\mathbb{Z}_2)$ has a non zero element. $\Rightarrow$ $H_{n-1}(M;\mathbb{Z}_2)$ has a nonzero element.

$$\mathbb{Z}_2 = H^n(M;\mathbb{Z}_2) = \mathrm{Hom}(H_n(\underset{\underset{0}{\|}}{M};\mathbb{Z}), \mathbb{Z}_2) \oplus \mathrm{Ext}(\underline{\underline{H_{n-1}(M;\mathbb{Z})}}, \mathbb{Z}_2)$$

$\Rightarrow$ $H_{n-1}(M;\mathbb{Z})$ has a $\mathbb{Z}_2$-subgroup as its torsion (Corollary 3.28)

$\underline{M^n} \subseteq \mathbb{R}^{n+1} \subseteq \mathbb{R}^{n+1} \cup \{\infty\} = S^{n+1}$

Since M is compact and locally contractible by the above corollary $H_{n-1}(M;\mathbb{Z})$ is torsion free, which is a contradiction.

Hence, M^n cannot be embedded into $\mathbb{R}^{n+1}$.

Example $M^2 = \mathbb{RP}^2$ compact non-orientable surface cannot be embedded into $\mathbb{R}^3$. (Boğaz Univ. Sem.)

Proof of the corollary: $X \subseteq \mathbb{R}^n \subseteq S^n$.

Alex. Duality $\Rightarrow H^i(X;\mathbb{Z}) \simeq H_{n-i-1}(S^n \setminus X;\mathbb{Z})$

If $i \geq n$ then $n-i-1 < 0$ and hence $H_{n-i-1}(S^n \setminus X;\mathbb{Z}) = 0$.

$i = n-1 \Rightarrow H^{n-1}(X;\mathbb{Z}) \simeq H_0(S^n \setminus X;\mathbb{Z})$ free abel.

$H^{n-1}(X;\mathbb{Z}) \cong \text{Fr}\, H_{n-1}(X;\mathbb{Z}) \oplus \text{Tor}\, H_{n-2}(X;\mathbb{Z})$

Since $H^{n-1}(X;\mathbb{Z})$ is free $\text{Tor}(H_{n-2}(X;\mathbb{Z})) = 0$
$\Rightarrow H_{n-2}(X;\mathbb{Z})$ is torsion free.

$0 = H^n(X;\mathbb{Z}) = \text{Fr}\, H_n(X;\mathbb{Z}) + \underbrace{\text{Tor}\, H_{n-1}(X;\mathbb{Z})}_{\overset{\shortparallel}{0}}$

$\Rightarrow H_{n-1}(X;\mathbb{Z})$ has no torsion.

Here, we use U.C.T. since X has finitely generated homology groups.

X compact + locally contractible.

$x \in X$, $x \in U \subseteq X$, $U \simeq *$

$X = U_1 \cup U_2 \cup \dots \cup U_k$, U_i contractible.
$\Rightarrow X$ has finitely generated homology.

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Proposition: If K is a compact, locally contractible subspace of an orientable n-manifold M, then there are isomorphisms

$$H_i(M, M \setminus K; \mathbb{Z}) \simeq H^{n-i}(K; \mathbb{Z}), \text{ for all } i.$$

Additional Topics

Universal Coefficient Theorem for Homology:

U.C.T. for cohomology uses Ext functor. For homology we need so called Tor-functor.

Tor: H abelian group

$$(F) \qquad \cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

(exact sequence = free resolution)

F_n: free abelian group

Let G be any other abelian group and consider the sequence

$$\cdots \to F_2 \otimes G \xrightarrow{f_2 \otimes id} F_1 \otimes G \xrightarrow{f_1 \otimes id} F_0 \otimes G \xrightarrow{f_0 \otimes id} H \otimes G \to 0,$$

which is not necessarily exact.

Lemma: If F and F' are two free resolutions for H there are canonical isomorphism

$$H_n(F \otimes G) \simeq H_n(F' \otimes G), \text{ for all } n.$$

Definition: The abelian group $H_n(F \otimes G)$ is denoted $\operatorname{Tor}_n(H; G)$

In particular, Tor_1 is denoted simply by Tor.

Remark: Any abelian group H has a resolution of the form

$$0 \to F_1 \to F_0 \to H \to 0.$$

$$0 \to F_1 \otimes G \to F_0 \otimes G \to H \otimes G \to 0$$

$\operatorname{Tor}_n(H, G) = H_n(F \otimes G) = 0 \quad \forall\ n \geq 2.$

Ex $H = \mathbb{Z}_n$
$$0 \to \mathbb{Z} \xrightarrow[\ ]{\times n \quad f_1} \mathbb{Z} \xrightarrow{f_0} \mathbb{Z}_n \to 0$$
$$1 \longmapsto n$$

$$G \otimes_{\mathbb{Z}} \mathbb{Z} \to G \otimes_{\mathbb{Z}} \mathbb{Z} \to G \otimes_{\mathbb{Z}} \mathbb{Z}_n \to 0$$

$$G \xrightarrow{\times n} G \xrightarrow{\mathrm{id} \otimes f_0} G \otimes_{\mathbb{Z}} \mathbb{Z}_n \to 0$$
$$g \longmapsto ng$$

$\operatorname{Tor}(\mathbb{Z}_n, G) = H_1(G \otimes \mathbb{Z}_n) \qquad \underset{\text{is}}{G} \quad g \longmapsto g \otimes 1$

$$= \frac{\ker(\mathrm{id} \otimes f_0 : G \otimes \mathbb{Z} \to G \otimes \mathbb{Z}_n)}{nG}$$

$\operatorname{Tor}(\mathbb{Z}_5, \mathbb{Z}_{12}) = \ker(\mathbb{Z}_{12} \xrightarrow{\times 5} \mathbb{Z}_{12})$

$= 0$

$\operatorname{Tor}(\mathbb{Z}_2, \mathbb{Z}_6) = \ker(\mathbb{Z}_6 \xrightarrow{\times 2} \mathbb{Z}_6)$

$= \langle \bar{3} \rangle$

$\simeq \mathbb{Z}_2.$

Theorem If C is a chain complex of free abelian groups, then there are natural short exact sequences

$$0 \to H_n(C) \otimes G \to H_n(C;G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0$$

for all n and all G, and these sequences split, not necessarily naturally.

Corollary For each pair of spaces (X,A) there are split exact sequences

$$0 \to H_n(X,A) \otimes G \to H_n(X,A;G) \to \mathrm{Tor}(H_{n-1}(X,A), G) \to 0$$

for all n, and these sequences are natural with respect to maps of pairs $(X,A) \to (Y,B)$.

Proposition:

1) $\mathrm{Tor}(A,B) \simeq \mathrm{Tor}(B,A)$
2) $\mathrm{Tor}(\bigoplus_i A_i, B) \simeq \bigoplus_i \mathrm{Tor}(A_i, B)$

3) $\mathrm{Tor}(A,B) = 0$ if A or B is free, or more generally torsion free.

4) $\mathrm{Tor}(A,B) \simeq \mathrm{Tor}(T(A), B)$, where $T(A)$ is the torsion part of A.

5) $\mathrm{Tor}(\mathbb{Z}/n, A) \simeq \ker(A \xrightarrow{n} A)$

6) For any exact sequence $0 \to B \to C \to D \to 0$ we have an exact sequence
$$0 \to \mathrm{Tor}(A,B) \to \mathrm{Tor}(A,C) \to \mathrm{Tor}(A,D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0$$

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The General Künneth Formula

Cross Product: $H_i(X;R)\times H_j(Y;R) \xrightarrow{\times} H_{i+j}(X\times Y;R)$

Cross Product on cell level:

[$d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^i\,\omega\wedge d\eta$
derivative formula for product of differential forms.]

X, Y cell complexes e^i and e^j two cells of X and Y respectively. Then $e^i\times e^j$ is an $i+j$ cell of $X\times Y$. It turns out that the right definition of the boundary of $e^i\times e^j$ is as follows:

$$d(e^i\times e^j) = de^i\times e^j + (-1)^i e^i\times de^j$$

Proposition The boundary map in the cellular chain complex $C_*(X\times Y)$ is determined by the boundary map in the cellular complexes $C_*(X)$ and $C_*(Y)$ via the formula

$$d(e^i\times e^j) = de^i\times e^j + (-1)^i e^i\times de^j.$$

Example: $S^n\times S^m$, $m,n>0$

i) $n=m$, $X=S^n=e^0\cup e^n$, $Y=S^n=f^0\cup f^n$

$$\begin{tikzcd}
0 \arrow[r] & C_n(X) \arrow[r, "d"] \arrow[d, phantom, "\text{is}"] & 0 \cdots \arrow[r, "d"] & C_0(X) \arrow[r] \arrow[d, phantom, "\text{is}"] & 0 \\
 & \mathbb{Z} \arrow[d, phantom, "\text{is}"] & & \mathbb{Z} \arrow[d, phantom, "\text{is}"] & \\
0 \arrow[r] & C_n(Y) \arrow[r, "d"] & 0 -- \arrow[r, "d"] & C_0(Y) \arrow[r] & 0
\end{tikzcd}$$

$d=0$ for all d.

$$X\times Y = S^n\times S^n = e^0\times f^0 \cup e^0\times f^n \cup e^n\times f^0 \cup e^n\times f^n$$

$$\begin{tikzcd}
0 \arrow[r] & C_{2n}(X\times Y) \arrow[r] \arrow[d, phantom, "\text{is}"] & 0 \cdots 0 \arrow[r] & C_n(X\times Y) \arrow[r] \arrow[d, phantom, "\text{is}"] & 0 \cdots \arrow[r] & C_0(X\times Y) \arrow[d, phantom, "\text{is}"] \\
 & \mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}
\end{tikzcd}$$

$$d(e^n \times f^n) = \overset{0}{\overset{\shortparallel}{de^n}} \times f^n + (-1)^n e^n \times d\overset{0}{\overset{\shortparallel}{f^n}}$$
$$= 0.$$

Similarly, all boundary maps are trivial. ~~Then~~
$H_{2n}(S^n \times S^n) \simeq \mathbb{Z}$, $H_n(S^n \times S^n) \simeq \mathbb{Z} \oplus \mathbb{Z}$,

$H_0(S^n \times S^n) \simeq \mathbb{Z}$ and all other homologies are trivial.

ii) $\underset{X}{\underset{\shortparallel}{S^n}} \times \underset{Y}{\underset{\shortparallel}{S^m}}$, $n > m > 0$

$$0 \to \underset{\mathbb{Z}}{\underset{\wr\shortparallel}{C_{n+m}(X \times Y)}} \to \cdots \to \underset{\mathbb{Z}}{\underset{\wr\shortparallel}{C_n(X \times Y)}} \to 0 \to \underset{\mathbb{Z}}{\underset{\wr\shortparallel}{C_m(X \times Y)}}$$
$$\cdots \to \underset{\mathbb{Z}}{\underset{\wr\shortparallel}{C_0(X \times Y)}} \to 0$$

$S^n = e^0 \cup e^n$, $S^m = f^0 \cup f^m$,
$S^n \times S^m = e^0 \times f^0 \cup e^0 \times f^m \cup e^n \times f^0 \cup e^n \times f^m$

Since all boundaries are trivial as above we get
$H_{n+m}(S^n \times S^m) \simeq \mathbb{Z}$, $H_n(S^n \times S^m) \simeq \mathbb{Z}$, $H_m(S^n \times S^m) \simeq \mathbb{Z}$
$H_0(S^n \times S^m) \simeq \mathbb{Z}$ and other homologies are zero.

