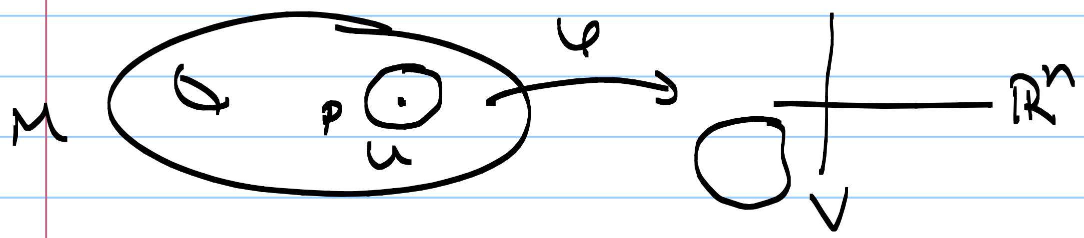
MATH 505 - Differentiable Manifolds

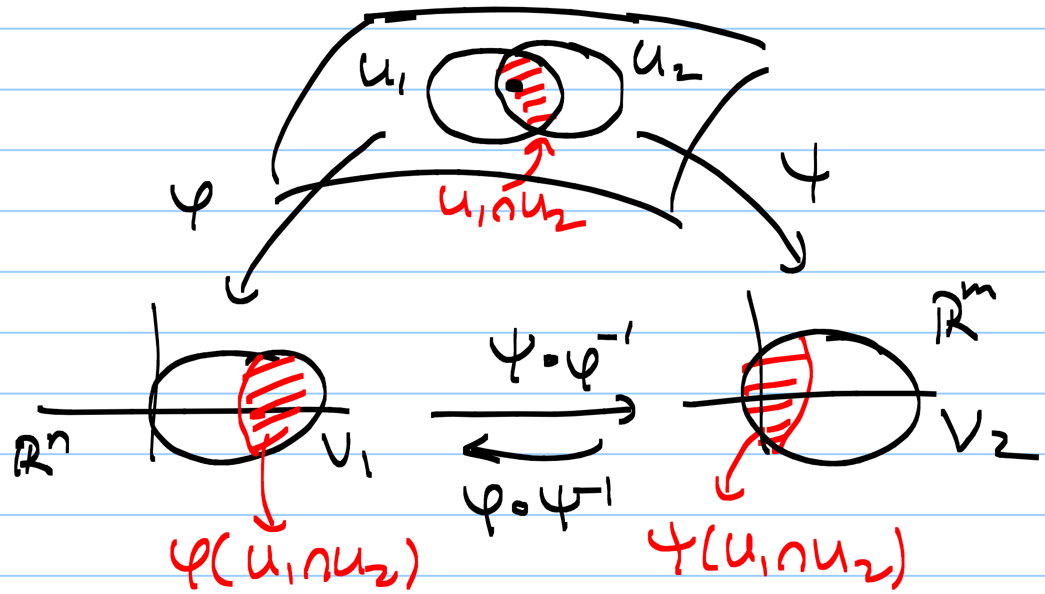
Note Title 8.03.2022

Definition: A topological manifold M is a Hausdorff second countable topological space with the following property: Each point $p \in M$ has a neighborhood $p \in U \subseteq M$ and there is a homeomorphism

$\varphi : U \to V$, where V is an open subset of some Euclidean space $\mathbb{R}^n$.



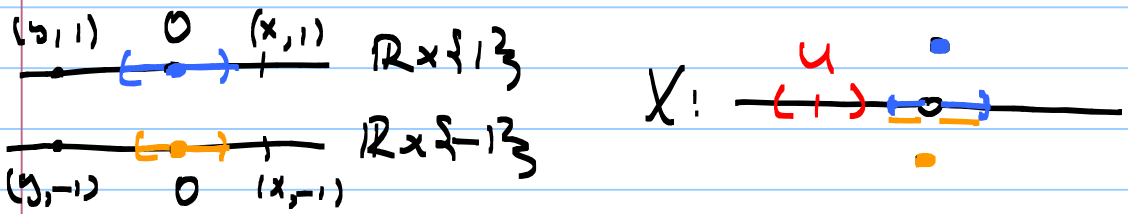
Assume that $p \in U_1 \cap U_2$ and $\varphi : U_1 \to V_1$ and $\psi : U_2 \to V_2$, homeomorphisms, where V_1, V_2 are open subsets of $\mathbb{R}^n$.



Remarks: 1) Any subspace of the Euclidean space is both Hausdorff and second countable. Indeed, we'll see later any manifold is homeomorphic to a subspace of some Euclidean space.

2) Consider the real line with double origin:

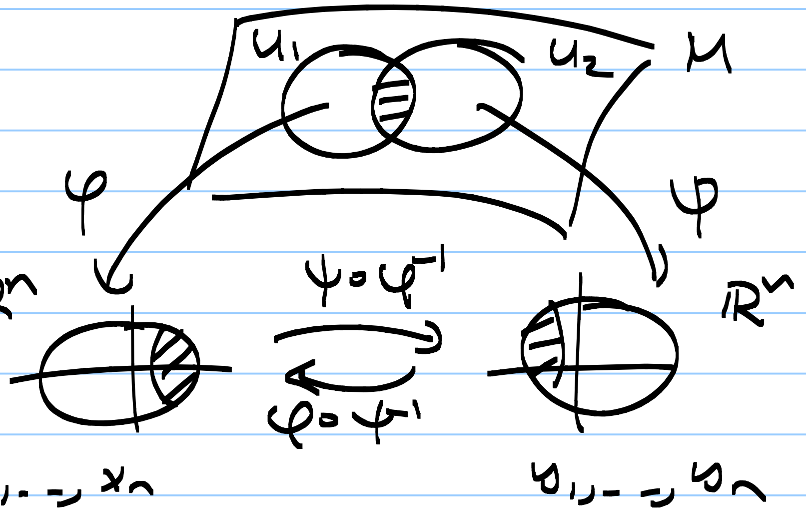
$X = \mathbb{R}\times\{-1,1\}\,/\,(x,-1)\sim(x,1),\ x\neq 0$



X is not Hausdorff but it is locally Euclidean.

Differentiable manifolds:

Definition: We will say that U_1, φ and U_2, ψ are C^∞-compatible if $U_1 \cap U_2$ nonempty implies that the compositions $\varphi\circ\psi^{-1}$ and $\psi\circ\varphi^{-1}$ are both smooth functions (i.e., they are infinitely many times differentiable).



$$(\psi\circ\varphi^{-1})(x_1,\dots,x_n) = (h_1(x_1,\dots,x_n),\dots,h_n(x_1,\dots,x_n))$$

$$(\varphi\circ\psi^{-1})(y_1,\dots,y_n) = (g_1(y_1,\dots,y_n),\dots,g_n(y_1,\dots,y_n))$$

All h_i's and g_j's are C^∞: $\dfrac{\partial^k h_i}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_k}}$ and $\dfrac{\partial^k g_j}{\partial y_{j_1}\partial y_{j_2}\cdots\partial y_{j_k}}$ exist for all k and $(i_1,\dots,i_k)$, $(j_1,\dots,j_k)$.

Video 2

Definition: A differentiable or C^∞ (or smooth) structure on a topological manifold M is a family $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}$ of coordinate neighborhoods such that

1) the collection $\{U_\alpha\}$ covers M: $M = \bigcup_\alpha U_\alpha$

2) for any α, β the neighborhoods U_α, φ_α and U_β, φ_β are C^∞-compatible.

3) Any coordinate neighborhood V, ψ compatible with every U_α, φ_α of $\mathcal{U}$ is itself in $\mathcal{U}$.

A C^∞-manifold is a topological manifold together with a C^∞-differentiable structure.

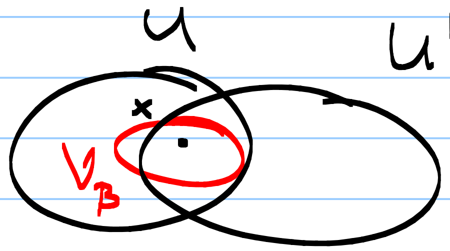
Remarks: 1) Not all topological manifolds admit a C^∞-structure.

2) A topological manifold may admit "different" C^∞-structures.

Theorem: Let M be a Hausdorff space with a countable basis of open sets. If $\{V_\beta, \psi_\beta\}$ is a covering of M by C^∞-compatible coordinate neighborhoods, then there is a unique C^∞-structure on M containing these coordinate neighborhoods.

Proof: Let $\mathcal{U}$ be the collection of all coordinate neighborhoods U, φ which are C^∞-compatible with each and every one of those in the given collection $\{V_\alpha, \psi_\beta\}$. Clearly, $\mathcal{U}$ contains $\{V_\alpha, \psi_\beta\}$. We need to check that different elements of $\mathcal{U}$ are compatible with each other.

Let U, φ and U', φ' be two elements from the collection $\mathcal{U}$. We may assume that $U \cap U' \neq \emptyset$. Take any $x \in U \cap U'$. Choose V_β, ψ_β so that $x \in V_\beta$.



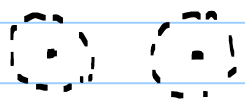
<u>must show:</u> $\varphi' \circ \varphi^{-1}$ is C^∞ at x.

$\varphi' \circ \varphi^{-1} = (\varphi' \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ \varphi^{-1})$ is C^∞ since both $\varphi' \circ \psi_\beta^{-1}$ and $\psi_\beta \circ \varphi^{-1}$ are C^∞.

Maximality of the family $\mathcal{U}$ follows from its definition.

<u>Some Examples:</u> 1) $\mathbb{R}^n$ is a smooth manifold.

$\mathbb{R}^n$ is clearly Hausdorff and second countable.



$\{B(p,r) \mid p \in \mathbb{Q}^n, r \in \mathbb{Q}^+\}$ a countable base for $\mathbb{R}^n$.

$\mathbb{R}^n$ is clearly (locally) Euclidean.

Consider the family $\{V_1 = \mathbb{R}^n, \psi_1 = \mathrm{Id}\}$.

$\psi_1 : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x$, is clearly a homeomorphism. This family has only one element and thus it is compatible. Finally, the above theorem states that this family extends to a unique C^∞-structure on $\mathbb{R}^n$.

2) <u>Open subsets of $\mathbb{R}^n$</u>.

Assume that $M \subseteq \mathbb{R}^n$ is an open subset. Choose $U = M$ and $\varphi = \mathrm{Id}: U \to U$. Then $\{(U, \varphi)\}$ extends to a smooth structure on M.

<u>Example</u>: $GL(n,\mathbb{R}) = \{ A \in M_{n\times n} \mid \det A \neq 0\}$.

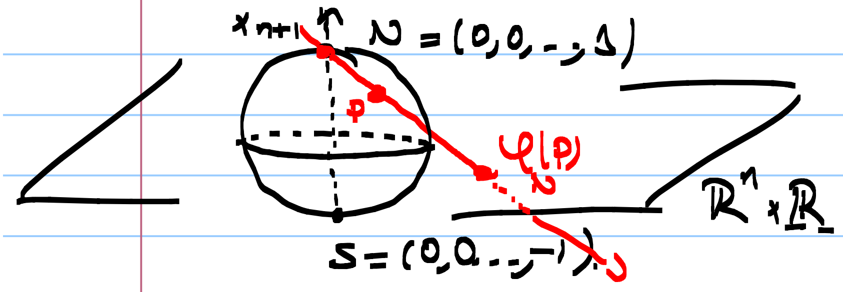
$GL(n,\mathbb{R}) \subseteq M_{n\times n} \simeq \mathbb{R}^{n^2}$. Consider the determinant function

$f: M_{n\times n} \to \mathbb{R}$, $f(A) = \det(A)$, which is continuous. Moreover,

$GL(n,\mathbb{R}) = f^{-1}(\mathbb{R}\setminus\{0\})$ and hence $GL(n,\mathbb{R})$ is an open subset of $M_{n\times n} = \mathbb{R}^{n^2}$.

3) <u>Spheres</u>: $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$.

Consider S^n with its subspace topology inherited from $\mathbb{R}^{n+1}$. Then S^n is also Hausdorff and second countable.



$\varphi_N : S^n \setminus \{N\} \longrightarrow \mathbb{R}^n$

$$P = (x_1, \ldots, x_n, x_{n+1}) \longmapsto \varphi_N(P) = \left(\frac{x_1}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right).$$

Similarly, define

$\varphi_S : S^n \setminus \{S\} \longrightarrow \mathbb{R}^n$

$$P = (x_1, \ldots, x_n, x_{n+1}) \longmapsto \varphi_S(P) = \left(\frac{x_1}{1+x_{n+1}}, \ldots, \frac{x_n}{1+x_{n+1}}\right).$$

φ_N and φ_S are homeomorphisms because they are clearly continuous and their inverses exist and are continuous.

$$\varphi_N^{-1} : \mathbb{R}^n \longrightarrow U_N = S^n \setminus \{N\}$$
$$(y_1, \dots, y_n) \longmapsto \left(\frac{2y_1}{1+\|y\|^2}, \dots, \frac{2y_n}{1+\|y\|^2}, \frac{\|y\|^2 - 1}{1+\|y\|^2} \right)$$

and

$$\varphi_S^{-1} : \mathbb{R}^n \longrightarrow U_S = S^n \setminus \{S\}$$
$$(y_1, \dots, y_n) \longmapsto \left(\frac{2y_1}{1+\|y\|^2}, \dots, \frac{2y_n}{1+\|y\|^2}, \frac{1 - \|y\|^2}{1+\|y\|^2} \right).$$

Hence, S^n is a topological manifold.

$\varphi_N \circ \varphi_N^{-1} = id$ ✓ $\varphi_S \circ \varphi_S^{-1} = id$ ✓

$$\varphi_N \circ \varphi_S^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$$
$$(y_1, \dots, y_n) \longmapsto \frac{1}{\|y\|^2}(y_1, \dots, y_n), \text{ which is } C^\infty.$$

Hence, $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ extends to a (uniquely determined) smooth structure on S^n.

§2. Further Examples

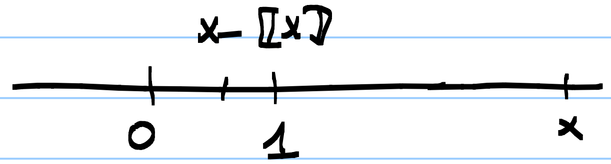
Let X be a topological space and $\sim$ an equivalence relation on X.

$[x] = \{y \in X \mid x \sim y\}$, the equivalence class of x.

$X/\sim \; = \{[x] \mid x \in X\}$ the set of all equivalence classes, called also as the quotient set of X via the relation $\sim$.

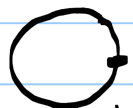
Let $\pi : X \to X/\sim$, $x \mapsto \pi(x) = [x]$, be the quotient map. We put so called the quotient topology on $X/\sim$ as follows: A subset U of $X/\sim$ will be said to be open if $\pi^{-1}(U)$ is open in X. This defines a topology on $X/\sim$ so that $\pi : X \to X/\sim$ is continuous. Note that this is the strongest topology on $X/\sim$ which makes $\pi : X \to X/\sim$ is continuous. This topology is called the quotient topology on $X/\sim$ and $X/\sim$ is called a quotient space of X.

Example: $X = \mathbb{R}_{std}$ and define $\sim$ on $\mathbb{R}$ as follows: $x, y \in X$, $x \sim y$ if and only if $x - y \in \mathbb{Z}$.
Clearly, $\sim$ is an equivalence relation.



$\pi : X \to X/\sim$, $x \mapsto [x]$. Then $X/\sim \; = \pi([0,1])$

$\Rightarrow X/\sim$ is compact, $X/\sim \; = [0,1]/_{0 \sim 1}$



Exercise: Show that $X/\sim$ is homeomorphic to $S^1 \subseteq \mathbb{R}^2$.

Definition: An equivalence relation $\sim$ on a space X is called open if whenever a subset $A \subseteq X$ is open, then $[A] = \{[x] \mid x \in A\}$ is open in $X/\sim$.

Lemma: An equivalence relation $\sim$ on X is open if and only if π is an open mapping (i.e., $\pi(U)$ is open whenever $U \subseteq X$ is open). When $\sim$ is open and X has a countable basis of open sets, then $X/\sim$ has a countable basis.

Proof: The first statement is almost trivial since $[U] = \pi(U)$. For the second statement assume $\sim$ is open and X has a countable basis $\{U_i\}_{i \in I}$ open subsets. If W is open in $X/\sim$ then the open subset $\pi^{-1}(W)$ in X can be written as

$$\pi^{-1}(W) = \bigcup_{j \in J} U_j, \text{ for some } J \subseteq I.$$

Then
$$W = \pi(\pi^{-1}(W)) = \pi(\cup U_j) = \bigcup_{j \in J} \pi(U_j), \text{ where}$$
each $\pi(U_j)$ is open in $X/\sim$, since π is an open map. Hence, $\{\pi(U_i)\}_{i \in I}$ is a countable basis for $X/\sim$. ∎

Lemma: Let $\sim$ be an open equivalence relation on a topological space X. Then
$R = \{(x,y) \mid x \sim y\}$ is a closed subset of the space $X \times X$ if and only if the quotient space $X/\sim$ is Hausdorff.

Proof: Suppose $X/\sim$ is Hausdorff.
must show: R is closed.

Take any $(x,y) \notin R$. Then $x \not\sim y$ so that $[x] \neq [y]$. Since $X/\sim$ is Hausdorff there are disjoint neighborhood U of $\pi(x)$ and V of $\pi(y)$. Then for any $\tilde{x} \in \tilde{U} = \pi^{-1}(U)$ and $\tilde{y} \in \tilde{V} = \pi^{-1}(V)$ we have $\tilde{x} \not\sim \tilde{y}$, because otherwise,

$$\pi(\tilde{x}) = \pi(\tilde{y}) \in U \cap V = \emptyset, \text{ a contradiction.}$$

Hence $(\tilde{U} \times \tilde{V}) \cap R = \emptyset$. Since $(x,y) \in \tilde{U} \times \tilde{V}$ and $(x,y) \notin R$ were arbitrary, we see that R is closed in $X \times X$.

Conversely, if R is closed, then for any given distinct points $\pi(x) = [x]$, $\pi(y) = [y]$ of $X/\sim$, $x \not\sim y$ then is an open set W with $(x,y) \in W$ and $W \cap R = \emptyset$. So there is an open set of the form $\tilde{U} \times \tilde{V}$ so that

$$(x,y) \in \tilde{U} \times \tilde{V} \subseteq W \subseteq X \times X \setminus R. \text{ So}$$

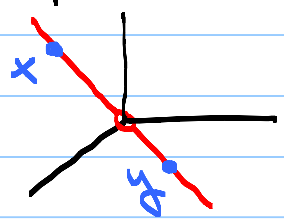
$$\pi(x) \in \pi(\tilde{U}),\ \pi(y) \in \pi(\tilde{V}) \text{ and } \pi(\tilde{U}) \cap \pi(\tilde{V}) = \emptyset.$$

Moreover, $\pi(\tilde{U})$ and $\pi(\tilde{V})$ are open since π is an open map. Hence, $X/\sim$ is Hausdorff. ■

<u>Example:</u> (The real projective space $\mathbb{P}^n(\mathbb{R}) = \mathbb{RP}^n$).

Let $X = \mathbb{R}^{n+1} \setminus \{0\}$. Define an equivalence relation on X as follows:

$x = (x_1, \dots, x_{n+1})$, $y = (y_1, \dots, y_{n+1})$, $x \sim y$ if and only if $x = ty$ for some $t \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$.



So $[x]$ is the line joining x to the origin.

The real projective space $\mathbb{RP}^n$ is defined to be the quotient space $X/\sim$.

Video 4

$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ is the space of lines through the origin in $\mathbb{R}^{n+1}$.

We'll show that $\mathbb{RP}^n$ is an n-dimensional differentiable manifold.

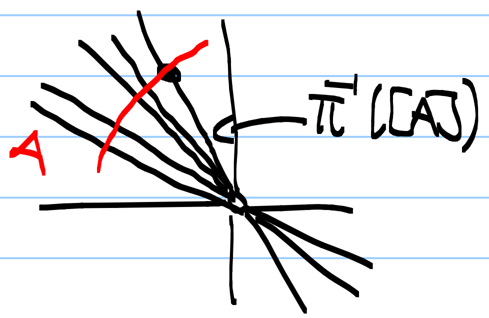
Step 1 Show that $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ is an open map.

It is enough to show that $\sim$ is an open relation. Let $A \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ is open. $[A] = \{[x] \mid x \in A\}$ must be open:

By the definition quotient topology we need to show that $\pi^{-1}([A])$ is open in $X = \mathbb{R}^{n+1} \setminus \{0\}$.

$\pi^{-1}([A]) = \bigcup_{t \in \mathbb{R} \setminus \{0\}} \varphi_t(A)$, $\varphi_t : X \to X, x \mapsto tx$.

$A \subseteq X$ open. φ_t is clearly continuous with continuous inverse $\varphi_{t^{-1}}$. Hence φ_t is a homeomorphism. Since $A \subseteq X = \mathbb{R}^{n+1} \setminus \{0\}$ is open each $\varphi_t(A)$ is open in X. It follows that $\pi^{-1}([A])$ is open.



Since $X = \mathbb{R}^{n+1} \setminus \{0\}$ is second countable and π is an open mapping its image $\mathbb{RP}^n = \pi(X)$ is also a second countable space.

Step 2: $\mathbb{RP}^n$ is Hausdorff.

We must show that $R = \{(x,y) \mid x,y \in X,\ x\sim y\}$ is closed in $X \times X = (\mathbb{R}^{n+1}\setminus\{0\}) \times (\mathbb{R}^{n+1}\setminus\{0\})$.

Note that $x \sim y$ if and only if $x = ty$ for some $t \in \mathbb{R}\setminus\{0\}$.

$x = (x_1, \ldots, x_{n+1})$
$y = (y_1, \ldots, y_{n+1})$

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_i & x_{i+1} & \cdots & x_{n+1} \\ y_1 & y_2 & \cdots & y_i & y_{i+1} & \cdots & y_{n+1} \end{bmatrix}$$

$$F : X \times X \longrightarrow \mathbb{R},\quad F(x,y) = \sum_{1 \le i < j \le n+1} \left(\det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}\right)^2$$

Now $x \sim y$ if and only if $F(x,y) = 0$.

Now $R = \{(x,y) \mid x \sim y\} = F^{-1}(0)$ is closed in $X \times X$ because F is clearly continuous on $X \times X$.

Hence, $\mathbb{RP}^n$ is Hausdorff.

Step 3: $\mathbb{RP}^n = \{[x] \mid x \in X\}$ $\quad x = (x_1, x_2, \ \ , x_{n+1})$ $x_i \neq 0$ for some i.

$= U_1 \cup U_2 \cup \cdots U_{n+1}$, where

$U_i = \{[x] \mid x_i \neq 0\}$. Note that each U_i is open since $\pi^{-1}(U_i) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}\setminus\{0\} \mid x_i \neq 0\}$ is open in $\mathbb{R}^{n+1}$ and hence in X.

Now define

$\phi_i : U_i \longrightarrow V_i = \mathbb{R}^n$ by $\phi_i([x]) = \left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \ldots, \widehat{\underbrace{\frac{x_i}{x_i}}_{1}}, \ldots, \frac{x_{n+1}}{x_i}\right)$

ϕ_i is clearly 1-1 and onto.

To show that ϕ_i is continuous we may use a general fact:

<u>Lemma</u>: Let $\pi : X \longrightarrow X/\sim$ be a quotient space and $f : X \longrightarrow Y$ a continuous map. Then there is a map $g : X/\sim \longrightarrow Y$ with $f = g\circ\pi$ iff f is constant on each equivalence class. Moreover, g is continuous if and only if f is continuous.

<u>Proof</u>

$$\begin{tikzcd} x \arrow[d, mapsto] \\ {[x]} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ X/\sim \arrow[ur, dashed, "g"'] & \end{tikzcd} \qquad f = g\circ\pi$$

Note that the condition $f(x) = g(\pi(x))$ for all $x \in X$ implies that g exists if and only if $f(x) = f(y)$ for all x, y with $x \sim y$.

For the second part assume g exists. If g is continuous then $f = g\circ\pi$ is clearly continuous because π is continuous. If f is continuous then for any open set U in Y we have

$$\pi^{-1}(g^{-1}(U)) = (g\circ\pi)^{-1}(U) = f^{-1}(U) \text{ is open in } X$$

and hence $g^{-1}(U)$ is open in $X/\sim$. Hence, g is continuous. ■

Back to the continuity of ϕ_i:

$$\begin{tikzcd} Z = \mathbb{R}^{n+1} \setminus \{(x_1, \dots, \overset{i\text{th}}{0}, \dots, x_{n+1}) \in \mathbb{R}^{n+1}\} \arrow[r, "\tilde{\phi}_i"] \arrow[d] & \mathbb{R}^n = Y \\ Z/\sim \, = U_i = \{[x] \mid x_i \neq 0\} \arrow[ur, "\phi_i"'] & \end{tikzcd}$$

Video 5

Define $\tilde{\phi}_i : Z \longrightarrow \mathbb{R}^n = Y, \quad (x_1,\dots,x_i,\dots,x_{n+1}) \longmapsto \left(\frac{x_1}{x_i},\dots,\widehat{\frac{x_i}{x_i}},\dots,\frac{x_{n+1}}{x_i}\right)$

Clearly, $\tilde{\phi}_i$ is continuous
and $\tilde{\phi}_i = \phi_i \circ \pi$.

<u>Exercise:</u> $\phi_i^{-1} : \mathbb{R}^n \longrightarrow U_i \subseteq \mathbb{RP}^n$ is also continuous.

$$\mathbb{R}^n \xrightarrow[j]{\text{inclusion}} X \xrightarrow{\quad\pi\quad} U_i \subseteq \mathbb{RP}^n$$
$$(y_1,\dots,y_n) \longmapsto (y_1,\dots,y_{i-1},1,y_i,\dots,y_n) \longmapsto [y_1,\dots,1,\dots,y_n].$$

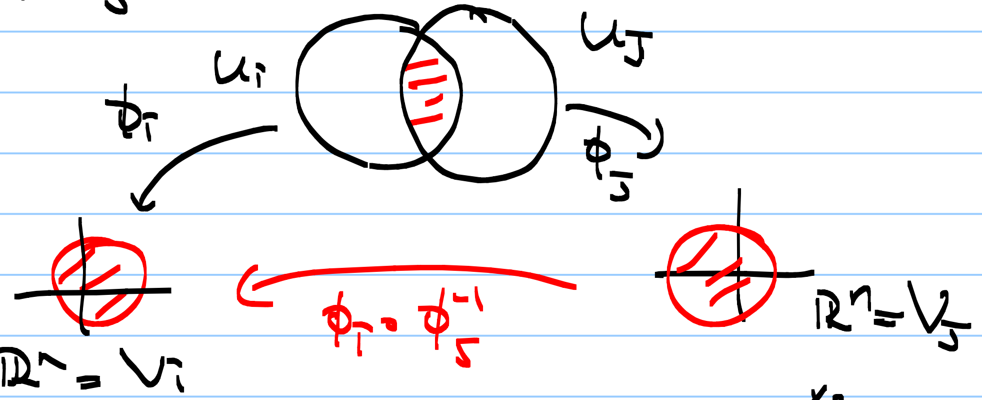
Check that ϕ_i and ϕ_i^{-1} are inverses of each other.
Note that $\phi_i^{-1} = \pi \circ j$ is continuous.

Hence, $\mathbb{RP}^n$ is a topological manifold.

$$\mathbb{RP}^n = U_1 \cup U_2 \cup \dots \cup U_i \cdots \cup U_{n+1}$$

$$\phi_i : U_i \longrightarrow V_i = \mathbb{R}^n$$

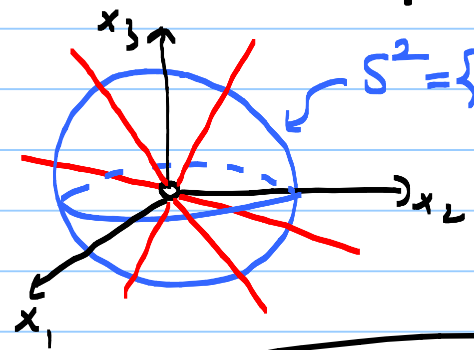
To show that the coordinate functions ϕ_i define a C^∞-structure we must check that each $\phi_i \circ \phi_j^{-1}$ are C^∞-maps.



$$(\phi_i \circ \phi_j^{-1})(x_1,\dots,x_n) = \phi_i\left([x_1,\dots,x_{j-1},1,x_j,\dots,\overset{x_i}{\downarrow},\dots,x_n]\right)$$
$$= \left(\frac{x_1}{x_i},\dots,\frac{x_{j-1}}{x_i},\frac{1}{x_i},\frac{x_j}{x_i},\dots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\dots,\frac{x_n}{x_i}\right)$$

Note that $\phi_i \circ \phi_j^{-1}$ is clearly C^∞. Hence, $\mathbb{RP}^n$ is a differentiable (smooth) manifold.

<u>Ex</u> $\mathbb{RP}^2 = \mathbb{R}^3 \setminus \{0\} \big/ (x_1, x_2, x_3) \sim t(x_1, x_2, x_3)$, $t \neq 0$.



$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$.

$\mathbb{RP}^2 = S^2 \big/ (x_1, x_2, x_3) \sim_2 (-x_1, -x_2, -x_3)$

$$\begin{tikzcd} S^2 \arrow[r, "inc."] \arrow[d, "\pi_2"'] \arrow[rr, bend left, "cont.\ f"] & X = \mathbb{R}^3 \setminus \{0\} \arrow[r, "\pi_1"] & X/\sim_1 = \mathbb{RP}^2 = Y \\ S^2/\sim_2 \arrow[urr, "g"'] & & \end{tikzcd}$$

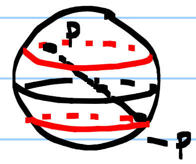
Since f is cont. so is g.