Ex: $\underset{X}{\underset{\shortparallel}{\mathbb{RP}^2}} \times \underset{Y}{\underset{\shortparallel}{\mathbb{RP}^3}}$

$\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$
$d : \underset{\mathbb{Z}}{\underset{\wr\shortparallel}{C_k}} \longrightarrow \underset{\mathbb{Z}}{\underset{\wr\shortparallel}{C_{k-1}}}$

$$d = \begin{cases} \times 2 & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}$$

$X = \mathbb{RP}^2$ $$0 \to C_2(X) \xrightarrow{d_2} C_1(X) \xrightarrow{d_1} C_0(X) \to 0$$
$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$Y = \mathbb{RP}^3$ $$0 \to C_3(Y) \to C_2(Y) \to C_1(Y) \to C_0(Y) \to 0$$
$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$X = \mathbb{RP}^2 = e^0 \cup e^1 \cup e^2$

$Y = \mathbb{RP}^3 = f^0 \cup f^1 \cup f^2 \cup f^3$

$X \times Y = (e^0 \times f^0) \cup (e^0 \times f^1 \cup e^1 \times f^0) \cup (e^0 \times f^2 \cup e^1 \times f^1 \cup e^2 \times f^0)$

$\cup (e^0 \times f^3 \cup e^1 \times f^2 \cup e^2 \times f^1) \cup (e^1 \times f^3 \cup e^2 \times f^2)$

$\cup (e^2 \times f^3)$

$$0 \to C_5(X \times Y) \xrightarrow{d_5} C_4(X \times Y) \xrightarrow{d_4} C_3(X \times Y) \xrightarrow{d_3} C_2(X \times Y) \xrightarrow{d_2} C_1(X \times Y) \xrightarrow{d_1} C_0(X \times Y) \to 0$$

$$0 \to \underset{e^2 \times f^3}{\mathbb{Z}} \to \underset{e^1 \times f^3}{\mathbb{Z}} \oplus \underset{e^2 \times f^2}{\mathbb{Z}} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

$1 \longmapsto (2, 0)$

$$d_5(e^2 \times f^3) = d(e^2) \times f^3 + (-1)^2 e^2 \times df^3$$
$$= 2e^1 \times f^3 + e^2 \times 0$$
$$= 2(e^1 \times f^3)$$

For the computations of the homology group we may Smith Normal Form.

The Algebraic Künneth Formula

There is a cross product as follows:

$$\bigoplus_i (H_i(X; \mathbb{Z}) \otimes H_{n-i}(Y; \mathbb{Z})) \longrightarrow H_n(X \times Y; \mathbb{Z})$$

Theorem. If X and Y are CW complexes and R is a principal ideal domain then there are natural short exact sequences

$$0 \to \bigoplus_i (H_i(X; R) \otimes_R H_{n-i}(Y; R)) \to H_n(X \times Y) \to \bigoplus_i \mathrm{Tor}_R(H_i(X; R), H_{n-i}(Y; R)) \to 0$$

and these sequences split.

<u>Remark</u>: $R = \mathbb{F}$ field $\Rightarrow \mathrm{Tor}_R(\_\_,\_\_) = 0$

and thus $\bigoplus_i \left(H_i(X;\mathbb{F}) \otimes_R H_{n-i}(Y;\mathbb{F})\right) \simeq H_n(X \times Y;\mathbb{F})$.

<u>H-Space</u>: Let X be a topological space and assume that X has a group structure whose group operations are continuous. Then X is called a topological group.

<u>Example</u>: 1) $X = (\mathbb{R}^n, +)$ $\quad +: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$(x,y) \longmapsto x+y$

$-: \mathbb{R}^n \to \mathbb{R}^n$, $x \longmapsto -x$

continuous.

2) $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, $(S^1, \cdot)$ is a topological group.

3) $S^3 = \{q \in \mathbb{H} \mid |q| = 1\}$ $\quad q = x + iy + jz + kw$

$x,y,z,w \in \mathbb{R}$

$\bar{q} = x - iy - jz - kw$

$|q| = (q\bar{q})^{1/2} = (x^2+y^2+z^2+w^2)^{1/2}$.

$(S^3, \cdot)$ is a topological group.

4) $SL(n,k)$, $GL(n,k)$, $O(n)$, $U(n)$ $\quad (k = \mathbb{R}, \mathbb{C})$ are all topological groups.

5) $S^1 \times S^1 \times \cdots \times S^1$

<u>Definition</u>: A space X is called an H-space (H stands for Hopf) if there is a continuous multiplication $\mu: X \times X \to X$ and an "identity"

element $e \in X$ such that the two maps $X \longrightarrow X$, $x \longmapsto \mu(x,e)$, and

$X \longrightarrow X$, $x \longmapsto \mu(e,x)$, are both homotopic to the identity map $id: X \longrightarrow X$.

Fact: $\pi_1(X)$ is abelian if X is an H-space.

Example: 1) $\mathbb{RP}^\infty = \varinjlim_n \mathbb{RP}^n = S^\infty / \mathbb{Z}_2$

$e = [1]$ $\quad = \mathbb{R}[x] \setminus \{0\} \Big/ \begin{smallmatrix} p(x) \sim \lambda p(x) \\ \lambda \in \mathbb{R}^* \end{smallmatrix}$

$$\mu: \mathbb{RP}^\infty \times \mathbb{RP}^\infty \longrightarrow \mathbb{RP}^\infty$$
$$([p(x)],[q(x)]) \longmapsto \mu([p],[q]) = [p(x)q(x)]$$

2) $\mathbb{CP}^\infty = \mathbb{C}[z] \setminus \{0\} \Big/ \begin{smallmatrix} p(z) \sim \lambda p(z) \\ \lambda \in \mathbb{C}^* \end{smallmatrix}$.
$e = [1]$

Note that the multiplication in the above examples are associative and commutative.

Remark: 1) For the H-space $\mathbb{RP}^\infty$ the inverse of an element $[p(x)]$ is itself.

$\mathbb{RP}^\infty \longrightarrow \mathbb{RP}^\infty$, $[p(x)] \longmapsto [p(x) \cdot p(x)]$ is homotopic to the constant functor $[p(x)] \longmapsto [1]$.

2) Similarly, for $\mathbb{CP}^\infty$ the inverse element of $[p(z)]$ is $[\overline{p(z)}]$. The map

$\mathbb{CP}^\infty \longrightarrow \mathbb{CP}^\infty$, $[p(z)] \longmapsto [p(z) \cdot \overline{p(z)}]$ is homotopic to the constant map $[p(z)] \longmapsto [1]$.

3) $n > 1$, $n \in \mathbb{Z}$, $k \geq 1$.

$\mathbb{Z}_n \times S^{2k-1} \longrightarrow S^{2k-1}$

$S^{2k-1} = \{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid \sum |z_i|^2 = 1\}$

$\mathbb{Z}_n = \langle \sigma \rangle$, $\sigma : S^{2k-1} \longrightarrow S^{2k-1}$, given by

$\sigma(z_1, \ldots, z_k) = (\xi z_1, \ldots, \xi z_k)$, $\xi = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$

$S^{2k-1}/\mathbb{Z}_n$ is called a lens space.

$L(n) = \lim_{k} S^{2k-1}/\mathbb{Z}_n$ $\qquad S^1 \subseteq S^3 \subseteq S^5 \subseteq \ldots$

$L(n)$ is also an H-space.

Note that if $n=2$ then $L(2) = \mathbb{RP}^\infty$ and if we replace $\mathbb{Z}_n$ by S^1 we get $\mathbb{CP}^\infty$.

$\mathbb{RP}^\infty = e^0 \cup e^1 \cup e^2 \cup \ldots \cup e^n$

$e^n = \{[p(x)] \mid \deg p(x) = n\}$. $\quad e^k \times e^l \longrightarrow e^{k+l}$

Bockstein Homomorphisms

Let $0 \to G \to H \to K \to 0$ be an exact sequence of abelian groups.

Let X be a space, then $C_n(X)$ is a free abelian group. Therefore both sequences below are still short exact:

$$0 \to G \otimes_{\mathbb{Z}} C_n(X) \to H \otimes_{\mathbb{Z}} C_n(X) \to K \otimes_{\mathbb{Z}} C_n(X) \to 0$$

and

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(C_n(X), G) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mathrm{Hom}(C_n(X), H) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mathrm{Hom}(C_n(X), K) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & 0 \\
0 \arrow[r] & C^n(X; G) \arrow[r] & C^n(X; H) \arrow[r] & C^n(X; K) \arrow[r] & 0.
\end{tikzcd}$$

Thus we get

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_n(X) \otimes G \arrow[r] \arrow[d] & C_n(X) \otimes H \arrow[r] \arrow[d] & C_n(X) \otimes K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(X) \otimes G \arrow[r] \arrow[d] & C_{n-1}(X) \otimes H \arrow[r] \arrow[d] & C_{n-1}(X) \otimes K \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

$\Rightarrow$ l.e.s. in homology

$$\to H_n(X; G) \to H_n(X; H) \to H_n(X; K) \xrightarrow{\beta} H_{n-1}(X; G) \to \cdots$$

Similarly, for cohomology we get

$$\to H^n(X; G) \to H^n(X; H) \to H^n(X; K) \xrightarrow[\beta]{} H^{n+1}(X; G) \to \cdots$$

β is called a Bockstein homomorphism.

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Example: $0 \to \underset{\overset{\shortparallel}{G}}{\mathbb{Z}} \xrightarrow{\times 2} \underset{\overset{\shortparallel}{H}}{\mathbb{Z}} \to \underset{\overset{\shortparallel}{K}}{\mathbb{Z}_2} \to 0$

Bockstein Homomorphisms

$$\to H_n(X;\mathbb{Z}) \xrightarrow{\times 2} H_n(X;\mathbb{Z}) \to H_n(X;\mathbb{Z}_2) \to \cdots$$

$$\beta \longmapsto 2\beta = \alpha \longmapsto \alpha = 0$$

and

$$\to H^n(X;\mathbb{Z}) \to H^n(X;\mathbb{Z}) \to H^n(X;\mathbb{Z}_2) \to H^{n+1}(X;\mathbb{Z}) \to \cdots$$

Example 1.