Note that 1) g is 1-1

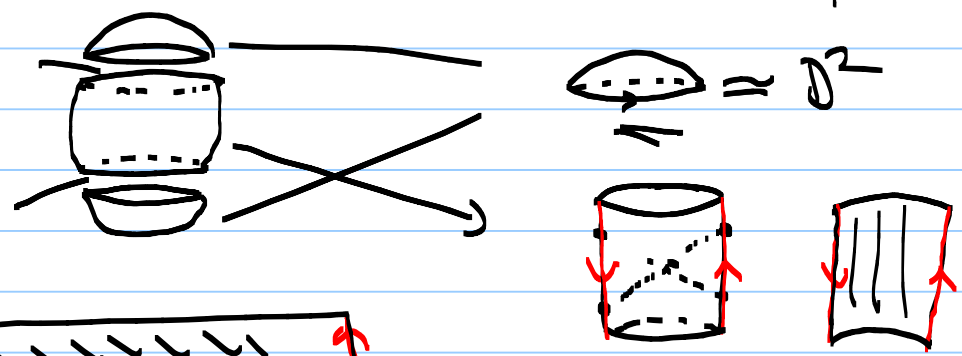
2) g is onto

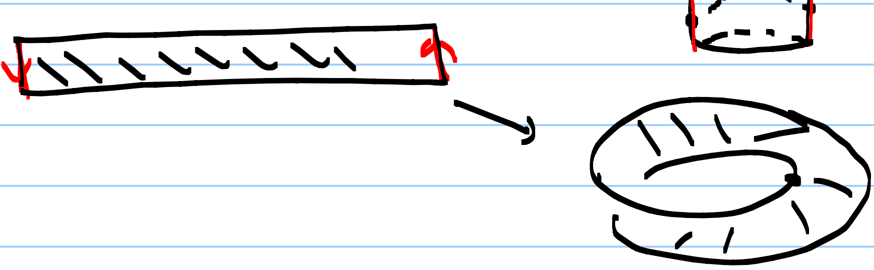
3) S^2 is compact and hence $S^2/\sim_2$ is compact and Y is Hausdorff. So by the homeomorphism theorem g is also a homeomorphism.

<u>Remark</u>: $\mathbb{RP}^2 = S^2 \big/ p \sim -p$



S^2





Möbius Band

$\mathbb{RP}^2 = S^2/_{p\sim -p} = D^2 \dot{\cup}_{\partial} MB$ ← glue them along their boundaries.

Example (Grassmann Manifolds)

$G(k,n)$: the space of all k-dim'l subspaces of $\mathbb{R}^n$

$G(2,4)$: the space of planes in $\mathbb{R}^4$ passing through the origin.

$G(1,n+1) = \mathbb{RP}^n$.

To show that $G(k,n)$ is a differentiable manifold first we consider so called the manifold of k-frames in $\mathbb{R}^n$ denoted $F(k,n)$.

$F(k,n) = \{ (v_1, \dots, v_k) \mid v_i \in \mathbb{R}^n, v_1, \dots, v_k \text{ are lin. ind.} \}$.

Note that $F(k,n) = \{ A \in M_{kn}(\mathbb{R}) \mid \operatorname{rank} A = k \}$.

$$A = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}_{k \times n}$$

Note that $F(k,n)$ is an open subset of $M_{kn}(\mathbb{R}) = \mathbb{R}^{kn}$ because $F(k,n) = \phi^{-1}(\mathbb{R} \setminus \{0\})$ where

$$\phi : M_{kn}(\mathbb{R}) \longrightarrow \mathbb{R}, \quad \phi(A) = \sum_{B \text{ } k\text{-minor of } A} (\det(B))^2$$

$F(2,4)$: $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$ In particular, $F(k,n)$ is a kn-dimensional (open) manifold in $\mathbb{R}^{kn} = M_{kn}(\mathbb{R})$.

Video 6

Note that two elements $(v_1, \ldots, v_k)$ and $(u_1, \ldots, u_k)$ of $F(k,n)$ represent the same subspace if and only if

$$\underset{k\times k}{P}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}_{k\times n} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}$$ for some invertible $k\times k$ matrix P.

Hence, $G(k,n)$ can be defined as the quotient space

$$G(k,n) = F(k,n)/\sim, \text{ where}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} \sim \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix} \text{ iff } P\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}, \text{ for}$$

some invertible $k\times k$-matrix P.

__Theorem:__ $G(k,n)$ has a structure of $k(n-k)$ dimensional differentiable manifold.

__Proof:__ (Homework)

__Step 1__ Show that $\sim$ is an open relation.
$\Rightarrow G(k,n)$ is second countable.

__Step 2__ Show that "R" as in the case of $\mathbb{RP}^n$ is closed in $F(k,n)\times F(k,n)$.
$\Rightarrow G(k,n)$ is Hausdorff.

__Step 3__ $G(k,n)$ is locally Euclidean.

__Idea:__ $x = (v_1, \ldots, v_k) \in F(k,n)$, $[x] \in G(k,n)$.
Note that each $[x]$ has a representative $y \in F(k,n)$ (i.e., $[x]=[y]$), where y has the form

$$y^T = (u_1, \dots, u_k)^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix} = \begin{array}{c} \begin{array}{cccccccccc} & & i_1 & & i_2 & & i_k & & & \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \cdots & 0 & \cdots & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & & & & & \\ \vdots & \vdots & \vdots & & 0 & & & & & \\ 0 & 0 & 0 & 0 \cdots & 0 & & 1 & * & * & * \end{bmatrix} \end{array}$$

$U_{i_1 \dots i_k} = \{[y] \in G(k,n) \mid$ the colums $i_1, i_2, \dots, i_k$ are linearly independent $\}$

$1 \leq i_1 < \quad < i_k \leq n$

If $y \in F(k,n)$ then there is a unique $k \times k$ invertible $\underline{P}$ s.t.

$$\underline{\underline{P_y}} = \begin{array}{c} \begin{array}{ccccc} & i_1 & i_2 & i_k & \end{array} \\ \begin{bmatrix} \underline{\underline{*}} & 1 & 0 & 0 & * \\ & 0 & 1 & \vdots & \\ & \vdots & \vdots & \vdots & \\ \underline{\underline{*}} & 0 & 0 & 1 & \underline{\underline{*}} \end{bmatrix} \end{array}.$$

Indeed, $P_y = [c_{i_1} \, c_{i_2} \quad c_{i_k}]^{-1}_{k \times k}$ where c_j is the j-th colum of y.

$$U_{i_1 i_2 \dots i_k} \xrightarrow{\varphi_{i_1 \dots i_k}} \mathbb{R}^{k(n-k)}$$

$$y \longmapsto (P_y y)^0 \doteq (c_1, c_2, \dots, \widehat{c_{i_1}}, \dots, \widehat{c_{i_k}}, \dots, c_n)$$

$$[(x_1, x_2, \dots, x_i, \dots x_n)] \mapsto \left(\frac{x_1}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots \frac{x_n}{x_i}\right)$$
$$\mathbb{RP}^n \supseteq U_i \longrightarrow \mathbb{R}^n$$

must show: each $\varphi_{i_1 \dots i_k}$ is a homeomorphism.
$\Rightarrow G(k,n)$ is a topological manifold.

Step 4 $\{\varphi_{i_1 \dots i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$ defines a differentiable structure on $G(k,n)$.

§ Differentiable Functions and Mappings:

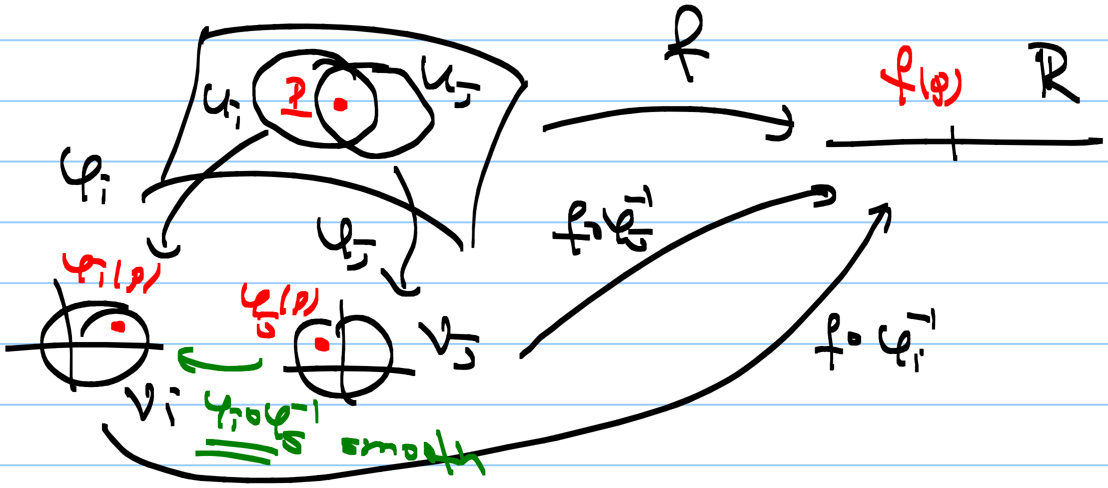
Let M be a differentiable manifold with a cover coordinate charts $\{\varphi_i : U_i \to V_i\}_{i\in I}$, $U_i \subseteq M$ open with $M = \cup U_i$, $V_i \subseteq \mathbb{R}^n$ open.

Let $f : M \to \mathbb{R}$ be a function. We'll say that f is a differentiable (or smooth) function if the composition

$$f\circ\varphi_i^{-1} : V_i \longrightarrow \mathbb{R} \quad \text{is (smooth)}$$

differentiable for any i.

Remark: If $p \in U_i \cap U_j$ then $f\circ\varphi_i^{-1}$ is differentiable at p if and only if $f\circ\varphi_j^{-1}$ is differentiable at p.



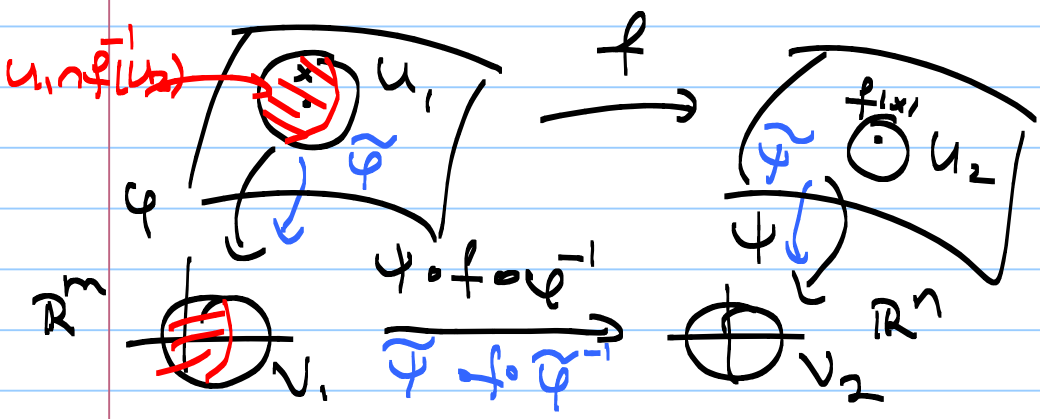
$$f\circ\varphi_j^{-1} = (f\circ\varphi_i^{-1})\circ(\varphi_i\circ\varphi_j^{-1})$$

Since $\varphi_i\circ\varphi_j^{-1}$ is smooth, $f\circ\varphi_j^{-1}$ is smooth whenever $f\circ\varphi_i^{-1}$ is smooth.

Similarly, a function $F : M^n \to \mathbb{R}^m$, where $F(p) = (f_1(p), \ldots, f_m(p))$ is called smooth (or differentiable) if each f_i is smooth.

Video 7

Let $f: M^m \to N^n$ be a function between two differentiable manifolds of dimension m and n, respectively.



f is called differentiable if $\psi \circ f \circ \varphi^{-1}$ is differentiable for any $x \in M$ and coordinate systems $\varphi: U_1 \to V_1$ and $\psi: U_2 \to V_2$ as above.

<u>Remark</u>: As above definition, indeed it is enough to find some $\varphi: U_1 \to V_1$ and $\psi: U_2 \to V_2$ with $\psi \circ f \circ \varphi^{-1}$ differentiable. This is because if $\tilde{\varphi}$ and $\tilde{\psi}$ are some other coordinate systems then

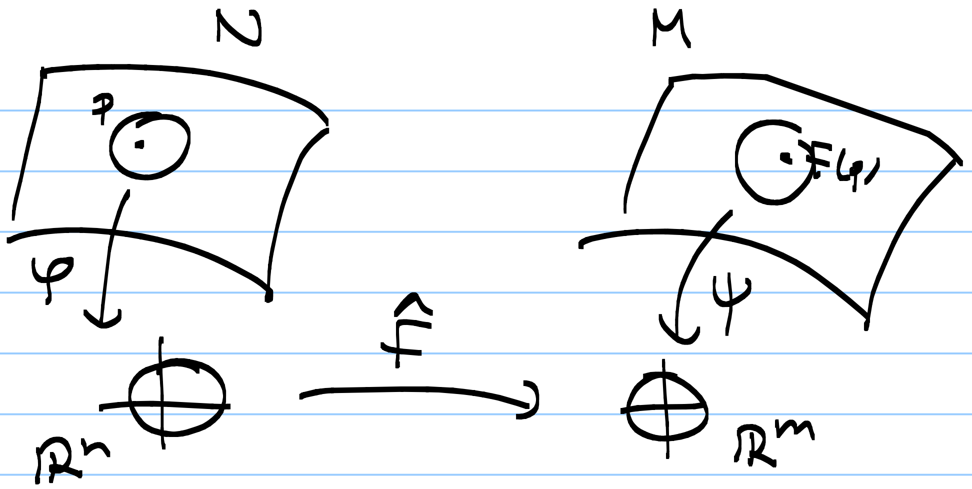
$$\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \tilde{\varphi}^{-1}),$$ where

$\tilde{\psi} \circ \psi^{-1}$, $\varphi \circ \tilde{\varphi}^{-1}$ and $\psi \circ f \circ \varphi^{-1}$ are all differentiable.

Rank of a mapping, Immersions and Submersions:

Let $F: N \to M$ be a differentiable mapping of C^∞-manifolds and let $p \in N$. If U, φ and V, ψ are coordinate neighborhoods of p and $F(p)$, respectively and $F(U) \subseteq V$, then we may consider the following function

$$\tilde{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V).$$



<u>Definition:</u> The rank of F at p is defined to be the rank of $\hat{F}$ at $\varphi(p)$:

$$\hat{F} = (f_1, \dots, f_m) \qquad f_i : \mathbb{R}^n \to \mathbb{R}$$
$$f_i = f_i(x_1, \dots, x_n)$$

The rank of $\hat{F}$ at $\varphi(p)$ is the rank of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

<u>Remark:</u> The rank of F at p is well defined because if we choose different coordinate neighborhoods say $\tilde{\varphi}$ and $\tilde{\psi}$ then we have

$\hat{\hat{F}} = G \circ \hat{F} \circ H$, where G and H are diffeomorphisms of open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively and for a diffeomorphism the Jacobian is invertible.

$$\Big(\quad L \circ L^{-1} = Id \implies DL \circ DL^{-1} = D\,Id = Id$$
$$\implies DL \text{ and } DL^{-1} \text{ are both invertible.} \Big)$$

Definition: A differentiable map $f: N \to M$ is said to be an immersion at a point $p \in N$ if the rank of F at p is $n = \dim N$. If F is an immersion at each point then we simply say that f is an immersion.

Definition: A differentiable map $F: N \to M$ is said to be a submersion at a point $p \in N$ if the rank of F at p is $m = \dim M$. If F is a submersion at each point then F is called a submersion.

Example: 1) The maps $F: \mathbb{R}^n \to \mathbb{R}^m$, when $n \leq m$,

given by $F(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0)$ and

$G: \mathbb{R}^n \to \mathbb{R}^m$, when $n \geq m$, given by

$G(x_1, \dots, x_m, \dots, x_n) = (x_1, \dots, x_m)$, are called the standard immersion and submersion, respectively.

Example 2) $F: \mathbb{R} \to \mathbb{R}^3$, $F(t) = (\underline{\cos 2\pi t}, \sin 2\pi t, t)$

and $G: \mathbb{R} \to \mathbb{R}^2$, $G(t) = (\cos 2\pi t, \sin 2\pi)$ are both immersions.

$DF(t) = (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t, 1)$ has rank 1 at all points.

Similarly, $DG(t) = (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t)$ has also rank 1 $(\sin^2 2\pi t + \cos^2 2\pi t = 1)$

Example 3) $F: \mathbb{R} \to \mathbb{R}^2$, $F(t) = (2\cos(t - \frac{\pi}{2}), \sin 2(t - \frac{\pi}{2}))$

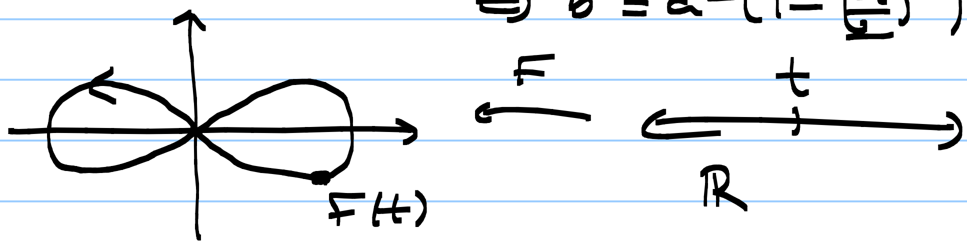
Video P

Let $a = 2\cos(t - \frac{\pi}{2})$ and $b = \sin 2(t - \pi/2)$.

Then $b = 2\cos(t-\pi/2)\sin(t-\pi/2)$

$b^2 = a^2(1-(a/2)^2)$

$\Rightarrow b^2 = a^2(1-(\frac{a}{2})^2)$



$F(t+2\pi) = F(t)$

$F'(t) = (-2\sin(t-\pi/2), 2\cos 2(t-\pi/2)) \neq (0,0)$

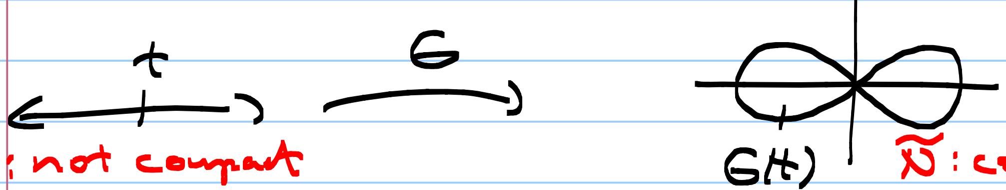
and hence F is an immersion, but not an imbedding since F is not 1-1.

Example 4) $G : \mathbb{R} \to \mathbb{R}^2$

$G(t) = F(g(t)) = (2\cos(g(t)-\frac{\pi}{2}), \sin 2(g(t)-\pi/2))$,

where $g(t) = \pi + 2\tan^{-1}t$ so that

$\lim_{t\to-\infty} g(t) = 0$, $\lim_{t\to\infty} g(t) = 2\pi$ and $g(0) = \pi$.



N: not compact

$\tilde{N}$: compact

G is a 1-1 immersion! G is also 1-1 but it is not an imbedding since G is not a homeomorphism onto its image!

Rank Theorems:

Theorem (Immersion)

Let $f : M \to N$ be a differentiable map and $p \in M$ so that f is an immersion at p. Then one can find coordinate neighborhoods around $p \in M$ and $f(p) \in N$, say $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$, $p \in U_1 \subseteq M$, $f(p) \in U_2 \subseteq N$, $V_1 \subseteq \mathbb{R}^m$, $V_2 \subseteq \mathbb{R}^n$ open subsets, so that

$$\begin{tikzcd}
M \supseteq U_1 \ni p \arrow[r, "f"] \arrow[d, "\varphi_1"'] & f(p) \in U_2 \subseteq N \arrow[d, "\varphi_2"] \arrow[dr, bend left, "\tilde{\varphi}_2"] & \\
\mathbb{R}^m \supseteq V_1 \arrow[r, "\hat{f}"', "\varphi_2 \circ f \circ \varphi_1^{-1}"] & V_2 \arrow[r, "H"'] & \tilde{V}_2 \subseteq \mathbb{R}^n
\end{tikzcd}$$

$x_1, \dots, x_m$ $\qquad$ $y_1, \dots, y_n$

$$(\varphi_2 \circ f \circ \varphi_1^{-1})(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$$

<u>Proof</u>: $\hat{f} = \varphi_2 \circ f \circ \varphi_1^{-1} = (f_1, f_2, \dots, f_n)$.

$$f_i = f_i(x_1, \dots, x_m) : \mathbb{R}^m \to \mathbb{R}$$

Since f is an immersion at p, $\hat{f}$ has rank m at $\varphi_1(p)$.

$$D\hat{f}(\varphi_1(p)) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}_{n \times m} (\varphi(p)) \qquad n \geq m$$

The row rank of the above matrix is m and thus by permuting the coordinates y_i's or equivalently f_i's we may assume that the first m rows are linearly independent.

Now define another differentiable map

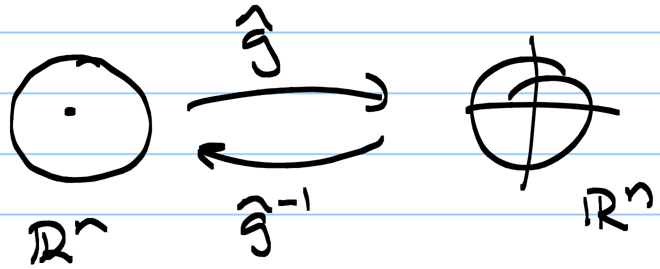
$$\hat{g} : \overset{\mathbb{R}^m}{U_1} \times \mathbb{R}^{n-m} \to \mathbb{R}^n \text{ given by}$$

$$\hat{g}(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = \hat{f}(x_1, \dots, x_m) + (0, \dots, 0, x_{m+1}, \dots, x_n)$$

$$\Rightarrow \hat{g}(x_1, \dots, x_n) = (f_1, f_2, \dots, f_m, f_{m+1} + x_{m+1}, \dots, f_n + x_n)$$

Then $D\hat{g}(\varphi_1(p)) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & & & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} & 0 & 0 & \cdots & 0 \\ * & * & * & 1 & 0 & \cdots & 0 \\ * & * & * & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & & & \\ * & * & * & 0 & \cdots & 0 & 1 \end{pmatrix}_{n \times n}$ $\quad I_{n-m}$

Note that since the first m rows are linearly independent we see that $D\hat{g}(\varphi_1(p))$ is invertible. Since $D\hat{g}(\varphi_1(p))$ is an isomorphism by the Inverse Function Theorem $\hat{g}$ is a local diffeomorphism.



Now we have the following picture:

$$\begin{tikzcd} M \ni p \arrow[r, "f"] \arrow[d, "\varphi_1"'] & f(p) \in N \arrow[d, "\varphi_2"] \arrow[dr, bend left, "g^{-1}\circ\varphi_2 = \tilde{\varphi}_2"] & \\ \mathbb{R}^m \ni \varphi_1(p) \arrow[r, "\hat{f}"'] \arrow[rr, bend right, "g^{-1}\circ\hat{f}"'] & \hat{f}(\varphi_1(p)) \in \mathbb{R}^n \arrow[r, "g^{-1}"'] & \mathbb{R}^n \end{tikzcd}$$

$(x_1,\dots,x_m) \longmapsto (f_1,\dots,f_n) \longmapsto g^{-1}(f_1,\dots,f_n)$, where

$(x_1,\dots,x_m,0,\dots,0)$

$(x_1,\dots,x_m,x_{m+1},\dots,x_n) \underset{g^{-1}}{\overset{g}{\longmapsto}} (f_1,\dots,f_m,f_{m+1}+x_{m+1},\dots,f_n+x_n)$

Finally, since $g(x_1,\dots,x_m,0,\dots,0) = (f_1,\dots,f_m,f_{m+1},\dots,f_n)$

Video 9

we have
$g^{-1}(f_1(x_1,\ldots,x_m),\ldots,f_n(x_1,\ldots,x_m)) = (x_1,\ldots,x_m,0,\ldots,0)$.

A similar result holds for submersions.

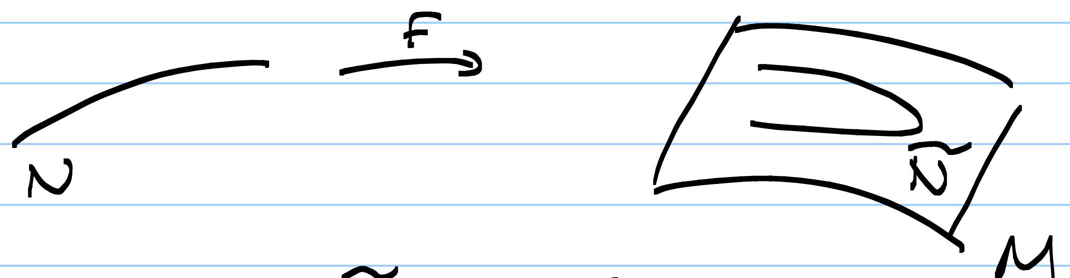
Theorem (Submersion)

Let $f: M \to N$ be a differentiable map and $p \in M$ so that f is a submersion at p. Then we can coordinate neighborhoods as in the above theorem so that

$$(\varphi_2 \circ f \circ \varphi_1^{-1})(x_1,\ldots,x_n,\ldots,x_m) = (x_1,\ldots,x_n) \text{ at all}$$

points of the range of φ_1.

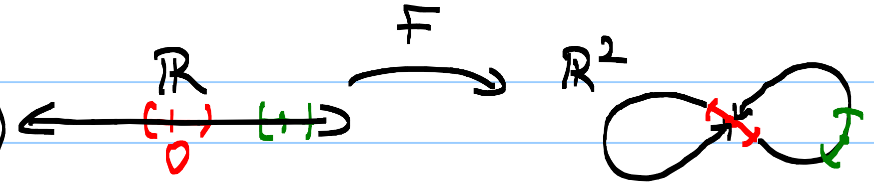
Definition: An imbedding is a one to one immersion $F: N \to M$, which is a homeomorphism of N into M, that is, F is a homeomorphism of N onto its image $\tilde{N} = F(N)$, with its topology as a subspace of M. The image of an imbedding is called an imbedded submanifold.



$F: N \to \tilde{N} \subseteq M$ is a homeomorphism.

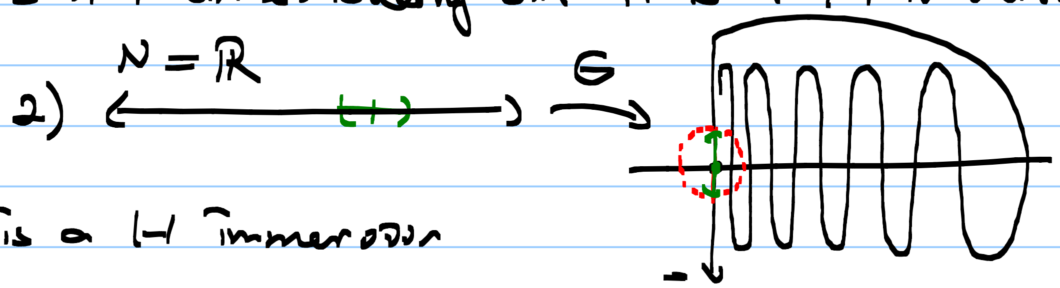
Theorem: Let $F: N \to M$ be an immersion. Then each $p \in N$ has a neighborhood U s.t. that $F|_U : U \to M$ is an embedding of U in M.

Example: 1)



F is not an embedding but it is a 1-1 immersion

2)



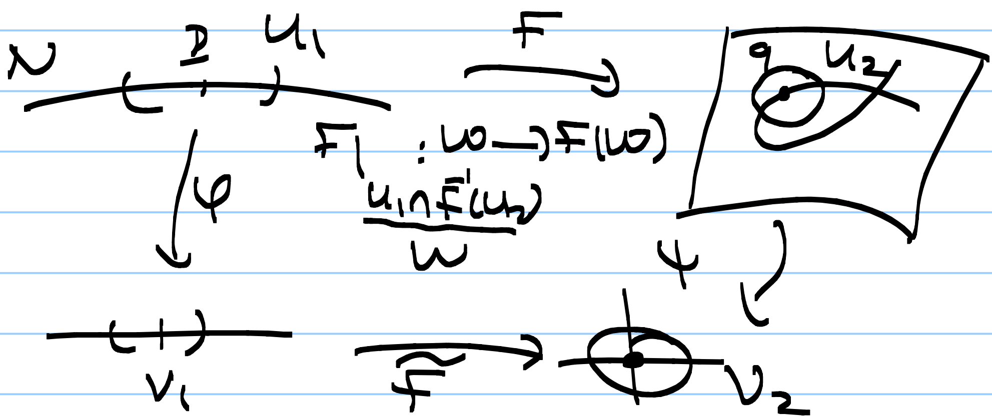
G is a 1-1 immersion

$G(\mathbb{R}) = \tilde{N}$. $N = \mathbb{R}$ is locally connected while $\tilde{N}$ is <u>not</u>. Hence, $G: N \to G(N) = \tilde{N}$ is not a homeomorphism. G is locally an embedding.

Indeed, we have the following result:

<u>Proof</u>: Let $p \in N$. Then by the theorem proved last time there are neighborhoods around $p \in N$ and $q = F(p) \in M$, say $\varphi: U_1 \to V_1$ and $\psi: U_2 \to V_2$ so that

$$(\psi \circ F \circ \varphi^{-1})(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0).$$



Since φ and ψ are homeomorphisms and $\tilde{F} = \psi \circ F \circ \varphi^{-1}$ is a homeomorphism onto its image we are done. ∎

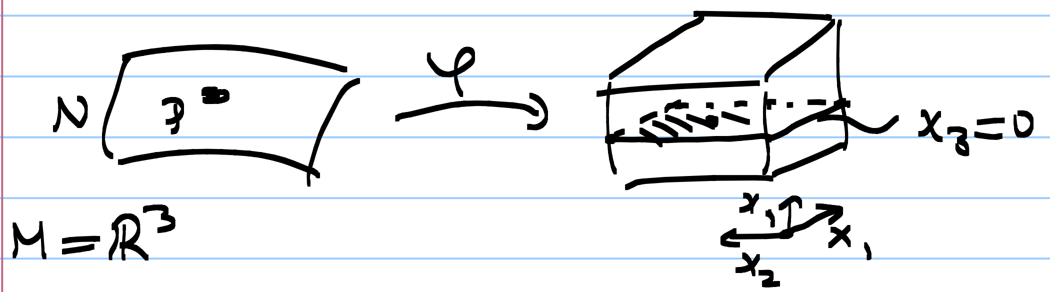
Submanifolds

Definition: A subset N of a C^∞-manifold M is said to have the n-submanifold property if each $p \in N$ has a coordinate neighborhood U, φ on M with local coordinates $x_1, \dots, x_m$ such that

i) $\varphi(p) = (0, \dots, 0)$

ii) $\varphi(U) = C^m_\epsilon(0) = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid |x_i| < \epsilon, \forall i\}$.

iii) $\varphi(U \cap N) = \{x \in C^m_\epsilon(0) \mid x^{n+1} = \dots = x^m = 0\}$.