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \simeq S^1 \to 0$$

$$\to H^n(X;\mathbb{Z}) \to H^n(X;\mathbb{R}) \to H^n(X, S^1) \to \cdots$$

$$0 \to H^0(X;\mathbb{Z}) \to H^0(X;\mathbb{R}) \to H^0(X, S^1) \to H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{R}) \to$$
$$H^1(X;S^1) \xrightarrow[\beta]{} H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R}) \to \cdots$$

Assume X is path connected.

$$0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{\text{onto}} S^1 \xrightarrow{\overset{0}{\shortparallel}} H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{R}) \to H^1(X;S^1) \xrightarrow{\beta}$$
$$H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R}) \to H^2(X;S^1) \to \cdots$$

$$0 \to H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{R}) \to H^1(X;S^1) \xrightarrow{\beta} H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R})$$

$$H^1(X;S^1) = \operatorname{Hom}(\underset{*}{\underline{H_1(X)}}, S^1) \oplus \underbrace{\operatorname{Ext}(\overset{\mathbb{Z}}{\overset{\wr}{H_0(X)}}, S^1)}_{\simeq 0}$$

$$\operatorname{Hom}(\mathbb{Z}, S^1) \simeq S^1, \quad \operatorname{Hom}(\mathbb{Z}_{p^k}, S^1) \simeq \mathbb{Z}_{p^k}$$

$$\Rightarrow 0 \to \underset{\underset{\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{b_1 - \text{times}}}{\wr}}{H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})} \to \underline{H^1(X;S^1)} \to \operatorname{Tor} H^2(X;\mathbb{Z}) \to 0$$

Example: $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.

$$0 \to H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{Q}) \to H^1(X;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Q})$$

$$\begin{tikzcd}
0 \arrow[r] & H^1(X;\mathbb{Q})/H^1(X;\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\cong"] & H^1(X;\mathbb{Q}/\mathbb{Z}) \arrow[r, "\beta"] & \operatorname{Tor} H^2(X;\mathbb{Z}) \arrow[r] & 0. \\
& \bigoplus_{b_1} \mathbb{Q}/\mathbb{Z} & & &
\end{tikzcd}$$

Remark: $\mathbb{Q}/\mathbb{Z} = \varinjlim_{n} \mathbb{Z}_n$, where the direct limit is over the directed set $\mathbb{N}$ with order $n > m \Leftrightarrow m \mid n$, and $\varphi_{mn}: \mathbb{Z}_m \to \mathbb{Z}_n$, sending $[1]$ to $[\frac{n}{m}]$. In other words, $\mathbb{Q}/\mathbb{Z}$ is the smallest abelian group that contains all torsion (finite) abelian groups.

HOMOTOPY THEORY

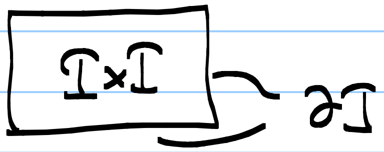
(X, x_0) based topological space

$\pi_1(X,x_0) = \{f: I \to X \mid f \text{ cont. } f(0)=f(1)=x_0\}/\sim$

$\sim$ homotopy relation, s.t. $f_t(0)=f(1)=x_0$, for all $t\in[0,1]$.

Let $n\geq 2$, $I^n = I\times I\times\cdots\times I$, $\partial I^n = \{(s_1,\dots,s_n) \mid s_i=0$ or $s_i=1$, for some $i\}$.

$I^2 = I\times I$, $\partial I^2 = \partial I\times I \cup I\times\partial I$



Definition: $\pi_n(X,x_0) = \{f:(I^n,\partial I^n)\to(X,x_0) \mid f \text{ cont.}\}/\sim$

where $\sim$ is the homotopy relation so that for each $t\in[0,1]$, $f_t(\partial I^n)=x_0$.

Hence, $\pi_n(X,x_0)$ is the set of equivalence classes of continuous maps of the pairs $(I^n,\partial I^n)$ to (X,x_0).

Multiplication (Addition)

If $[f],[g]\in\pi_n(X,x_0)$ then define

$$f+g: I^n \to X,\quad (f+g)(s_1,\dots,s_n)=\begin{cases} f(2s_1,s_2,\dots,s_n), & s_1\leq 1/2 \\ g(2s_1-1,s_2,\dots,s_n), & s_1\geq 1/2\end{cases}$$

must check $[f]+[g]\doteq[f+g]$ is well defined

If $f'\sim f$ and $g'\sim g$ then $f+g\sim f'+g'$.

i) Addition is associative: $([f]+[g])+[h]=[f]+([g]+[h])$.

ii) Inverse: $-[f]=[\bar{f}]$, where

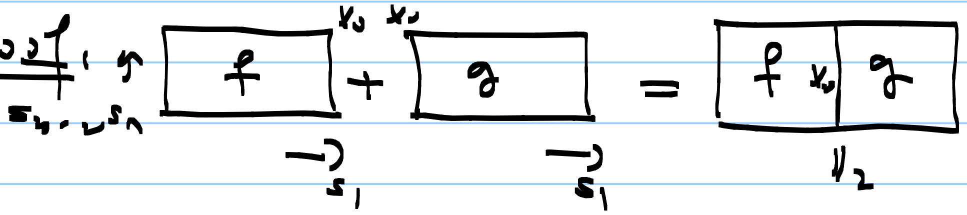
$\bar{f}(s_1,\dots,s_n)=f(1-s_1,\dots,s_n)$.

must check: $[f]+[\bar{f}]=[e]$, where e is the constant map at x_0.

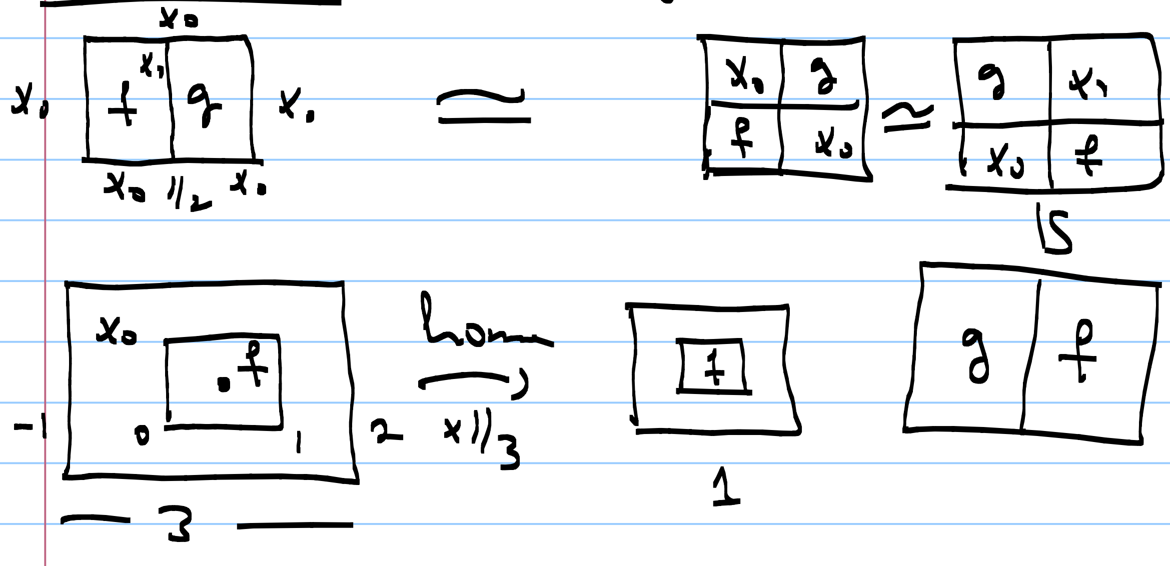
Hence, $\pi_n(X,x_0)$ is a group.

Proposition: If $n\geq 2$ then $\pi_n(X,x_0)$ is abelian.

Proof:



must show: $f+g\sim g+f$.



$\Rightarrow [f]+[g]=[g]+[f]$. ∎

Proposition: If $\varphi : (X, x_0) \to (Y, y_0)$ is a continuous map then the induced map

$$\varphi_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0), \ [f] \longmapsto [\varphi \circ f]$$

is a group homomorphism.

Moreover, if $\varphi \simeq_h \psi$ as maps from (X, x_0) to (Y, y_0) then $\varphi_* = \psi_*$.

In particular, if $\varphi \sim \mathrm{id}_{(X, x_0)}$ $(\varphi : (X, x_0) \to (X, x_0))$ then $\varphi_* = \mathrm{id}_{\pi_n(X, x_0)}$.

Proposition: If $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering projection then $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for $n \geq 2$.

Proof: p_* is surjective: $[f] \in \pi_n(X, x_0)$

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\
(S^n, s_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
f : (I^n, \partial I^n) \arrow[r] \arrow[d] & (X, x_0) \\
(S^n, s_0) \arrow[ur, "f"'] &
\end{tikzcd}$$

$n \geq 2 \Rightarrow \pi_1(S^n, s_0) = (e)$ and thus

$f_*(\pi_1(S^n, s_0)) = (e) \leq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ so that

f lifts to some $\tilde{f} : (S^n, s_0) \to (\tilde{X}, \tilde{x}_0)$.

Clearly, $f = p \circ \tilde{f}$ and hence

$[f] = [p \circ \tilde{f}] = p_*([\tilde{f}])$.

Thus, p_* is onto.

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For injectivity we use homotopy lifting: Assume that $p_*([\tilde{f}]) = 0$, for some $[\tilde{f}] \in \pi_n(\tilde{X}, \tilde{x}_0)$.

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_1) \arrow[d, "p"] \\
(S^n, s_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f = p\circ\tilde{f}"'] & (X, x_0)
\end{tikzcd}$$

$[f] = [p\circ\tilde{f}] = p_*([\tilde{f}]) = 0$ and thus there is a homotopy $F: S^n \times I \to (X, x_0)$

$F(s,0) = f(s)$ and $F(s,1) = x_0$, for all $s \in S^n$.

$F_t(s) = F(s,t)$

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_1) \arrow[d, "p"] \\
S^n \times I \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & (X, x_0)
\end{tikzcd}$$

So, we have $F: S^n \times I \longrightarrow (X, x_0)$ and $f = F_0$ has a lift $\tilde{f}$ to $(\tilde{X}, \tilde{x}_0)$. Now by the homotopy lifting property F lifts to some $\tilde{F}: S^n \times I \to \tilde{X}$ covering F:

$p\circ\tilde{F} = F$. Since $F_1 : S^n \to X$ is the constant map, $\tilde{F}_1 : S^n \to \tilde{X}$ is the constant map to $\tilde{x}_0$. Hence, $[\tilde{f}] = 0$ in $\pi_n(\tilde{X}, \tilde{x}_0)$.