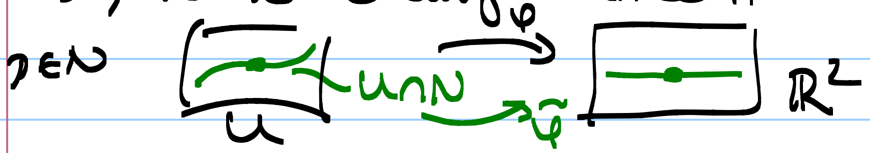


If N has this property, coordinate neighborhoods of this type are called preferred coordinates.

Lemma: Let $N \subseteq M$ have the n-submanifold property. Then N with the relative topology is a topological n-manifold and each preferred coordinate system U, φ of M (relative to N) defines a local coordinate neighborhood $V, \tilde{\varphi}$ on N by $V = U \cap N$, $\tilde{\varphi} = \varphi|_V$. These local coordinates on N are C^∞-compatible whenever they overlap and determine a C^∞-structure on N relative to which $i: N \to M$ is an imbedding.

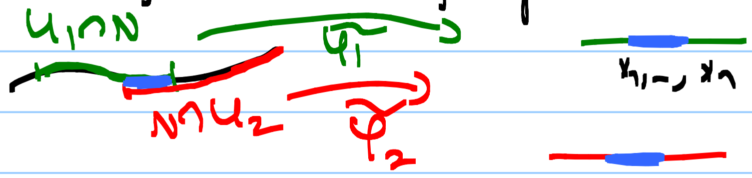
Sketch of proof: 1) Since M is Hausdorff and second countable so is the subspace N.

2) N is locally Euclidean:



Video 10

Hence, N is a topological manifold.



$\varphi_1(x_1, \dots, x_n) = (x_1, \dots, x_n, 0)$
$\varphi_2(y_1, \dots, y_n) = (y_1, \dots, y_n, 0)$

$$(\tilde{\varphi}_1 \circ \tilde{\varphi}_2^{-1}) = (\varphi_1 \circ \varphi_2^{-1})\big|: W \longrightarrow W'$$

$$W \subset C^m_\varepsilon(0) \text{ with } y_{n+1} = \dots = y_n = 0, \quad W' \subset C^m_\varepsilon(0)$$

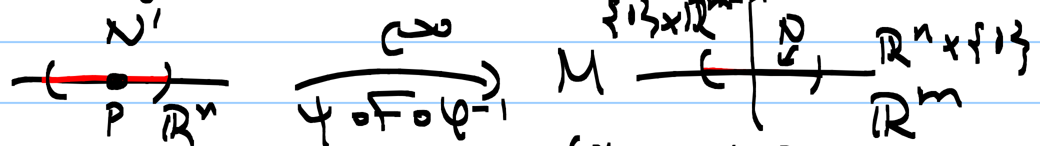
$$y_1, \dots, y_m \qquad\qquad x_1, \dots, x_m$$

The rest is exercise!

Definition: A regular submanifold of a C^∞-manifold M is any subspace N with the submanifold property and with the C^∞ structure that the corresponding preferred coordinate neighborhood determine on it. (Note: N is trivially a submanifold and $i: N \hookrightarrow M$ is an embedding).

Theorem: Let $F: N' \to M$ be an embedding of a C^∞ manifold N' of dimension n in a C^∞-manifold M of dimension m. Then $N = F(N')$ has the n-submanifold property and thus N is a regular submanifold. As such it is diffeomorphic to N' with respect to the mapping $F: N' \to N$.

Sketch of Proof Since F is a local immersion, for any $p \in N'$ there is a neighborhood ψ, V around $q = F(p)$ and φ, U around $p \in N'$ so that F is given as in those coordinates:



$$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_n, 0, \dots, 0)$$

$\overset{\longleftarrow}{C^\infty}$

$\Rightarrow N$ has the n-submanifold property.
$\Rightarrow$ The C^∞-structure of M gives a C^∞-str. on N.
$\Rightarrow N$ is a regular submanifold and $F|_{N'}: N \to N'$ is a diffeomorphism.

Since F is a homeomorphism and locally a diffeomorphism (from N' to N) F is a diffeomorphism from N' to N. $\blacksquare$

<u>Corollary</u> If F is a one to one immersion and N' is compact then $N = F(N')$ is a regular submanifold.
Thus a $\underbrace{\text{submanifold}}_{\text{Image of a 1-1 immersion.}}$ of M, if compact, it is regular.

<u>Proof</u>: We just need to show that F is an embedding (topological). Since $F: N' \to M$ is one to one, continuous, N' is compact and M is Hausdorff we see that F is a homeomorphism onto its image. $\blacksquare$

<u>Example</u>: Consider $\mathbb{RP}^2$ as the quotient

$$\mathbb{RP}^2 = S^2 / p \sim -p \,, \; p \in S^2$$

$\mathbb{RP}^2$ is C^∞-manifold and $\pi: S^2 \to \mathbb{RP}^2$ is a C^∞-map.

$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$.

$$\begin{array}{ccccc} (x,y,z) & & & & \\ p & & S^2 & \xrightarrow{\tilde{f}} & \mathbb{R}^5 \\ \downarrow & & \pi \downarrow & \nearrow f & \\ \{\pm p\} = [p] & & \mathbb{RP}^2 & & \end{array}$$

Let $\tilde{f}(x,y,z) = (x^2, y^2, xy, xz, yz)$.
$\tilde{f}$ is cont. and even. Hence, $\tilde{f}$ induces a cont. map $f: \mathbb{RP}^2 \to \mathbb{R}^5$.

Video 11

Note that f is one to one. Since S^2 is compact so is $\mathbb{RP}^2 = \pi(S^2)$. It follows that f is an imbedding.

<u>f is C^∞</u>: Consider the chart $\mathbb{R}^2 \to \mathbb{RP}^2$ given by

$(u,v) \longmapsto [u:v:1]$. Then f is given in this chart as

$$\underline{(u,v)} \longmapsto \underset{\pi(x,y,z)=[x:y:z]}{[u:v:1]} \longmapsto \underset{\underline{(u^2,v^2,uv,u,v)}}{(x^2,y^2,xy,xz,yz)}$$

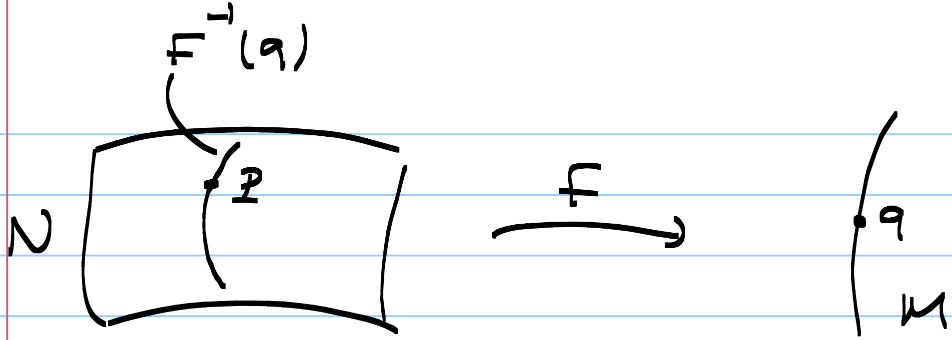
$u = \frac{x}{z}$, $v = \frac{y}{z}$, $\underline{1 = z}$

f is C^∞. Hence, $f(\mathbb{RP}^2)$ is a regular submanifold of $\mathbb{R}^5$. Indeed, $\mathbb{RP}^2$ can be embedded into $\mathbb{R}^4$.

<u>Remark</u>: $\mathbb{RP}^2$ cannot be embedded into $\mathbb{R}^3$.

<u>Theorem</u>: Let N be a C^∞-manifold of dimension n, M a C^∞-manifold of dimension m. Suppose that $F: N \to M$ is a smooth map of constant rank m at all points. Then $F^{-1}(q)$ is a closed, regular submanifold of N of dimension $n-m$.

Indeed a stronger version holds: Let $F: N \to M$ C^∞-map. A point $q \in M$ is called a regular value of F if $F^{-1}(q)$ has no critical points, i.e., a point $p \in F^{-1}(q) \subseteq N$ so that F has rank less than m at p.

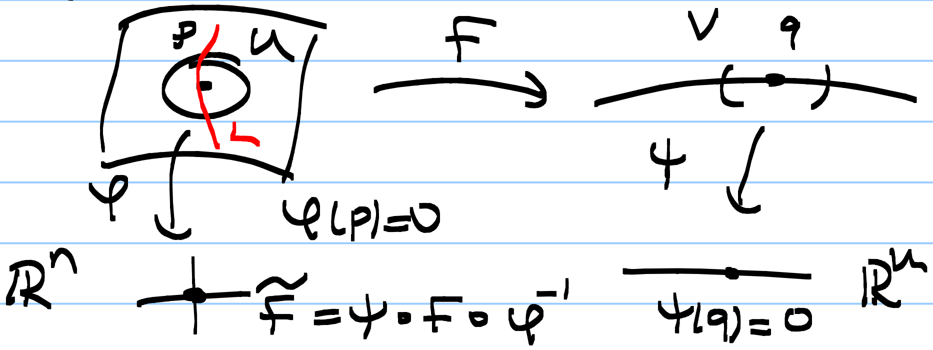


F has rank m at each $p \in F^{-1}(q)$.
$DF(p)$ has rank m.

Lemma: If $F: N \to M$ is a C^∞-map and $q \in M$ is a regular value then $F^{-1}(q)$ is a regular submanifold of N of dimension $n-m$.

Proof: Let $L = F^{-1}(q)$. We need to show that L is a regular submanifold of N of dimension $n-m$.

Let $p \in L$, then by the local description of submersions there are coordinate neighborhoods U, φ around $p \in N$ and V, ψ around $q \in M$ so that



$(x_1, \dots, x_m, \dots, x_n) \longmapsto (x_1, \dots, x_m)$

Then $L \cap U = F^{-1}(q) \cap U$ will look like

$$\tilde F^{-1}(0) = \{(0, \dots, 0, x_{m+1}, \dots, x_n) \mid x_i \in \mathbb{R}\}.$$

Hence, L has the submanifold property of dimension $n-m$. Thus, L is a regular submanifold of N of dimension $n-m$. ∎

Example: 1) $F: \underset{N}{\mathbb{R}^3} \longrightarrow \underset{M}{\mathbb{R}}$, $F(x,y,z) = x^2+y^2+z^2-1$.

$$DF(p) = [F_x \; F_y \; F_z] : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$p=(x,y,z)$
$$= [2x \; 2y \; 2z]$$

If $p \in S^2$, at least one of the coordinates is non zero. Hence, $DF(p)$ is onto. Then, $0 \in \mathbb{R}$ is a regular value.

$$S^2 = \{(x,y,z) \mid x^2+y^2+z^2-1=0\} = F^{-1}(0).$$

Hence S^2 is a regular submanifold of $\mathbb{R}^3$ of dimension $m-n = 3-1 = 2$.

2) $SL(n,\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det A = 1\}$.

Claim: $SL(n,\mathbb{R})$ is a regular submanifold of $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ of dimension n^2-1.

Proof Consider the smooth map

$$F: \mathbb{R}^{n^2} = M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$A \longmapsto F(A) = \det A.$$

$$SL(n,\mathbb{R}) = F^{-1}(1).$$

It is enough to show that 1 is a regular value.
Let $A \in F^{-1}(1) = SL(n,\mathbb{R})$.

$$DF(A) : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$$

Exercise: $DF_{I}(B) = \operatorname{tr} B$.

$$F_{\underline{p}}(v) = \lim_{t\to 0} \frac{F(p+tv)-F(p)}{t}$$

$$DF_{Id}(B) = \lim_{t\to 0} \frac{F(Id+tB)-F(Id)}{t}$$

$$= \lim_{t\to 0} \frac{\det(Id+tB)-\det(Id)}{t}$$

$$= \lim_{t\to 0} \frac{\det(Id+tB)-1}{t} \overset{?}{=} tr(B).$$

Example: $O(n) = \{A \in M_{n\times n}(\mathbb{R}) \mid A^T A = Id\}$.

$O(n)$ is a regular submanifold of $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ of dimension $n(n-1)/2$.

$O(3) = \{A \in M_{3\times 3}(\mathbb{R}) \mid A^T A\} \subseteq \mathbb{R}^9 = M_{3\times 3}(\mathbb{R})$ is a regular submanifold of dimension $3\cdot 2/2 = 3$.

Consider the C^∞ map $\left[\begin{smallmatrix} \cdot & \cdot & \cdot \\ & \cdot & \cdot \\ & & \cdot \end{smallmatrix}\right]$ $\mathbb{R}^{1+2+\dots+n} = \mathbb{R}^{\frac{n(n+1)}{2}}$

$$F : M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2} \longrightarrow \underset{\text{is}}{Sym_{n\times n}(\mathbb{R})} = \{A \in M_{n\times n}(\mathbb{R}) \mid A^T = A\}.$$
$$X \longmapsto X^T X$$

$$F^{-1}(Id) = \{X \in M_{n\times n}(\mathbb{R}) \mid F(X) = X^T X = Id\} = O(n)$$

Hence, it is enough to show that Id is a regular value of F.

$$\dim F^{-1}(Id) = n^2 - \frac{n(n+1)}{2} = n(n-1)/2.$$

Let $X \in F^{-1}(Id)$. We need to compute

$$DF_X : M_{n\times n}(\mathbb{R}) \longrightarrow Sym_{n\times n}(\mathbb{R})$$
$$Y \longmapsto DF_X(Y) = \lim_{t\to 0} \frac{F(X+tY)-F(X)}{t}$$

$$D_XF(Y) = \lim_{t\to 0} \frac{(X+tY)^T(X+tY) - X^TX}{t}$$

$$= \lim_{t\to 0} \frac{\cancel{X^TX} + \cancel{t}X^TY + \cancel{t}Y^TX + t^{\cancel{2}}Y^TY - \cancel{X^TX}}{\cancel{t}}$$

$$= X^TY + Y^TX.$$

<u>must show:</u> D_XF is onto. In other words,

If S is a symmetric $n\times n$ matrix then $S = X^TY + Y^TX$ for some $Y \in M_{n\times n}(\mathbb{R})$.

<u>Exercise</u>: Show that D_XF is onto!

Video 12

Tangent Space of a Manifold

M C^∞-manifold of dimension n. If $U \subseteq M$ is an open subset let $C^\infty(U)$ be the collection of all C^∞-functions on U. For any point $p \in M$ let

$$C^\infty(p) = \bigcup_{\substack{p \in U \subseteq M \\ U \text{ open}}} C^\infty(U) \Big/ f \sim g, \quad \begin{array}{l} f \in C^\infty(U) \\ g \in C^\infty(V) \end{array}$$

$f(x) = g(x)$ for all $x \in W \subseteq U \cap V$, $p \in W$ open

Each equivalence class $[f] \in C^\infty(p)$ is called a germ of a smooth function at p.