This finishes the proof. ∎

Action of π_1 on π_n:

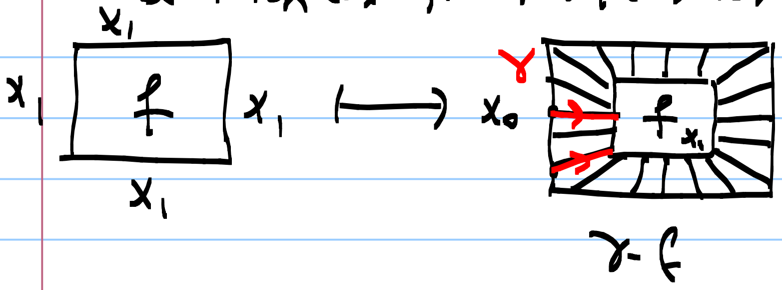
Let $\gamma : [0,1] \to X$ be any path and $[f] \in \pi_n(X, x_1)$,

$\gamma : [0,1] \longrightarrow X$, $\gamma(0) = x_0$, $\gamma(1) = x_1$

$f : I^n \longrightarrow X$, $\gamma(\partial I) = x_1$

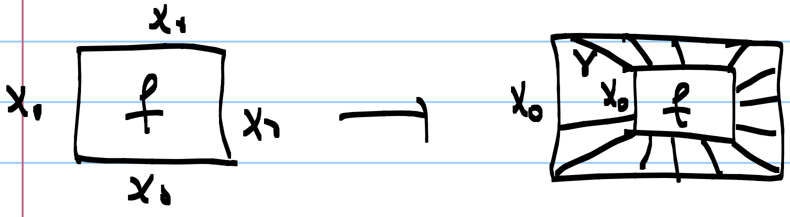
Then we have a homomorphism

$\gamma_* : \pi_n(X, x_1) \longrightarrow \pi_1(X, x_0)$ defined as



This defines an action of $\pi_1(X, x_1)$ on $\pi_n(X, x_1)$.

If this action is trivial, i.e. $[f] = [\gamma \cdot f]$ for all $[f]$ and $[\gamma]$, then we say that (X, x_1) is n-simple.

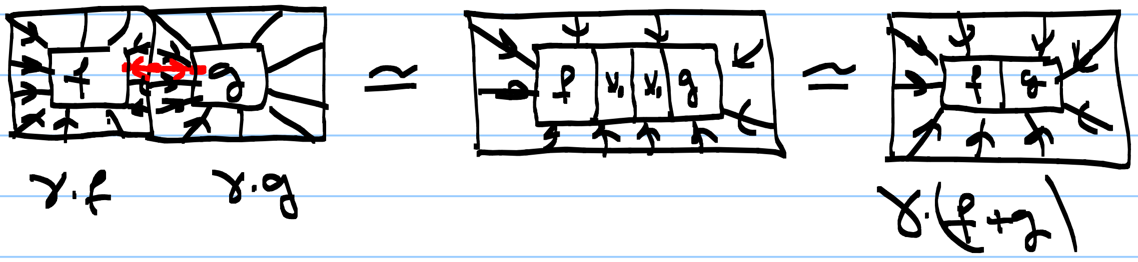


$\gamma \cdot f$

$\gamma : [0,1] \longrightarrow X$

$\gamma(0) = \gamma(1) = x_1$

<u>must show:</u> $\gamma([f] + [g]) = [\gamma \cdot f] + [\gamma \cdot g]$



Since $\pi_1(X)$ acts on $\pi_n(X)$ we may regard the abelian group as an $\underline{\mathbb{Z}[\pi_1(X)]}$-module.

$(3g_1 + 2g_2)[f] = 3[g_1 \cdot f] + 2[g_2 \cdot f]$

Proposition For a product of spaces $\prod_\alpha X_\alpha$ we have $\pi_n(\prod_\alpha X_\alpha) \simeq \prod_\alpha \pi_n(X_\alpha)$

$f : I^n \to \prod_\alpha X_\alpha$, $f = (f_\alpha)$, $f_\alpha : I^n \to X_\alpha$

Proposition If $\gamma : [0,1] \to X$ is a path with $\gamma(0) = x_0$ and $\gamma(1) = x_1$ then

$\gamma_\cdot : \pi_n(X, x_1) \longrightarrow \pi_n(X, x_0)$, $[f] \mapsto [\gamma \cdot f]$ is an isomorphism.

Proof We've seen that $\gamma_\cdot$ is a homomorphism.

This is an isomorphism since it has inverse given by $\bar{\gamma}_\cdot$ ∎

Corollary: $\pi_i(T^n) = 0$ if $i \geq 2$.

Proof $\mathbb{R}^n \to T^n$ is the universal cover of T^n.

Hence, $\pi_i(T^n) = \pi_i(\mathbb{R}^n) = 0$ since $\mathbb{R}^n$ is contractible ($\Rightarrow$ any $f : (I^n, \partial I^n) \to (\mathbb{R}^n, x_0)$ is homotopic to the constant function at x_0).

Similarly, $\pi_i(\Sigma_g) = 0$ if $i \geq 2$ and $g \geq 1$.

$g = 1$, $\Sigma_g = T^2$ ✓

$g \geq 2$ then the universal cover of Σ_g is the Poincaré disk which is again contractible.

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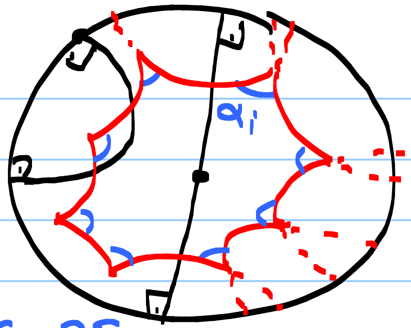
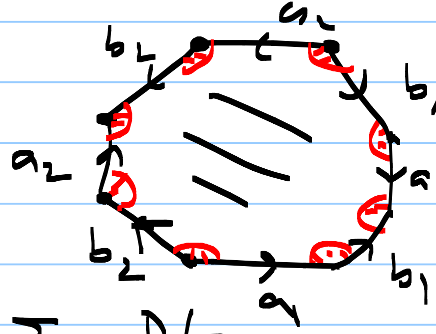
$\Gamma \leq \mathrm{Aut}(D)$

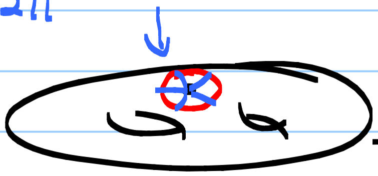


$\Sigma_g,\ g \geq 2$



$\sum \alpha_i = 2\pi$

$\Gamma = \pi_1(\Sigma_g)$



$= \Sigma_g = D/\Gamma$

(universal cover)

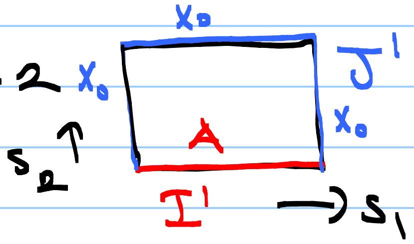
Relative Homotopy Groups

$x_0 \in A \subseteq X$, where A is a subspace. The $\pi_n(X, A, x_0)$ is defined to be the set of homotopy classes of maps

$$(I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, x_0), \text{ where}$$

$J^{n-1} = \partial I^n \setminus I^{n-1}$, where $I^{n-1} = \{(s_1, \ldots, s_n) \in I^n \mid s_n = 0\}$.

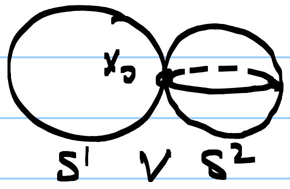
Ex $n = 2$



The group operation on $\pi_n(X, x_0)$ induces a group on $\pi_n(X, A, x_0)$.

An example of the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

$X = S^1 \vee S^2$,

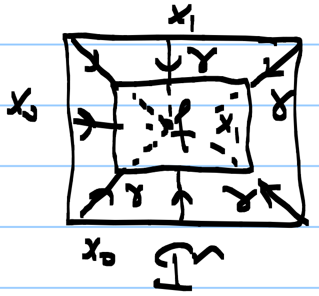


$\pi_1(X, x_0) = \pi_1(S^1, s_0) \simeq \mathbb{Z}$ since $\pi_1(S^2) = 0$.

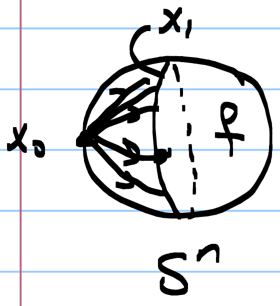
$\gamma : [0,1] \longrightarrow X$, $\gamma(0) = x_0$, $\gamma(1) = x_1$

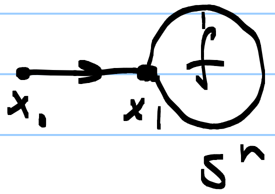
$[f] \in \pi_n(X, x_1)$. $f : (I^n, \partial I^n) \to (X, x_0)$

$\gamma \cdot [f] = [\gamma \cdot f]$

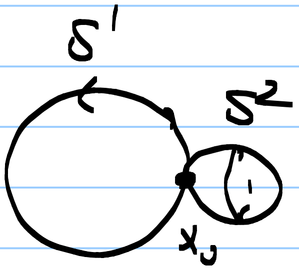


$\gamma_* : \pi_n(X, x_1) \to \pi_n(X, x_0)$



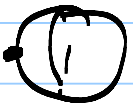


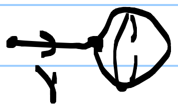
$\boxed{\gamma} = 5 \in \pi_1(S^1 \vee S^2)$



$f : \underline{\underline{S^2}} \longrightarrow \underline{\underline{S^2}}$ id.

$\gamma \cdot [f]$

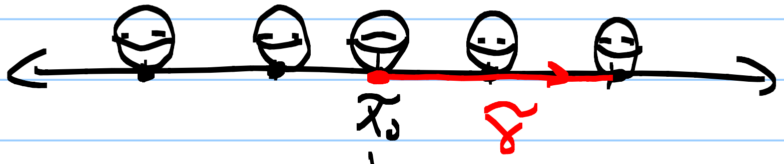




We'll see that $\pi_n(S^n) \cong \mathbb{Z}$

$\pi_2(S^1 \vee S^2) \simeq ?$

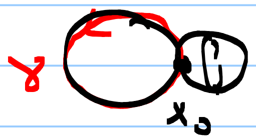
$\tilde{X}$:



$\underset{h.e.}{\simeq} \bigvee_{\mathbb{Z}} S^2$

$\downarrow$

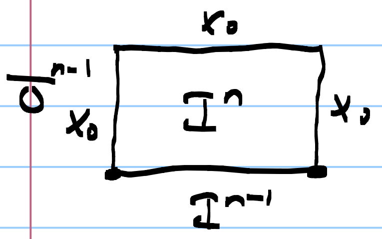
$X = \underline{\underline{S^1}} \vee \underline{\underline{S^2}}$



$[\gamma] = 2 \in \mathbb{Z} \cong \pi_1(X)$

$\pi_2(X) \cong \pi_2(\tilde{X}) = \bigoplus_{\mathbb{Z}} \mathbb{Z}$

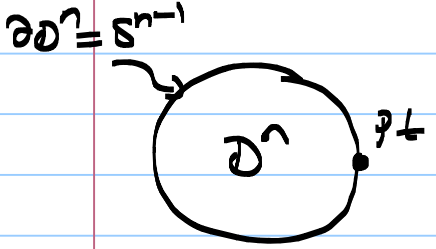
Back to the $\pi_n(X, A, x_0)$



$f : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, x_0)$

$$\begin{tikzcd}
(I^n, \partial I^n, J^{n-1}) \arrow[r, "f"] \arrow[d, "q"] & (X, A, x_0) \\
\left(I^n / J^{n-1}, \partial I^n / J^{n-1}, pt\right) \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

$I^n/J^{n-1} \cong D^n$, $\quad \partial I^n/J^{n-1} \cong I^{n-1}/\partial I^{n-1} \simeq S^{n-1} = \partial D^n$



$\xrightarrow{f} (X, A, x_0)$

$\| $

$(D^n, \partial D^n = S^{n-1}; pt)$

This gives us a sequence of homotopy groups:

$$\cdots \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots$$

$i : (A, x_0) \to (X, x_0)$, $i_*([f]) = [i \circ f]$.