Remark: $\mathbb{C}$ $p \in \mathbb{C}$,

$$\mathcal{A}(p) = \bigcup_{p \in U \subseteq \mathbb{C} \text{ open}} \mathcal{O}(U) \Big/ f \sim g, \quad f \in \mathcal{O}(U),\ g \in \mathcal{O}(V)$$

$f(x) = g(x)$, $\forall x \in W \subseteq U \cap V$, $p \in W$

$\mathcal{O}(U)$: analytic function on U.

In this case any equivalence class $[f]$ can be represented by the power series

$$[f] = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (z-p)^n \qquad f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

$\uparrow$ Weierstrass

smooth (C^∞) but has no infinite Taylor expansion

Remark: $C^\infty(p)$ is the ring of local C^∞-functions at $p \in M$. Indeed, $C^\infty(p)$ is an $\mathbb{R}$-algebra.

$$[f] + [g] \doteq [f+g], \quad [f]\cdot[g] = [fg].$$

Definition: We define the tangent space $T_p(M)$ of M at p to be set of all mappings

$X_p : C^\infty(p) \longrightarrow \mathbb{R}$ satisfying for all $\alpha, \beta \in \mathbb{R}$,

$f, g \in C^\infty(p)$ the two conditions:

i) $X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g)$ (linearity)

ii) $X_p(fg) = (X_p f) g(p) + f(p)(X_p g)$ (Leibnitz Rule)

A tangent vector to M at p is any X_p.

Example: $M = \mathbb{R}^n$, take any $v \in \mathbb{R}^n$. Define

$V_p : C^\infty(p) \longrightarrow \mathbb{R}$ as $V_p(f) = \nabla f(p) \cdot v$, the directional derivative of f at p along the vector v. V_p satisfies both conditions of a tangent vector at $p \in \mathbb{R}^n$. Hence, $V_p \in T_p\mathbb{R}^n$.

Consider a coordinate system on $\mathbb{R}^n$, say $x_1, \dots, x_n$. Define the tangent vectors $\partial/\partial x_i|_p \in T_p\mathbb{R}^n$ as follows:

$$\frac{\partial}{\partial x_i}\Big|_p : C^\infty(p) \longrightarrow \mathbb{R}, \quad f \longmapsto \frac{\partial f}{\partial x_i}(p)$$

$$= \lim_{h \to 0} \frac{f(p + h e_i) - f(p)}{h}$$

Note that $\frac{\partial}{\partial x_i}\Big|_p (x_j) = \delta_{ij}$

$x_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ $\qquad$ $e_i = (0, \dots, 1, \dots, 0)$

$p = (p_1, \dots, p_n) \longmapsto x_j(p) = p_j$

Note that this implies that the vectors $\partial/\partial x_i|_p$, $i = 1, \dots, n$, are linearly independent.

If $v = \sum c_i \frac{\partial}{\partial x_i}|_p = 0$ the zero vector

then $0 = v(x_j) = \left(\sum c_i \frac{\partial}{\partial x_i}|_p\right)(x_j) = \sum c_i \delta_{ij} = c_j$
$\Rightarrow c_j = 0 \ \forall j = 1 \to n.$

<u>Lemma:</u> The set $\{\partial/\partial x_i|_p\}$ is a basis for $T_p\mathbb{R}^n$.

<u>Proof:</u> Let $\underline{\underline{v}} \in T_p\mathbb{R}^n$ be any tangent vector.

Let $c_i \doteq v(x_i)$ and let $u = \sum_{i=1}^{n} c_i \, \partial/\partial x_i|_p$.

<u>Claim</u> $v = u$.

Note that the claim finishes proof of the lemma.

<u>Proof of the claim.</u> Let $f \in C^\infty(p)$ be any germ.

Then f has a Taylor expansion

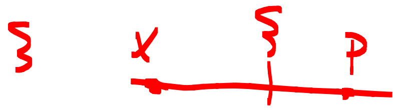
$$f(x) = \underset{\substack{\downarrow \\ 0}}{f(p)} + \sum_i \frac{\partial f}{\partial x_i}(p)\underset{\substack{\downarrow}}{(x_i - p_i)} + \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi)\,(x_i - p_i)(x_j - p_j)$$

$$\sum_i \frac{\partial f}{\partial x_i}(p)\, c_i \qquad \text{for some } \xi.$$

Note that for a derivation X (i.e. a function on $C^\infty(p)$ satisfying (i) and (ii)) we have

$$X(1) = X(1 \cdot 1) = \underbrace{1(p)}_{=1} X(1) + X(1) \underbrace{1(p)}_{=1} = 2X(1)$$
$$\Rightarrow X(1) = 0.$$

Then $X(c) = c\,X(1) = c \cdot 0 = 0.$



Note that $v(f) = \sum_i \frac{\partial f}{\partial x_i}(p)\, c_i + v\Big(\sum_{i,j} \underbrace{\frac{\partial f}{\partial x_i \partial x_j}(\xi)(x-p_i)}_{C^\infty}(x-p_j)\Big)$

$$= \sum_i \frac{\partial f}{\partial x_i}(p)\, c_i + \underbrace{\sum_{i,j} v\Big(\frac{\partial f}{\partial x_i \partial x_j}(\xi)(x-p_i)\Big) \underbrace{(x_j-p_j)(p)}_{0} + \frac{\partial f}{\partial x_i \partial x_j}(\xi)\underbrace{(p_i-p_i)}_{0}\, v(x_j-p_j)}_{=0}$$

$$= \Big(\sum_i c_i \cdot \frac{\partial}{\partial x_i}\Big|_p\Big)(f)$$

$$= u(f). \quad \blacksquare$$

Video 13

Last time: $p \in M$ T_pM: the vector space of all derivations on the $\mathbb{R}$-algebra of germs of smooth functions defined near p.

$C^\infty(p) = \{ f : f: U \to \mathbb{R}, f \in C^\infty(U) \} / \sim$

$(f, U) \sim (g, V) \iff f = g$ on some open $p \in W \subseteq U \cap V$.

$v \in T_pM$, $v : C^\infty(p) \to \mathbb{R}$ st. v is $\mathbb{R}$-linear and satisfies Leibniz Rule.

Ex: $U \subseteq \mathbb{R}^n$ open set $x_1, \ldots, x_n$ coordinate functions on U. $p \in U$.

$\frac{\partial}{\partial x_i}\Big|_p : C^\infty(p) \to \mathbb{R}$, $[f] \longmapsto \frac{\partial f}{\partial x_i}(p)$ is a derivation.

Also we've seen $\{ \frac{\partial}{\partial x_i}\big|_p \mid i = 1, \ldots, n \}$ is a basis for T_pU.

We may define the tangent bundle T_*U as the set

$$T_*U \doteq \bigcup_{p \in U} T_pU \cong U \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n \text{ open subset}$$

$$\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\Big|_p \longleftrightarrow (p, v), \quad v = (v_1, \ldots, v_n)$$

where (p, v) is identified with the directional derivative at p along the vector v.

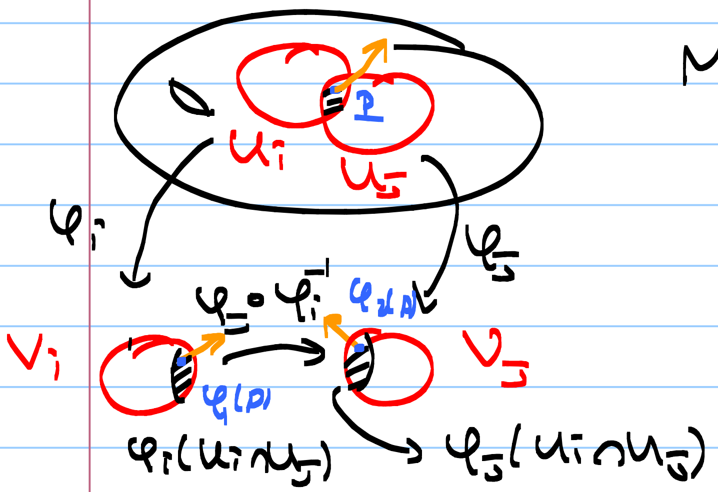
$\underline{\underline{(p, v)}} \longmapsto D_{v,p} : C^\infty(p) \longrightarrow \mathbb{R}$

$\boxed{f} \longmapsto D_v(f)(p) = \nabla f(p) \cdot v$

So, $D_v(f)(p) = \sum_{i=1}^n \left(\frac{\partial}{\partial x_i}\Big|_p (f) \right) v_i = \left(\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\Big|_p \right)(f)$.

Hence we may put the product topology on $U \times \mathbb{R}^n$ to $T_* U$ via the above correspondence. Here, $T_* U$ is homeomorphic to $U \times \mathbb{R}^n$, which is a $2n$-dimensional C^∞-manifold.

Let M be a C^∞ manifold with an atlas
$\{\varphi_i : U_i \to V_i\}_{i \in I}$, $U_i \subseteq M$, $V_i \subseteq \mathbb{R}^n$ open subsets so that $M = \bigcup_{i \in I} U_i$.



$$M = \bigcup_{i \in I} U_i \simeq \dot{\bigcup_{i \in I}} V_i \Big/ p \sim \overbrace{\varphi_j \circ \varphi_i^{-1}}^{\varphi_{ij}}(p)$$
$\forall\, p \in \varphi_i(U_i \cap U_j)$

<u>Exercise</u>: The topology on $\dot{\bigcup} V_i / \sim$ is the quotient topology on the disjoint union of V_i's. Show that it is homeomorphic to M.

We can use this approach to describe manifold structure on $T_* M$.

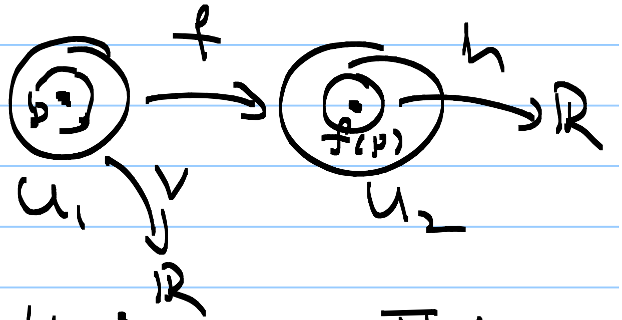
$$T_* M = \bigcup_{p \in M} T_p M \quad \text{as a set.}$$

$$M = \dot{\bigcup} V_i \Big/ \sim \qquad p \in V_i, \; p \sim \varphi_{ij}(p)$$

$$T_* M = \dot{\bigcup} T_* V_i \Big/ (p, v) \sim (\varphi_{ij}(p), D\varphi_{ij}(p)(v))$$
$v \in T_p V_i$

Remark: $\underline{\underline{f: U_1 \to U_2}}$ smooth map of open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

$p \in U_1$, $\underline{\underline{v}} \in T_pU_1$



$$Df_p(v)(h) \doteq v(h\circ f)$$

Here $Df_p: T_pU_1 \to T_{f(p)}U_2$ is a map. This map is linear and if $x_1,\dots,x_n$ and $y_1,\dots,y_m$ are coordinates on U_1 and U_2, respectively, then the matrix representation of the linear map Df_p in the bases $\underbrace{\{\partial/\partial x_i|_p\}}_{=B}$ and $\underbrace{\{\partial/\partial y_j|_{f(p)}\}}_{``B'"}$ is the Jacobian matrix.

$$[Df_p]_B^{B'} = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \qquad f=(f_1,\dots,f_m), \quad f_i:\mathbb{R}^n\to\mathbb{R}$$

$$h: \underset{\substack{\cap\\ \mathbb{R}^m}}{U_2} \to \underline{\underline{\mathbb{R}}}, \quad (h\circ f)(x_1,\dots,x_n) = h(f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n))$$

$$v=\frac{\partial}{\partial x_i}\Big|_p, \quad Df_p(v)(h) = v(\underline{\underline{h\circ f}})$$

$$= \frac{\partial}{\partial x_i}\Big|_p \big(h(\overset{y_1}{\overbrace{f_1(x_1,\dots,x_n)}},\dots,\overset{y_m}{\overbrace{f_m(x_1,\dots,x_n)}})\big)$$

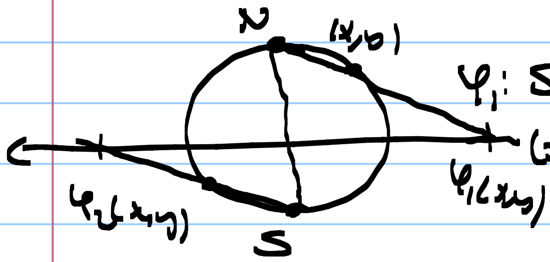
$$= \sum_j \frac{\partial h}{\partial y_j}(f(p)) \; \frac{\partial y_j}{\partial x_i}(p)$$

$$= \sum_j \frac{\partial h}{\partial y_j}(f(p)) \; \frac{\partial f_j}{\partial x_i}(p)$$

$$= \left(\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \vdots & & \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{(p)} \begin{bmatrix} \partial h/\partial y_1 \\ \vdots \\ \partial h/\partial y_m \end{bmatrix}_{f(p)} \right)_i$$

$$\begin{bmatrix} \partial h/\partial y_1 \\ \vdots \\ \partial h/\partial y_m \end{bmatrix}_{f(p)} = \begin{bmatrix} \partial/\partial y_1|_{f(p)} \\ \vdots \\ \partial/\partial y_m|_{f(p)} \end{bmatrix}(h)$$

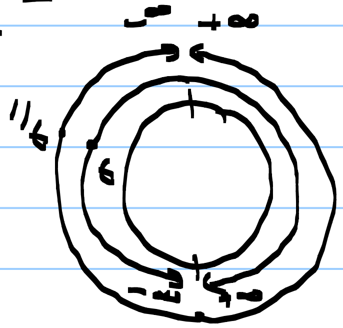
<u>Example:</u> $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$.



$$\varphi_1 : S^1 \setminus N \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \frac{x}{1-y} = t$$

$$\varphi_2 : S^1 \setminus S \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \frac{x}{1+y} =$$

$$\begin{array}{ccc} S^1 \setminus \{S,N\} & \xrightarrow{\varphi_2 \circ \varphi_1^{-1}} & S^1 \setminus \{S,N\} \\ \wr\!\wr & & \wr\!\wr \\ \mathbb{R}\setminus\{0\} & & \mathbb{R}\setminus\{0\} \\ t & \longmapsto & 1/t \end{array}$$



$$\frac{x}{1+y} = t \qquad x^2+y^2=1 \Rightarrow x^2 = 1-y^2 = (1-y)(1+y)$$

$$t = \varphi_1(x,y) = \frac{x}{1-y} = \frac{1+y}{x} = \frac{1}{\varphi_2(x,y)} = \frac{1}{t}$$

$$S^1 = \mathbb{R} \,\dot\cup\, \mathbb{R} \Big/ t \sim \frac{1}{t},\ t \neq 0$$

$$T_* S^1 = T_*\mathbb{R} \,\dot\cup\, T_*\mathbb{R} \Big/ (t,v) \sim \left(1/t, -\frac{1}{t^2}v\right)$$
$$\wr\!\wr \qquad\quad \wr\!\wr$$
$$\mathbb{R}\times\mathbb{R} \qquad \mathbb{R}\times\mathbb{R}$$