$j : (X, x_0, x_0) \longrightarrow (X, A, x_0)$

$\partial : \pi_n(X, A, x_0) \longrightarrow \pi_{n-1}(A, x_0)$

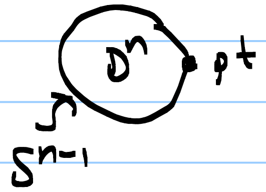
$[f] \longmapsto [f|_{S^{n-1} = \partial D^n}]$

Theorem: The above sequence of groups is exact.

$$\to \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \to \cdots$$

$$\underline{\underline{[\tilde f]}} \longmapsto [f] \longmapsto [f|_{S^{n-1}}] = 0$$

$\underset{\searrow x_0}{J^{n-1}}, \underline{\underline{I^{n-1}}}$



$f(\partial D^n) = x_0$

$$\Rightarrow \tilde f : \underset{\substack{\text{is} \\ S^n}}{D^n/\partial D^n} \longrightarrow X$$

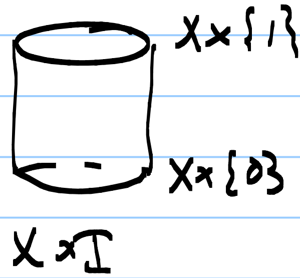
Indeed more is true:

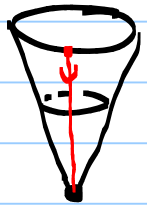
Theorem: Let $x_0 \in B \subseteq A \subseteq X$. Then we have the following exact sequence:

$$\cdots \to \pi_n(A,B,x_0) \xrightarrow{i_*} \pi_n(X,B,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,B,x_0) \to$$

Example: X path connected space. Let CX be the cone on X:

$$CX = X\times I \big/ X\times\{0\}$$





Clearly CX is contractable.
Consider the homotopy exact sequence of the pair (CX, X):

$$\xrightarrow{\partial} \pi_n(X,x_0) \xrightarrow{i_*} \underset{\substack{\text{is} \\ (0)}}{\pi_n(CX,x_0)} \xrightarrow{j_*} \pi_n(CX,X,x_0) \xrightarrow{\partial} \pi_{n-1}(X,x_0) \to$$

since CX is contractible

Hence, $\partial : \pi_n(CX, X, x_0) \to \pi_{n-1}(X,x_0)$ is an

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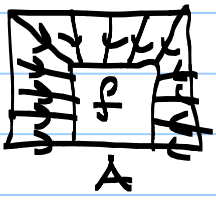
an isomorphism for all $n \geq 1$.

In particular, $\pi_2(CX, X, x_0) \simeq \pi_1(X, x_0)$ and thus $\pi_2(CX, X, x_0)$ may <u>not</u> be abelian.

Action of $\pi_1(X, x_0)$ on $\pi_n(X, A, x_0)$:



$\gamma \cdot [f]$

Definition: A space X with base point x_0 is called n-connected if $\pi_i(X, x_0) = 0$ for $i \leq n$.

Remark: It is easy to check that the following conditions are equivalent:

1) Every map $S^i \to X$ is homotopic to a constant map

2) Every map $S^i \to X$ extends to a map $D^{i+1} \to X$, $S^i = \partial D^{i+1}$.

3) $\pi_i(X, x_0) = 0$ for all $x_0 \in X$.

Whitehead's Theorem:

Theorem: If a map $f: X \to Y$ between connected CW-complexes induces isomorphisms

$$f_*: \pi_n(X) \to \pi_n(Y), \text{ for all } n, \text{ then}$$

f is a homotopy equivalence. In case, f is the inclusion of a subcomplex $f: X \hookrightarrow Y$, the conclusion is stronger: X is a deformation retract of Y.

Remark: $\pi_k(S^1) = \begin{cases} \mathbb{Z} & \text{if } k=1 \\ 0 & \text{otherwise} \end{cases}$

$\pi_k(S^2)$:	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
	1	2							

Theorem: (Cellular Approximation Theorem)

Every map $f: X \to Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subseteq X$, the homotopy may be taken to be stationary on A.

Example: $S^n = e^0 \cup e^n$, $S^m = f^0 \cup f^m$ $m > n$.

If $f: (S^n, e^0) \to (S^m, f^0)$ is continuous then

then f is homotopic to a cellular map

$\tilde{f}: (S^n, e^0) \to (S^m, f^0)$ is the constant map onto f^0 because S^n maps into the n-skeleton of S^m which is just f^0. Thus we have

Corollary $\pi_n(S^m) = 0$ if $m > n$.

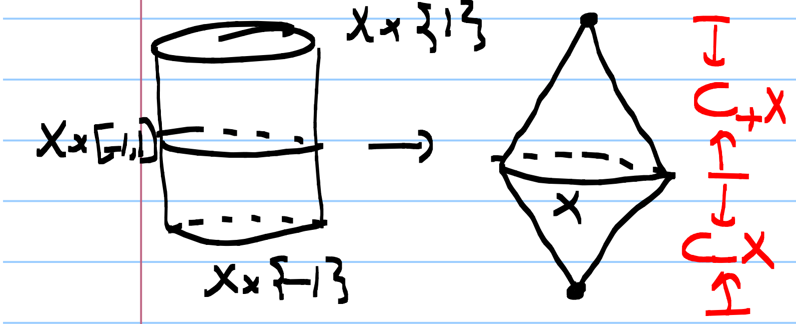
Excision for Homotopy Groups

Theorem: Let X be a CW-complex decomposed as the union of subcomplexes A and B with nonempty connected intersection $C = A \cap B$. If (A,C) is m-connected and (B,C) is n-connected, $m,n \geq 0$, then the map $\pi_i(A,C) \to \pi_i(X,B)$ induced by inclusion is an isomorphism for $i < m+n$ and a surjection for $i = m+n$.

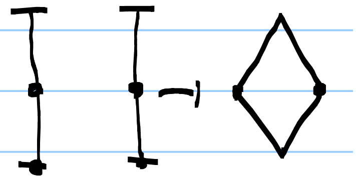
The following corollary is called the Freudenthal Suspension Theorem:

The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$. More generally, this holds for the suspension $\pi_i(X) \to \pi_i(SX)$, whenever X is an $(n-1)$-connected CW-complex.

Definition: $SX = X \times [-1,1] \Big/ \begin{matrix} (x,1) \sim (y,1) \\ (x,0) \sim (y,0) \\ \forall\, x,y. \end{matrix}$



$SS^0 \overset{}{=} S^1$



Ex $SS^n \cong S^{n+1}$

Proof of the corollary $SX = \underset{\substack{\shortparallel \\ A}}{C_+X} \cup \underset{\substack{\shortparallel \\ B}}{C_-X}$

$C_+X \cap C_-X = X \times \{0\} = X = C$

$\pi_i(A,C) \xrightarrow{\cong} \pi_i(X,B)$

$$\begin{tikzcd}
\pi_{i+1}(X) \arrow[d] & \pi_{i+1}(C_-X)\simeq 0 \arrow[d] \\
0\simeq\pi_{i+1}(C_+X) \arrow[d] & \pi_{i+1}(SX) \arrow[d, "\simeq"] \\
\pi_{i+1}(C_+X,X) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \pi_{i+1}(SX, C_-X) \arrow[d] \\
\pi_i(X) \arrow[d] & \pi_i(C_-X)\simeq 0 \arrow[d] \\
0\simeq\pi_i(C_+X) & {}
\end{tikzcd}$$

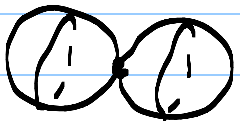
Since C_+X is contractible.

In particular we have
$\pi_n(S^n) = \pi_n(SS^{n-1}) = \pi_{n-1}(S^{n-1})$ and thus

$\pi_n(S^n) \simeq \pi_1(S^1) \simeq \mathbb{Z}$.

Corollary $\pi_n(S^n) \simeq \mathbb{Z}$.

Example: $\underbrace{S^n}_{A} \vee \underbrace{S^n}_{B}$ $C = A\cap B = \{x_0\}$

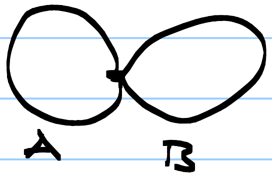


$$\pi_i(A,C) \xrightarrow{\simeq} \pi_i(A\vee B, B) \qquad i\leq n$$

$$\| $$

$$\pi_i(S^n,x_0) = \begin{cases} \mathbb{Z} & i=n \\ 0 & i<n. \end{cases}$$

$$\begin{tikzcd}[column sep=small]
\pi_{n+1}(A\vee B,B) \arrow[r, "0"] & \pi_n(B) \arrow[r, "\text{inj.}"] \arrow[d, phantom, sloped, "\cong"] & \pi_n(A\vee B) \arrow[r] & \pi_n(A\vee B, B) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \pi_{n-1}(B) \arrow[r] \arrow[d, equal] & \cdots \\
 & \mathbb{Z} & & \mathbb{Z} & 0 &
\end{tikzcd}$$



$\Rightarrow \pi_n(S^n\vee S^n) \simeq \mathbb{Z}\oplus\mathbb{Z}$.

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More generally, $\pi_n(\bigvee_\alpha S^n_\alpha) \simeq \bigoplus_\alpha \pi_n(S^n_\alpha)$

Eilenberg-MacLane Spaces:

G group, $n \in \mathbb{Z}^+$. We require G is abelian if $n \geq 2$.

Definition: A space X is an Eilenberg-MacLane space for (G,n) if

$$\pi_k(X) = \begin{cases} G & \text{if } k=n \\ (0) & \text{otherwise.} \end{cases}$$

Proposition: Such X always exists and unique upto homotopy equivalence.

Notation: The unique homotopy type of the space in the above proposition will be denoted as $K(G,n)$.

Example: 1) $K(\mathbb{Z},1) = S^1$ because $\pi_1(S^1) \cong \mathbb{Z}$

and $\pi_k(S^1) = \pi_k(\mathbb{R}) = 0$ if $k \geq 2$, because

$\mathbb{R} \to S^1$, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is the universal cover and thus $\pi_k(S^1) \simeq \pi_k(\mathbb{R}) = (0)$ if $k \geq 2$.

2) $K(\mathbb{Z},2) = ?$ $X_2 = S^2$; $\pi_1(X_2) = (0)$ and $\pi_2(X_2) \simeq \mathbb{Z}$.

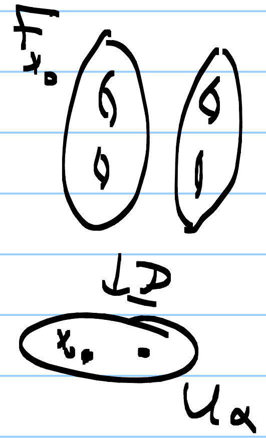
To obtain $K(\mathbb{Z},2)$ we'll attach cells to kill higher homotopy groups.

Fact (Homotopy Exact Sequence of a Fibration)

$F \to E \xrightarrow{p} B$ is a locally trivial fibration with fiber F if $p : E \to B$ is a continuous surjection and B is covered with open subsets $\{U_\alpha\}_{\alpha\in\Lambda}$ so that there are homeomorphism

$\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ making the diagram below commutative:

$$\begin{tikzcd}
E \supseteq p^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha"] \arrow[d, "p"'] & U_\alpha \times F \arrow[dl, "Pr_1(x,y)=x"] \\
U_\alpha &
\end{tikzcd}$$



Remark: If F has the discrete topology then $E \xrightarrow{p} B$ is a covering space.

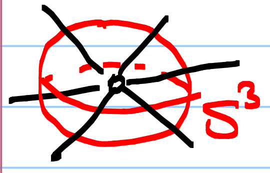
Homotopy Exact Sequence of a fibration:

$$F \xrightarrow{i} E \xrightarrow{p} B$$

$$\to \pi_k(F) \xrightarrow{i_*} \pi_k(E) \xrightarrow{p_*} \pi_k(B) \to \pi_{k-1}(F) \to \cdots$$

Example: 1) Hopf fibration:

$$S^2 = \mathbb{CP}^1 = \mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{C}^* \simeq S^3/S^1$$



$(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$ $\qquad \|\lambda\| = 1$

$\lambda \neq 0$

$$S^3/S^1 \simeq S^2 \Rightarrow S^1 \to S^3 \xrightarrow{P} S^2$$

$$(z_1, z_2) \longmapsto [z_1 : z_2]$$

$$\underset{\overset{\|}{0}}{\pi_k(S^1)} \to \pi_k(S^3) \to \pi_k(S^2) \to \underset{\overset{\|}{0}}{\pi_{k-1}(S^1)} \to \cdots$$

$$k \geq 3 \Rightarrow \pi_k(S^2) \simeq \pi_k(S^3)$$

$$\pi_2(S^1) \to \underset{\overset{\|}{0}}{\pi_2(S^3)} \to \underset{\overset{\wr\|}{\mathbb{Z}}}{\pi_2(S^2)} \to \underset{\overset{\wr\|}{\mathbb{Z}}}{\pi_1(S^1)} \to \underset{\overset{\|}{0}}{\pi_1(S^3)}$$

$$\pi_3(S^2) \simeq \pi_3(S^3) \simeq \mathbb{Z} = \langle P \rangle$$

$$P : S^3 \to S^2$$

To kill this generator we need to attach a 4-disk D^4 to S^2 so that the map $P : S^3 \to S^2$ extends to D^4.

$$\begin{tikzcd}
\partial D^4 = S^3 \arrow[r, "P"'] \arrow[d, hook] & S^2 \cup_P D^4 \\
D^4 \arrow[ur, "i"'] &
\end{tikzcd}$$

$i : D^4 \to S^2 \cup_P D^4$ Inclusion.

Note that the attaching space $S^2 \cup_P D^4$ is nothing but $\mathbb{CP}^2$.

Now $\pi_3(\mathbb{CP}^2) = (0)$.

$$S^1 \to \underset{\underset{\mathbb{C}^3 \setminus \{0\}}{\cap}}{S^5} \to \mathbb{CP}^2$$

$$\pi_k(\mathbb{CP}^2) = \begin{cases} 0 & k=1 \\ \mathbb{Z} & k=2 \\ 0 & k=3,4 \end{cases}$$

$$\underset{\underset{(0)}{\shortparallel}}{\pi_4(S^5)} \to \underset{\underset{0}{\shortparallel}}{\pi_4(\mathbb{CP}^2)} \to \underset{\underset{(0)}{\shortparallel}}{\pi_3(S^1)}$$

$$\underset{\underset{0}{\shortparallel}}{\pi_5(S^1)} \to \pi_5(S^5) \xrightarrow{\cong} \pi_5(\mathbb{CP}^2) \to \underset{\underset{0}{\shortparallel}}{\pi_4(S^1)} \to \cdots$$

$\pi_k(\mathbb{CP}^2) \cong \pi_k(S^5)$ if $k \geq 5$.

$\Rightarrow \pi_5(\mathbb{CP}^2) \simeq \pi_5(S^5) \simeq \mathbb{Z}$.

To kill the group $\pi_5(\mathbb{CP}^2)$ we attach a six disk D^6 to $\mathbb{CP}^2$ to extend the projection map

$$\begin{tikzcd}
\partial D^6 = S^5 \arrow[r, "P"] \arrow[d, hook] & \mathbb{CP}^2 \cup_P D^6 \\
D^6 \arrow[ur, "\tau"'] &
\end{tikzcd}$$

$(z_1, z_2, z_3) \mapsto [z_1 : z_2 : z_3]$

This gives us $\mathbb{CP}^3$. This process gives us inductively $K(\mathbb{Z}, 2)$ which is homotopy equivalent $\mathbb{CP}^\infty$.

$$K(\mathbb{Z},2) = \underset{\underset{\lim_n \mathbb{CP}^n}{\shortparallel}}{\mathbb{CP}^\infty} = S^\infty / S^1, \quad S^\infty = \lim_n S^{2n-1}$$

Note that S^∞ is contractible and thus the homotopy exact sequence yield

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$$\underset{\overset{\shortparallel}{0}}{\pi_k(S^1)} \to \overset{\overset{0}{\shortparallel}}{\pi_k(S^\infty)} \to \pi_k(\mathbb{CP}^\infty) \to \underset{\overset{\shortparallel}{0}}{\pi_{k-1}(S^1)} \to \cdots$$

$k \geq 3 \Rightarrow \pi_k(\mathbb{CP}^\infty) = 0.$

$$k=2 \Rightarrow \pi_2(\underset{\overset{\shortparallel}{(0)}}{S^\infty}) \to \pi_2(\mathbb{CP}^\infty) \xrightarrow[\cong]{} \pi_1(S^1) \to \pi_1(\underset{\overset{\shortparallel}{(0)}}{S^\infty})$$

$$\pi_2(\mathbb{CP}^\infty) \simeq \pi_1(S^1) \simeq \mathbb{Z}$$

$$\mathbb{CP}^\infty = S^2 \cup D^4 \cup D^6 \cup \cdots$$
$$\Rightarrow \pi_1(\mathbb{CP}^\infty) = (0).$$

Hence, $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.

Exercise: $\mathbb{H}$ Quaternions, $\mathbb{HP}^\infty$

$$\begin{tikzcd} S^\infty = \lim\limits_{n} S^{4n-1} \arrow[d] \\ \mathbb{HP}^\infty = \lim\limits_{n} S^{4n-1}/S^3 \end{tikzcd}$$

S^{4n-1} = unit vectors in $\mathbb{H}^n$.

S^3 = unit quaternions $= U\mathbb{H}$

$$S^3 \to S^\infty \to \mathbb{HP}^\infty$$

$$\Rightarrow \pi_k(\underset{\overset{\shortparallel}{0}}{S^\infty}) \to \pi_k(\mathbb{HP}^\infty) \to \pi_{k-1}(S^3) \to \underset{\overset{\shortparallel}{0}}{\pi_{k-1}(S^\infty)}$$

$$\Rightarrow \pi_k(\mathbb{HP}^\infty) \simeq \pi_{k-1}(S^3) = \pi_{k-1}(S^2)$$

Ex

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r] & S^\infty = \lim S^n \arrow[d] \\ & \mathbb{RP}^\infty = \lim_n \mathbb{RP}^n = \lim_n S^n/\mathbb{Z}_2 \end{tikzcd}$$

$\pi_k(\mathbb{RP}^\infty) \simeq \pi_k(S^\infty)$ if $k \geq 2$
$= (0)$

$\pi_1(\mathbb{RP}^\infty) \simeq \mathbb{Z}/_2$ since S^∞ is simply connected and $\mathbb{RP}^\infty = S^\infty/\mathbb{Z}_2$.

So, $\mathbb{RP}^\infty \simeq K(\mathbb{Z}_2, 1)$.

Exercise $L(n) = \lim S^{2n-1}/\mathbb{Z}_n$

$S^{2n-1} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \dots + |z_n|^2 = 1\}$

$\mathbb{Z}_n = \langle \sigma \rangle$, $\sigma(z_1, \dots, z_n) = (\xi z_1, \dots, \xi z_n)$, $\xi = e^{2\pi i/n}$.

$$S^\infty \longrightarrow S^\infty/\mathbb{Z}_n$$

$\Rightarrow L(n) \simeq K(\mathbb{Z}_n, 1)$

Remark How to construct $K(\mathbb{Z} \times \mathbb{Z}, 5)$.

$X_1 = S^5 \times S^5$, $\pi_5(X_1) \simeq \mathbb{Z} \oplus \mathbb{Z}$
$\pi_k(X_1) = 0$ if $k < 5$.

If $f : S^6 \to X_1$ is non trivial map:
$[f] \neq 0$ in $\pi_6(X_1)$.

$X_2 = X_1 \cup_f D^7$ $\quad [S] = (2, 0)$

$$\begin{tikzcd}
\partial D^7 = S^6 \arrow[r, "f"] \arrow[d, hook] & X_1 \cup_f D^7 \\
D^7 \arrow[ur, "i"] &
\end{tikzcd}$$

This way we kill $\pi_6(X_1)$.

Theorem: If X is a topological space then the set of homotopy classes of maps from X to $K(G,n)$ form an abelian group isomorphic to $H^n(X;G)$.

$$H^n(X;G) \doteq [X, K(G,n)]$$

Example: $H^1(X;\mathbb{Z}) = [X, K(\mathbb{Z},1)]$
$= [X, S^1]$

The group operation in $[X, S^1]$ is induced by the group operation on S^1:

$$f: X \to S^1, \quad g: X \to S^1, \quad f\cdot g : X \to S^1$$
$$x \mapsto f(x)\cdot g(x)$$

Example $H^1(X;\mathbb{Z}_2) = [X, \underset{\overset{\uparrow}{\text{H-space}}}{\mathbb{RP}^\infty}]$

$\mathbb{RP}^\infty = \mathbb{R}[t]\setminus\{0\}/\mathbb{R}^*$

$f: X \to \mathbb{RP}^\infty, \quad g: X \to \mathbb{RP}^\infty$

$f\cdot g: X \to \mathbb{RP}^\infty, \quad x \mapsto f(x)\cdot g(x)$

Example: $H^2(X;\mathbb{Z}) = [X, K(\mathbb{Z},2)]$

$$= [X, \mathbb{CP}^\infty]$$

$$= [X, \mathbb{C}[z]^*/\sim]$$

Remark: $K(\mathbb{Z},1)$, $K(\mathbb{Z},2)$, $K(\mathbb{Z}/n,1)$ have manifold structure.