Section 14

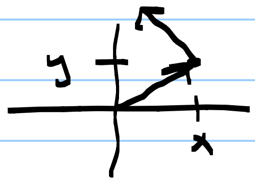
Sections of Tangent Bundle

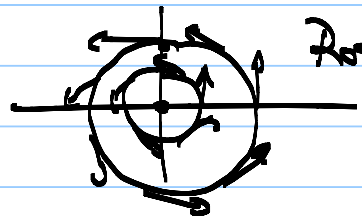
$\pi : T_* M \longrightarrow M$ tangent bundle projection
$(p,v) \longmapsto p$

A section of the $\pi : T_* M \to M$ is a C^∞-map $s : M \to T_* M$ so that $\pi \circ s = \mathrm{Id}_M$.

$$\begin{tikzcd} T_* M \arrow[d, "\pi"] \\ M \arrow[u, bend left, "s"] \end{tikzcd} \qquad s(p) = (p,v) \in T_p M$$

Ex: $M = \mathbb{R}^2$, $s(x,y) = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$



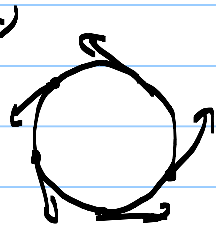


Rotational vector field

Ex $T_* S^1 = \mathbb{R} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{R} / (t,v) \sim (1/t, -\frac{1}{t^2} v)$, with $(t,v) = s_1(t)$, $(1/t, -\frac{1}{t^2}v) = s_2(t)$
$\pi \downarrow \uparrow s_1, s_2$
$S^1 = \underset{t}{\mathbb{R}} \cup \underset{t}{\mathbb{R}} / t \sim 1/t,\ t \neq 0$

$s_1(t) = (t, \underbrace{\tfrac{1+t^2}{2}}_{s_1(t)})$, $s_2(t) = (t, \underbrace{-\tfrac{1+t^2}{2}}_{s_2(t)})$

$$\begin{tikzcd} t \arrow[d, "s_1"'] \arrow[r, leftrightarrow] & 1/t \arrow[d, "s_2"] \\ s_1(t) & s_2(1/t) \end{tikzcd} \qquad s_1(t) \overset{?}{=} s_2(1/t)$$

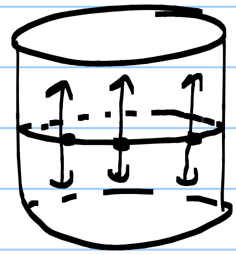


$$s_2(1/t) = -\frac{1+(1/t)^2}{2} = -\frac{1+t^2}{2t^2} = -\frac{1}{t^2} \frac{(1+t^2)}{2}$$
$$= -1/t^2\, s_1(t)$$

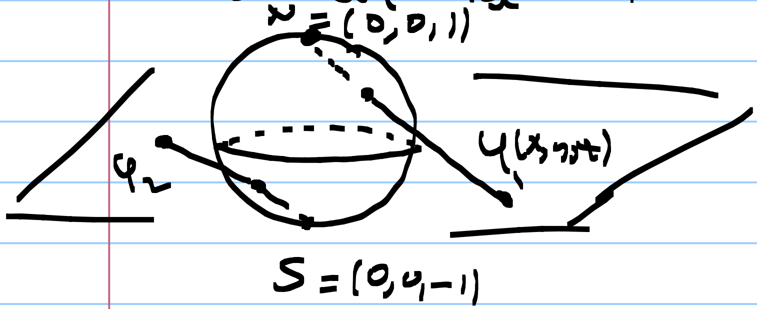
Indeed we get a map $T_x S^1$ to $S^1 \times \mathbb{R}$ as follows

$$T_x S^1 \xrightarrow{\cong} S^1 \times \mathbb{R} \quad \text{(Diffeomorphism)}$$
$$(p,v) \longmapsto (p,\ s_{x_0}(p)/v)$$

$T_x S^1$



What about $T_x S^2$? $S^2 = \mathbb{C}P^1 =$ Riemann Sphere



$$\varphi_1 : S^2 \setminus \{N\} \to \mathbb{R}^2 = \mathbb{C}$$
$$\varphi_2 : S^2 \setminus \{S\} \to \mathbb{R}^2 = \mathbb{C}$$

$$\varphi_1(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right), \quad \varphi_2(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$$
$$z \longmapsto 1/z$$

$$S^2 = \mathbb{C} \,\dot\cup\, \mathbb{C} \big/ z \sim 1/z,\ z \neq 0$$

$$T_x S^2 = \underline{\underline{T_x \mathbb{C}}} \,\dot\cup\, \underline{T_x \mathbb{C}} \big/ (z,w) \sim \left(1/z, -\tfrac{1}{z^2} w\right)$$

Sections: $s_1(z) = \dfrac{1+z^2}{2}, \quad s_2(z) = -\dfrac{1+z^2}{2}$

$s_i(z)$ has exactly two zeros, namely $\pm i$.
So we do not have an apparent diffeomorphism from $T_x S^2$ to $S^2 \times \mathbb{R}^2$.

Video 15

Remark: T_*S^2 is <u>not</u> isomorphic to the product (trivial) bundle $S^2 \times \mathbb{R}^2$.

$$\begin{tikzcd} T_*S^2 \arrow[rr, "F?"] \arrow[dr, "\pi"'] & & S^2 \times \mathbb{R}^2 \arrow[dl] \\ & S^2 & \end{tikzcd}$$

$$(p,v) \mapsto p \qquad (q,w) \mapsto q$$

T_*S^2 is locally trivial as any tangent bundle but <u>not</u> globally trivial.

Example: T_*S^4, $\quad S^1 = \mathbb{R} \,\dot\cup\, \mathbb{R} / t \sim 1/t,\ t \neq 0$

$$S^2 = \mathbb{C} \,\dot\cup\, \mathbb{C} / z \sim 1/z,\ z \neq 0$$

$S^4 = \mathbb{H} \,\dot\cup\, \mathbb{H} / p \sim 1/p,\ p \neq 0,\quad \mathbb{H} = \mathbb{R}^4 = \{a + bi + cj + dk \mid a,b,c,d \in \mathbb{R}\}$

$i \cdot j = k,\ j \cdot k = i,\ k \cdot i = j$
$j \cdot i = -k,\ k \cdot j = -i,\ i \cdot k = -j$

$T_*S^4 = T_*\mathbb{H} \,\dot\cup\, T_*\mathbb{H} / (p,v) \sim \left(\frac{1}{p},\ D\varphi_p(v) = -\frac{1}{p} v \frac{1}{p}\right),\ p \neq 0$

where $\varphi : \mathbb{H} \setminus \{0\} \to \mathbb{H} \setminus \{0\},\ \varphi(p) = 1/p$.

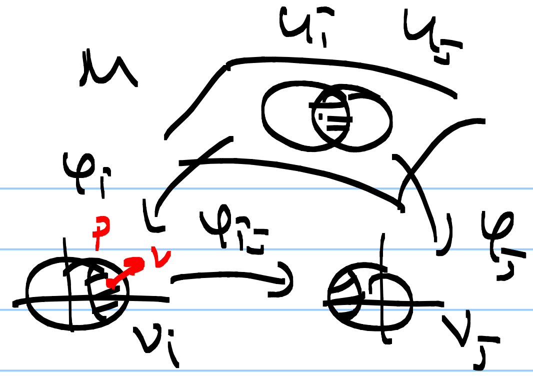
Exercise: Show that $D\varphi_p(v) = -1/p\, v\, 1/p$.

These examples can be generalized to vector bundles as follows:

$M = \cup U_i = \dot\cup V_i / p \sim \varphi_{ij}(p),\ p \in \varphi_i(U_i \cap U_j)$

$\varphi_i : U_i \to V_i$ coordinate maps, $U_i \subseteq M,\ V_i \subseteq \mathbb{R}^n$

and $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$



$$T_* M = \dot{\bigcup} T_* V_i \Big/ (p, v) \sim (\varphi_{ij}(p), D\varphi_{ij}(p)(v))$$

$$p \in \varphi_i(U_i \cap U_j),\ v \in T_p V_i$$

$$T_* V_i = V_i \times \mathbb{R}^n \qquad V_i \subseteq \mathbb{R}^n$$

$$\left(p,\ v_p = \sum a_i(p) \frac{\partial}{\partial x_i}\Big|_p\right)$$

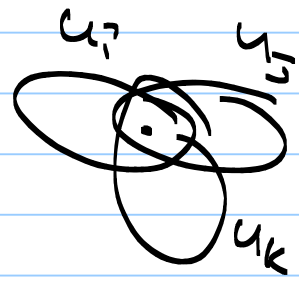
A vector bundle E on M is a map $\pi : E \to M$ so that

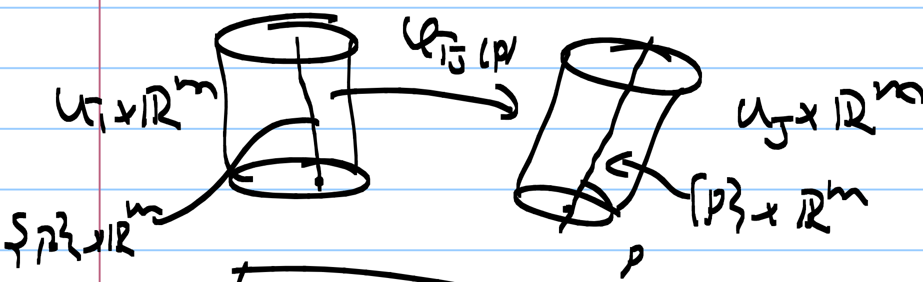
$$E = \dot{\bigcup_i} U_i \times \mathbb{R}^m \Big/ (p, v) \sim (p, \varphi_{ij}(p)(v))$$

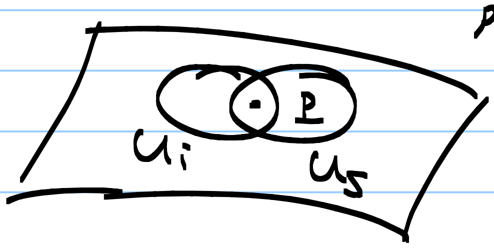
$$p \in U_i \cap U_j$$

$\varphi_{ij}(p) : \{p\} \times \mathbb{R}^m \to \{p\} \times \mathbb{R}^m$ linear isomorphism.

$\pi : E \to M$, $\pi(p, v) = p$.

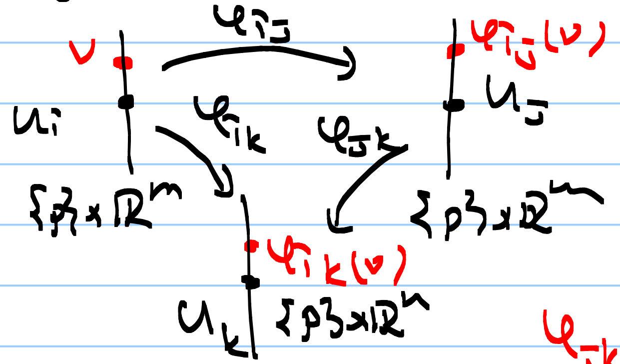






Remark: For the $T_* M$ construction $\varphi_{ij}(p)$ were $D\varphi_{ij}(p)$.

Here the transition functions φ_{ij}'s must satisfy the cocycle condition:



$$\varphi_{jk}(\varphi_{ij}(v)) = \varphi_{ik}(v)$$

Cocycle Condition: $\varphi_{jk}\circ\varphi_{ij} = \varphi_{ik}$ on $U_i\cap U_j\cap U_k$

Cotangent Bundle: This is the dual of the tangent bundle.

More generally, let $E \xrightarrow{\pi} M$ be a vector bundle.

$$E = \dot\bigcup U_i \times \mathbb{R}^m \Big/ (p,v)\sim(p,\underline{\underline{\varphi_{ij}}}(p)(v)) \quad p\in U_i\cap V_j$$

$$V=\mathbb{R}^m,\quad V^* = (\mathbb{R}^m)^* = \mathrm{Hom}(\mathbb{R}^m,\mathbb{R})$$

$L: V \longrightarrow W$ linear $\quad L^*: \mathrm{Hom}(W,\mathbb{R}) \to \mathrm{Hom}(V,\mathbb{R})$

$\varphi_{ij}: U_i\times\mathbb{R}^n \longrightarrow U_j\times\mathbb{R}^n \qquad (T: W\to\mathbb{R}) \mapsto (T\circ L: V\to\mathbb{R})$

$(\varphi_{ij}^{-1})^* = (\varphi_{ji})^*: U_j\times(\mathbb{R}^n)^* \longrightarrow U_i\times(\mathbb{R}^n)^*$

$$E^* = \mathrm{Hom}(E;\mathbb{R}) = \dot\bigcup U_i \times \underset{\mathbb{R}^m}{\underset{\shortparallel}{(\mathbb{R}^m)^*}} \Big/ (p,v^*)\sim(p,\underset{(\varphi_{ij}^{-1})^*}{\underset{\shortparallel}{\varphi_{ji}^*}}(p)(v^*))$$

Example: $T_* S^2 = \mathbb{C}\times\underset{\mathbb{R}^2}{\overset{\shortparallel}{\mathbb{C}}} \cup \mathbb{C}\times\underset{\mathbb{R}^2}{\overset{\shortparallel}{\mathbb{C}}} \Big/ (z,v)\sim(z,-1/z^2\, v) \quad z\in S^2\setminus\{S,N\}$

$S^2 = \mathbb{C}\cup\mathbb{C}$

$U_1 = S^2\setminus\{N\} \quad S^2\setminus\{S\} = U_2$

Video 16

$$\varphi_{12}$$

$$\{z\}\times\mathbb{C} = \{z\}\times\mathbb{R}^2 \longrightarrow \{z\}\times\mathbb{R}^2 = \{z\}\times\mathbb{C}$$

$$v \longmapsto -1/z^2 \cdot v$$

$$\mathbb{C} \cong (\{z\}\times\mathbb{C})^* \xrightarrow{(\varphi_{12}^*)^{-1} = \varphi_{21}^*} (\{z\}\times\mathbb{C})^* \cong \mathbb{C}$$

$$v^* \longmapsto z^2 v^*$$

$$\left(\overline{\frac{1}{z}} = \frac{1}{\bar z} = \frac{z}{\|z\|^2}, \quad \overline{\left(\frac{1}{z^2}\right)} = \frac{1}{\|z\|^4} z^2 \right)$$

This map is "homotopic" to $z \mapsto z^2$ and thus

$$T^*S^2 = \mathbb{C}\times\mathbb{C} \,\dot\cup\, \mathbb{C}\times\mathbb{C} \big/ (p,v)\sim(p, z^2 v)$$

In general, we have the bundle $\mathcal{O}(k) \to S^2 = \mathbb{CP}^1$ defined by

$$\mathcal{O}(k) = \mathbb{C}\times\mathbb{C} \,\dot\cup\, \mathbb{C}\times\mathbb{C} \big/ (z,v)\sim(z, \tfrac{1}{z^k} v)$$

It is known that any "oriented" $\mathbb{R}^2$ bundle (or complex line bundle) over $S^2 = \mathbb{CP}^1$ is $\mathcal{O}(k)$ for some k.