Fact: M^n smooth manifold. Poincaré dual of any cohomology class $\alpha \in H^2(M;\mathbb{Z})$ is represented by the fundamental class of a smooth submanifold $L \subseteq M^n$ of dimension $n-2$.

Proof $H^2(M;\mathbb{Z}) = [M, \mathbb{CP}^\infty]$

$\alpha \in H^2(M,\mathbb{Z})$, $f: M^n \longrightarrow \mathbb{CP}^\infty$

By cellular approximation we may assume that f is given as $f: M \to \mathbb{CP}^N$ for some N.

Let $a \in H^2(\mathbb{CP}^N) \simeq \mathbb{Z}$ be a generator.
$PD(a) = \mathbb{CP}^{N-1} \subseteq \mathbb{CP}^N$.

It follows that $\alpha = f^*(a)$, $f^*: H^2(\mathbb{CP}^N) \to H^2(M)$.
$a \longmapsto \alpha$

Choosing f transversal to $\mathbb{CP}^{N-1}$ we see that

$PD(\alpha) = f^{-1}(\mathbb{CP}^{N-1}) \subseteq M$ submanifold of codimension 2.

Remark This does not hold for $H^3(M;\mathbb{Z})$ because the above proof fails since $K(\mathbb{Z},3)$ is not a smooth manifold.

Definition Let G be any topological group. Then there is a contractible space called EG on which G acts freely with quotient

$$BG = EG/G.$$

So we have a fibration

$$\begin{tikzcd} G \arrow[r] & EG \arrow[d, "p"'] \\ & BG := EG/G \end{tikzcd}$$

called the universal G-fibration.

If $f: X \to BG$ is any continuous map. Then

$$\begin{tikzcd} G \arrow[r] & f^*(EG) \arrow[r] \arrow[d] & EG \arrow[d, "p"] \\ & X \arrow[r, "f"'] & BG \end{tikzcd}$$

$$\begin{tikzcd} f^*(EG) \arrow[d] := \{ & (x,e) \arrow[d, mapsto] \in X \times EG \mid f(x) = p(e)\}. \\ X & x \end{tikzcd}$$

Fact: Isomorphism classes of principal G-bundles over X is in 1-1 correspondence with the homotopy classes of maps $[X, BG]$.

BG = Classifying space for G-bundles.

Ex $B\mathbb{Z}_2 = \mathbb{RP}^\infty$, $B\mathbb{Z} = \mathbb{R}/\mathbb{Z} \simeq S^1$

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$BS^1 = \mathbb{C}P^\infty$

Theorem (Hurewicz Theorem)

If X is a $(n-1)$-connected space, $n \geq 2$, then $\tilde{H}_i(X) = 0$ for $i \leq n-1$ and $H_n(X) \cong \pi_n(X)$. If a pair (X, A) is $n-1$-connected, $n \geq 2$, with A simply connected and nonempty, then $H_i(X, A) = 0$ for $i \leq n-1$ and $H_n(X, A) \cong \pi_n(X, A)$.

Here we use the natural homomorphism

$\varphi : \pi_n(X) \longrightarrow H_n(X)$ given by

$\varphi([f]) = f_*([S^n])$, where $f: S^n \to X$ and

$[S^n] \in H_n(S^n)$ is a fixed generator.

Remark: Recall that we've already seen this homomorphism $\varphi : \pi_1(X) \to H_n(X)$, which is onto with kernel the commutator subgroup $[\pi_1(X), \pi_1(X)]$.

A crash Course on Vector Bundles and Characteristic Classes:

$k = \mathbb{R}, \mathbb{C}, \mathbb{H}$

A k-vector bundle over a space X is a map $p : E \to X$ so that

1) p is onto
2) For each $x \in X$ the fibre $F_x := p^{-1}(x)$ has a $\mathbb{F}$-vector space structure ($\mathbb{F} = \mathbb{H}$ consider it as an $\mathbb{R}$ or $\mathbb{C}$-vector space).

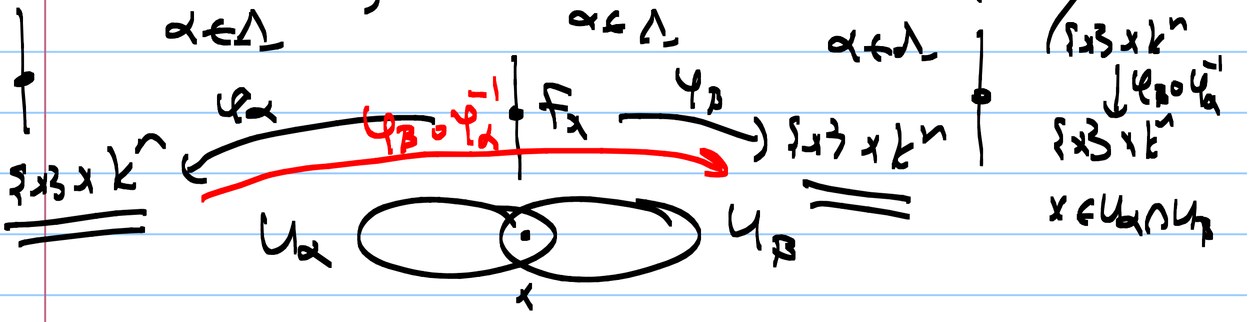
3) X is covered by open subsets $\{U_\alpha\}$ so that there are homeomorphisms $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times k^n$ so that

$$\begin{tikzcd}
p^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "p"'] & & U_\alpha \times k^n \arrow[dl, "pr_1"] \\
& U_\alpha &
\end{tikzcd}$$

is commutative and the restriction of φ_α to each fiber F_x is a vector space isomorphism.

$$\begin{tikzcd}
F_x \arrow[rr, "\varphi_\alpha"] \arrow[dr, "p"'] & & \{x\} \times k^n \arrow[dl, "pr_1"] \\
& x &
\end{tikzcd}$$

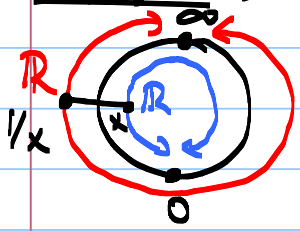
$$X = \bigcup_{\alpha \in \Lambda} U_\alpha \,, \quad E = \bigcup_{\alpha \in \Lambda} p^{-1}(U_\alpha) \cong \dot{\bigcup_{\alpha \in \Lambda}} U_\alpha \times k^n \Big/$$



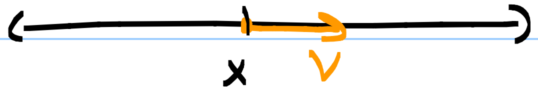
$$\psi_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1} : U_\alpha \cap U_\beta \longrightarrow GL(n,k)$$
$$x \longmapsto \varphi_\beta \circ \varphi_\alpha^{-1}(x) : k^n \to k^n$$

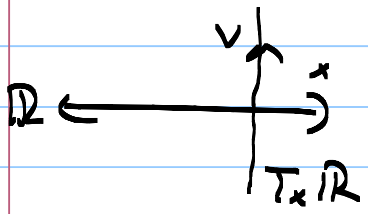
$\psi_{\beta\alpha}$ transition function of the bundle.

Example 1) $S^1 = \mathbb{R} \dot{\cup} \mathbb{R} / x \sim 1/x \; , \; x \neq 0$



$T_*\mathbb{R} = \{(x,v) \mid x \in \mathbb{R}, v \in T_x\mathbb{R}\}$





$S^1 = \mathbb{R} \dot{\cup} \mathbb{R} / x \sim 1/x \; , \; x \neq 0$

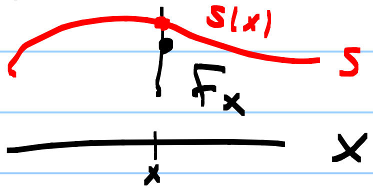
$T_x S^1 = T_x\mathbb{R} \dot{\cup} T_*\mathbb{R} / (x,v) \sim (\varphi(x), \varphi'(x)(v))$

$\varphi(x) : \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\} \; , \; \varphi'(x) : T_x\mathbb{R} \longrightarrow T_{1/x}\mathbb{R}$
$x \longmapsto 1/x \qquad v \longmapsto -1/x^2 \cdot v$

$T_* S^1 = \mathbb{R}\times\mathbb{R} \dot{\cup} \mathbb{R}\times\mathbb{R} / (x,v) \sim (1/x, -\frac{1}{x^2} \cdot v)$

Section of a vector bundle: $E \xrightarrow{p} X$

A section of a fiber bundle $E \xrightarrow{p} X$ is a map $s : X \longrightarrow E$ so that $p \circ s = \mathrm{id}_X$



$T_* S^1 = T_* \mathbb{RP}^1$ has two sections

$s_1(x) = (x, \frac{1+x^2}{2})$, $s_2(x) = (x, -\frac{1+x^2}{2})$

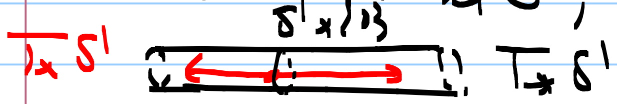
$S^1 = \mathbb{R} \dot\cup \mathbb{R} / x \sim 1/x$, $T_* S^1 = \mathbb{R}\times\mathbb{R} \dot\cup \mathbb{R}\times\mathbb{R} / (x,v) \sim (1/x, -1/x^2 v)$

$s_2(1/x) = -1/x^2 \, s_1(x)$ —

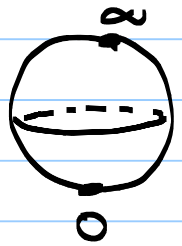
Note that $s_1(x) \neq 0$ for any $x \in S^1$.

Hence, $T_* S^1$ is isomorphic to the trivial bundle $S^1 \times \mathbb{R}$.

$S^1 \times \mathbb{R} \longrightarrow T_* S^1$, $(x,v) \longmapsto (x, s_1(x) v)$



2) $S^2 = \mathbb{CP}^1 = \mathbb{C} \dot\cup \mathbb{C} / z \sim 1/z$, $z \neq 0$ —



$T_* S^2 = T_* \mathbb{C} \dot\cup T_* \mathbb{C} / (z,v) \sim (1/z, -1/z^2 v)$

$s_i : \mathbb{C} \longrightarrow T_* S^2$, $s_1(z) = \frac{1+z^2}{2}$ and

$s_2(x) = -\frac{1+z^2}{2}$.

Note that $s_i(z)$ has two roots $z = \mp i$.
Moreover, it turns out that this bundle is not trivial.

Generalizing the above constructions

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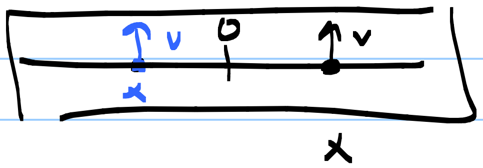
1) $T_* S^1 = \mathbb{R}\times\mathbb{R} \,\dot\cup\, \mathbb{R}\times\mathbb{R} \Big/ (x,v)\sim(x, -\tfrac{1}{x^2}v)$
$x \neq 0$ $=$

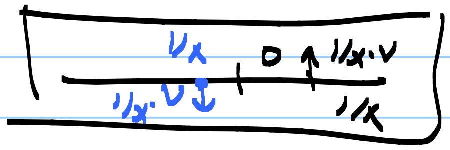
$E(k) = \mathbb{R}\times\mathbb{R} \,\dot\cup\, \mathbb{R}\times\mathbb{R} \Big/ (x,v)\sim(1/x, \pm \tfrac{1}{x^k})$ $k \in \mathbb{Z}$

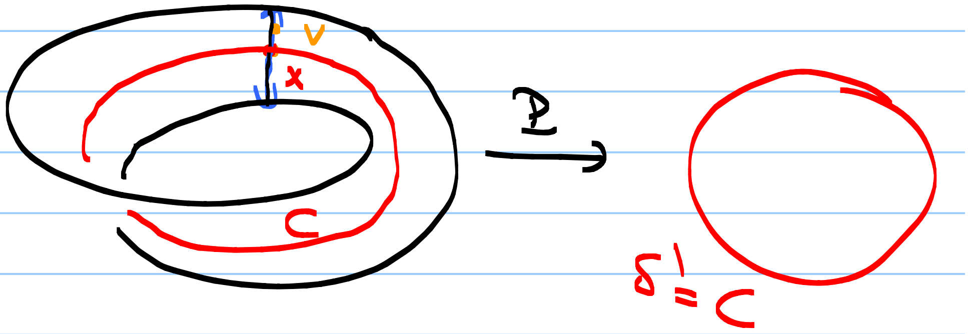
Fact $E(k) \simeq \begin{cases} E(0) & \text{if } k \text{ is even} \\ E(1) & \text{if } k \text{ is odd} \end{cases}$

$E(2) = T_* S^1 = E(0)$

$E(1) = \text{Möbius Band} = \mathbb{R}\times\mathbb{R} \,\dot\cup\, \mathbb{R}\times\mathbb{R} \Big/ (x,v)\sim(1/x, 1/x\, v)$







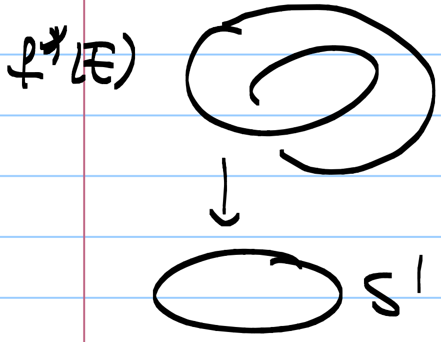
Let $E \longrightarrow X$ be a $\mathbb{R}$-vector bundle of rank 1:

$$\begin{tikzcd} \mathbb{R} \arrow[r] & E \arrow[d, "p"] \\ & X \end{tikzcd}$$

There is a cohomology class $w_1(E) \in H^1(X;\mathbb{Z}_2)$ given by
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$
$\qquad\qquad\qquad\qquad\qquad \mathrm{Hom}(H_1(X);\mathbb{Z}_2)$

If $f: S^1 \longrightarrow X$ is a map representing the homology class $f_*([S^1]) = \alpha \in H_1(X)$, then

$$w_1(\alpha) = \begin{cases} 0 & \text{if } f^*E \text{ is trivial} \\ 1 & \text{if } f^*E \text{ is nontrivial.} \end{cases}$$



$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\ S^1 \arrow[r, "f"'] & X \end{tikzcd}$$

$w_1(E)$: 1st Stiefel-Whitney class of E.

2) $T_x S^2 = \mathbb{C} \times \mathbb{C} \cup \mathbb{C} \times \mathbb{C} / (z,v) \sim (1/z, -\frac{1}{z^2} v)$

$k \in \mathbb{Z}$ $\qquad\qquad\qquad z \neq 0$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & \mathcal{O}(k) = \mathbb{C} \times \mathbb{C} \cup \mathbb{C} \times \mathbb{C} / (z,v) \sim (1/z, 1/z^k v) \arrow[d] \\ & S^2 = \mathbb{C}\mathbb{P}^1 \end{tikzcd}$$

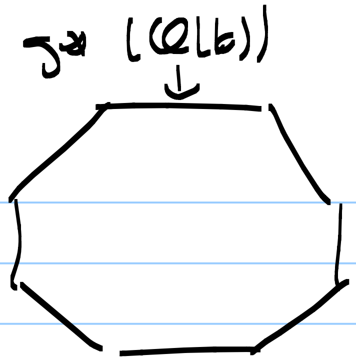
$c_1(\mathcal{O}(k)) = k.$

$c_1(E) \in H^2(X, \mathbb{Z})$, $E \xrightarrow{p} X$ cx. v.b.

$c_1(E)(\alpha) = k$ if $\alpha \in H_2(X)$ and $f : \Sigma \to X$

(Σ orientable surface) and $f^*(E) \to \Sigma$ is isomorphic to $\mathcal{O}(k)$

$\Sigma = \Sigma_0 \xrightarrow{g} S^2$ degree one map



$$\begin{tikzcd} & \omega(b) \arrow[d, mapsto] \\ \xrightarrow{g} \Sigma_g/\text{1-skeleton} = & S^2 \end{tikzcd}$$

$c_1(E)$: 1st Chern class of $E \to X$.

What about higher classes?

$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & E \arrow[d] \arrow[r, squiggly] & \bigoplus_{i=1}^{n} L_i \arrow[d] & \mathbb{R} \arrow[r] & L_i \arrow[d] \\ & X & X & & X \end{tikzcd}$$

w_k: kth Stiefel–Whitney class of E.

$w_k(E) \in H^k(X;\mathbb{Z}_2)$ is the kth elementary symmetric product of $w_1(L_1), \ldots, w_1(L_n)$.

$w_1(E) = w_1(L_1) + \cdots + w_1(L_n) \in H^1(X;\mathbb{Z})$
$w_2(E) = \sum_{i<j} w_1(L_i) \cup w_1(L_j) \in H^2(X;\mathbb{Z})$
$\vdots$
$w_n(E) = w_1(L_1)\, w_1(L_2) \cdots w_1(L_n) \in H^n(X;\mathbb{Z})$.

Similar argument works for Chern classes:

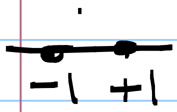
$$\begin{tikzcd} \mathbb{C}^n \arrow[r] & E \arrow[d] \arrow[r, squiggly] & L_1 \oplus \cdots \oplus L_n \arrow[d] \\ & X & X \end{tikzcd}$$

$c_i(E) =$ ith elementary symmetry product of $c_1(L_i)$'s.

$c_i(E) \in H^{2i}(X;\mathbb{Z}_2)$.

Relation to Classifying Spaces

$\mathbb{R} \longrightarrow L \longrightarrow X$ $\mathbb{R}$-bundle

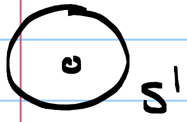


$\longrightarrow X \Longleftrightarrow f: X \longrightarrow B\mathbb{Z}_2 = \mathbb{RP}^\infty$

$\mathbb{Z}_2$-cover

$w_1(E) = f^*(a),\ a \in \underset{\mathbb{Z}/2}{\underset{\wr\|}{H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)}}$

$\mathbb{C} \longrightarrow L \longrightarrow X \Longleftrightarrow f: X \longrightarrow BS^1 = \mathbb{CP}^\infty$



$c_1(E) = f^*(a),\ a \in \underset{\mathbb{Z}}{\underset{\wr\|}{H^2(\mathbb{CP}^\infty, \mathbb{Z})}}$

Quaternion Bundles / Pontryagin Classes

$S^4 = \mathbb{H} \cup \mathbb{H} \big/ p \sim 1/p,\ p \neq 0$ $\quad \mathbb{H} = \mathbb{R}^4$

$p = x_1 + i x_2 + j x_3 + k x_4$

$T_* S^4 = \underset{\mathbb{H}\times\mathbb{H}}{\underset{\|}{T_*\mathbb{H}}} \cup \underset{\mathbb{H}\times\mathbb{H}}{\underset{\|}{T_*\mathbb{H}}} \big/ (p,v) \sim (1/p, \varphi_*(p)\cdot(v))$

$\varphi: \mathbb{H}\setminus\{0\} \longrightarrow \mathbb{H}\setminus\{0\},\ p \longmapsto 1/p$

$$\varphi_*(p)(v) = \lim_{h\to 0} \frac{\varphi(p+hv) - \varphi(p)}{h}$$

$$= -1/p \cdot v \cdot 1/p$$

$E_{n,j} = \mathbb{H}\times\mathbb{H} \cup \mathbb{H}\times\mathbb{H} \big/ (p,v) \sim (1/p, p^n v p^{\bar{j}})$

$T S^4 = E_{-1,-1}$

$E_{h,j}$ has two characteristic classes:

1) Euler Class: $e(E_{h,j}) = -(h+j)\nu \in H^4(S^4) \simeq \mathbb{Z}$

when $\nu \in H^4(S^4)$ generator.

2) 1st Pontryagin class: $P_1(E_{h,j}) = 2(h-j)\nu$

Relation to Classifying Space:

$$\begin{tikzcd}
\mathbb{H} \arrow[r] & E \arrow[d] \arrow[r, phantom, "\longleftrightarrow"] & f: X \longrightarrow BSU(2) \\
SU(2) \arrow[u, phantom, "\cup"] & X &
\end{tikzcd}$$

$\uparrow$

unit vectors in $\mathbb{H}$

$S^4 = \mathbb{H} \cup \mathbb{H}$

$E_i = \mathbb{H} \times \mathbb{H}$

$$\begin{tikzcd}
\mathbb{R}^4 \arrow[r] & E \arrow[d] & E_1 \arrow[d] \cup & E_2 \arrow[d] & \text{(Orientable)} \\
& S^4 = & \mathbb{H}_p \cup & \mathbb{H}_{1/p} / p \sim 1/p & p \neq 0
\end{tikzcd}$$

$\mathbb{H}^* \longrightarrow GL^+(4,\mathbb{R}) \simeq SO(4) \simeq S^3 \times SO(3)$

$\mathbb{H}^* = \mathbb{R}^4 \setminus \{0\} \underset{h.e.}{\simeq} S^3$

$\left[S^3, S^3 \times \underset{\underset{\mathbb{RP}^3}{is}}{SO(3)} \right] = \left[S^3, S^3 \times S^3 \right] \quad (f,g): S^3 \to S^3 \times S^3$

$\pi_3(S^3 \times S^3) \simeq \mathbb{Z} \times \mathbb{Z}$