We've seen that $T_* S^2 = \mathcal{O}(2)$ and $T^*S^2 = \mathcal{O}(-2)$.

<u>Exercise:</u> Dual of $\mathcal{O}(k)$ is $(\mathcal{O}(k))^* = \mathcal{O}(-k)$.

<u>Cotangent Bundle of a Smooth manifold</u>

$U \subseteq \mathbb{R}^n$ open subset, $x_1, \dots, x_n$ coordinate functions on U.

$T_* U = U \times \mathbb{R}^n$, $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_{i=1}^n$ is a basis for $T_p U$ for all p.

The dual vector to $\partial/\partial x_i|_p$ in T_pU will be denoted $dx_{i_p} \in T_p^*U$. So we have

$dx_{i_p}(\partial/\partial x_j|_p) = \delta_{ij}$. Moreover, we may write

$T^*U = U \times \mathbb{R}^n$, $\omega \in T^*U$, $\omega(p) = a_1\, dx_1|_p + \dots + a_n dx_n|_p$

$a_i : U \to \mathbb{R}$ a function. An object of this type is called a differential 1-form on U.

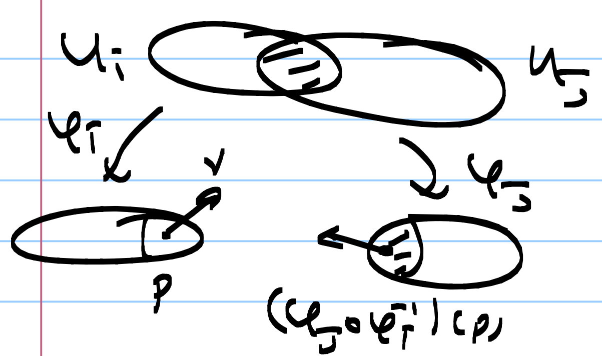
$$\begin{array}{c} T^*U \\ \pi \downarrow \; \uparrow \omega \\ U \end{array} \qquad \omega = \omega(p) = a_1(p)\, dx_1|_p + \dots + a_n(p)\, dx_n|_p.$$

If all $a_i : U \to \mathbb{R}$ are smooth then ω is called a smooth differential 1-form on U.

If M is a smooth manifold and $M = \bigcup U_i$, $\varphi_i : U_i \to V_i$ coordinate charts then

$$T_*M = \bigcup_i T_*U_i \cong \dot{\bigcup} T_*V_i \Big/ (p,v) \sim \left((\varphi_j \circ \varphi_i^{-1})(p), D(\varphi_j \circ \varphi_i^{-1})_p(v)\right)$$

$p \in \varphi_i(U_i \cap U_j)$.

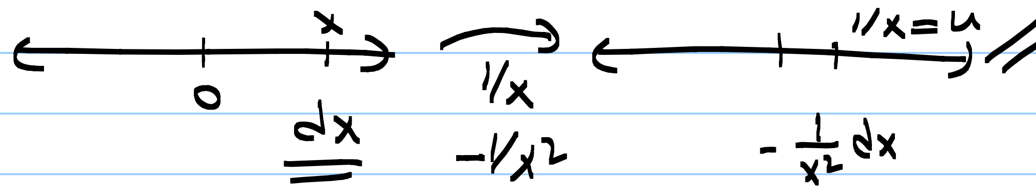


$$T^*M = \bigcup_i T^*U_i = \dot{\bigcup} T^*V_i \Big/ (p,\theta) \sim \left((\varphi_j \circ \varphi_i^{-1})(p), (D(\varphi_j \circ \varphi_i^{-1})_p)^{*}(\theta)\right)$$

Example: $S^1 = \mathbb{R} \cup \mathbb{R} \big/ x \sim 1/x,\ x \neq 0$

$T S^1 = T\mathbb{R} \cup T\mathbb{R} \big/ (x,v) \sim (1/x, -\tfrac{1}{x^2} v),\ x \neq 0$

$T^* S^1 = T^*\mathbb{R} \cup T^*\mathbb{R} \big/ (x,\theta) \sim (1/x, -\tfrac{1}{x^2} v),\ x \neq 0$



$$d(1/x) = -\frac{dx}{x^2} = du$$

We'll get back to differential forms later!

Sard's Theorem: Recall that if $f: \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function a point $p \in \mathbb{R}^n$ is called a critical point if $Df(p): T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ is not onto.

A point $q \in \mathbb{R}^m$ is called a critical value if $f^{-1}(q)$ contains a critical point.

A point $q \in \mathbb{R}^m$ is called a regular value of f if q is not a critical value.

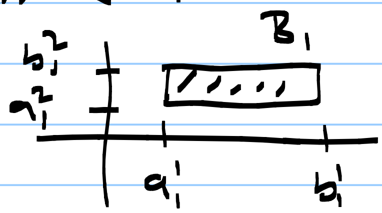
Hence, a point $q \in \mathbb{R}^m$ is a regular value if and only if $Df(p): T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ is onto for any $p \in f^{-1}(q)$.

Note that if $f^{-1}(q) = \emptyset$ (i.e., $q \notin f(\mathbb{R}^n)$) then q is a regular value.

Theorem (Sard) Given any smooth function $f: \mathbb{R}^n \to \mathbb{R}^m$ the set of critical values of f in $\mathbb{R}^m$ has measure zero in $\mathbb{R}^m$.

Let $C \subseteq \mathbb{R}^n$ be the set of critical points of f. Then $f(C)$, the set of critical values has measure zero in $\mathbb{R}^m$. In other words, for any $\epsilon > 0$ there are rectangular boxes $B_1, B_2, \ldots, B_n, \ldots$ in $\mathbb{R}^m$ so that

$$f(C) \subseteq \bigcup_{i=1}^{\infty} B_i \quad \text{and} \quad \sum_{i=1}^{\infty} \operatorname{vol}(B_i) < \epsilon.$$

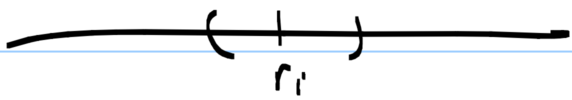


$$B_i = (a_i^1, b_i^1) \times \cdots \times (a_i^m, b_i^m)$$

$$\operatorname{vol}(B_i) = \prod_{k=1}^{m} (b_i^k - a_i^k).$$

Example: $\mathbb{Q}$ in $\mathbb{R}$ has measure zero.

Since $\mathbb{Q}$ is countable we may write $\mathbb{Q} = \{r_1, r_2, \ldots, r_n, \ldots\}$. Given $\epsilon > 0$ let $B_i = (r_i - \epsilon/2^{i+1}, r_i + \epsilon/2^{i+1})$



$$\mathbb{Q} \subseteq \bigcup_{i=1}^{\infty} B_i, \quad \operatorname{vol}(B_i) = \frac{\epsilon}{2^i} \quad i = 1, 2, \ldots$$

$$\sum_{i=1}^{\infty} \operatorname{vol}(B_i) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon.$$

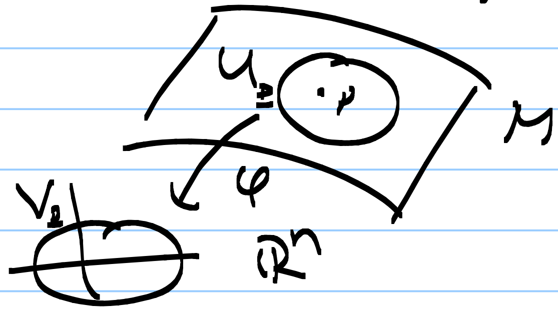
Exercise: Prove Sard's theorem for a function $f: \mathbb{R} \to \mathbb{R}$.

As an application of Sard's Theorem we may prove the following Embedding Theorem.

Theorem (Whitney Embedding Thm)
If M is a compact smooth manifold of dimension n then there is an immersion of M into $\mathbb{R}^{2n}$ and an embedding of M into $\mathbb{R}^{2n+1}$.

Proof: We first embedd M into $\mathbb{R}^N$ for some possible big N.
For any $p \in M$ choose a coordinate map

$\varphi : U_p \to V_p \subseteq \mathbb{R}^n$.



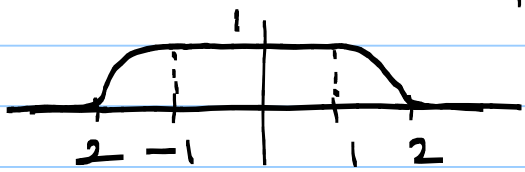
Clearly $\{U_p\}_{p \in M}$ is an open cover for M and thus M is covered by finitely many U_i's, say

$M = U_1 \cup \dots \cup U_k$, $\varphi_i : U_i \to V_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$.

Extend each φ_i to a smooth function on M.

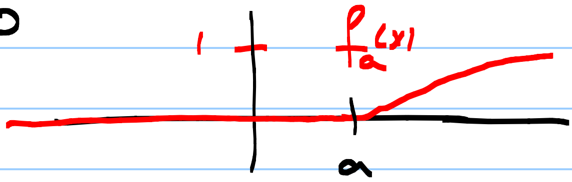
For this consider the smooth function

$\sigma : \mathbb{R} \to \mathbb{R}$

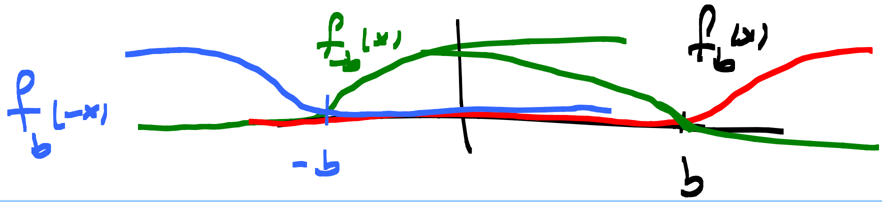


$$f : \mathbb{R} \to \mathbb{R}, \quad f_a(x) = \begin{cases} e^{\frac{-1}{x-a}} & a < x \\ 0 & \end{cases}$$

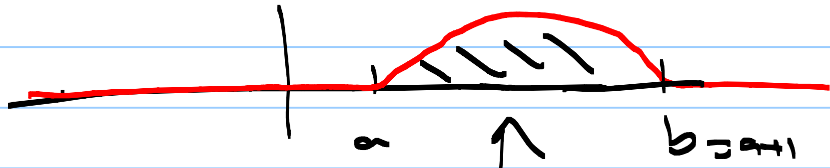
$f \in C^\infty(\mathbb{R})$



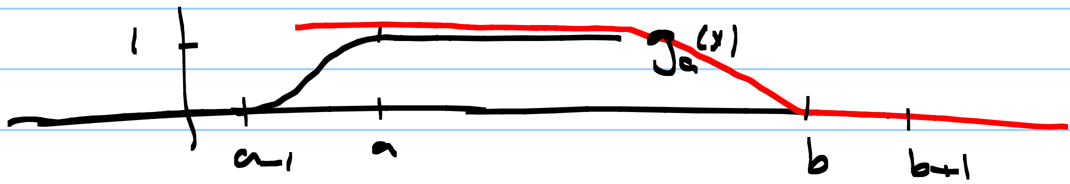
Video 17



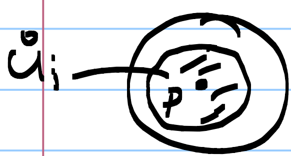
If $a<b$, let $f_{a,b}(x) = f_a(x)\, f_{-b}(-x)$ $\quad f_{-b}(-x)$



$$g_a(x) = \left(\int_{-\infty}^{x} f_{a,a+1}(t)\,dt\right) \Big/ \left(\int_{-\infty}^{\infty} f_{a,a+1}(t)\,dt\right)$$

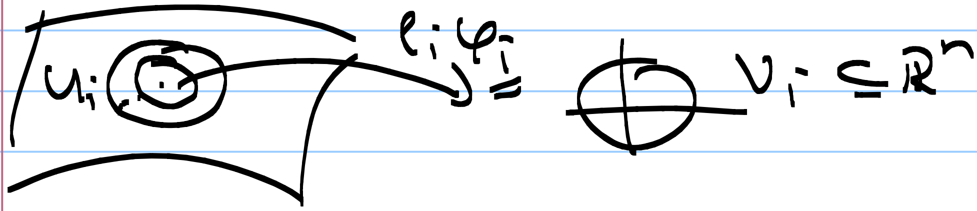


$\rho = g_{a-1}(x)\, g_{-(b+1)}(-x)$ (?)



U_i $\qquad \rho_i \equiv 1$ on $\mathring{U}_i$, $\rho_i \equiv 0$ outside U_i.

$\rho_i : M \longrightarrow \mathbb{R}$ $\qquad$ Then replace φ_i by $\rho_i \varphi_i$.



Now we have map $M \longrightarrow \mathbb{R}^{(n+1)k} = \mathbb{R}^{nk} \times \mathbb{R}^k$

$$p \overset{F}{\longmapsto} (\underset{\mathbb{R}^n}{\varphi_1(p)}, \dots, \underset{\mathbb{R}^n}{\varphi_k(p)}, \underset{\mathbb{R}}{\rho_1(p)}, \dots, \underset{\mathbb{R}}{\rho_k(p)})$$

Each φ_i is an immersion on U_i. Hence this map is an immersion on M. This map is also 1-1.

If $p, q \in M$ so that $p \neq q$. If $p, q \in U_i$ then $\varphi_i(p) \neq \varphi_i(q)$ since φ_i is a homeomorph on U_i. If $p \in U_i$ but $q \notin U_i$ for some i. Then $\rho_i(p) > 0$ but $\rho_i(q) = 0$.

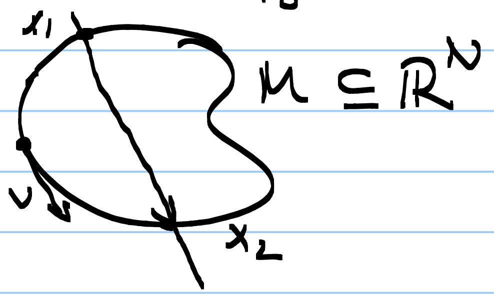
Hence we have an 1-1 immersion $F: M \to \mathbb{R}^N$. Since M is compact F is an embedding.

Now consider two functions

$\Psi_1 : \underline{M \times M} \times \mathbb{R} \longrightarrow \mathbb{R}^N, \ (x_1, x_2, \lambda) \longmapsto \lambda(x_1 - x_2)$

$\Psi_2 : T_x M \longrightarrow \mathbb{R}^N, \ (x, v) \longmapsto v, \ (x,v) \in T_x M \subseteq T_x \mathbb{R}^N \subseteq \mathbb{R}^N$

when we identify M with its image $F(M)$.



If $N \leq 2n+1$ then we are done.

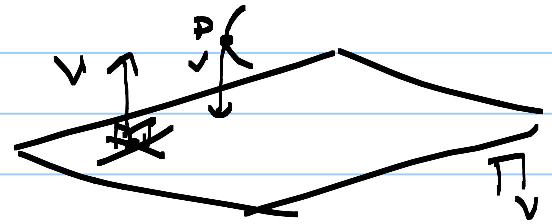
If $\underline{\underline{N}} > 2n+1$ then both tangent maps

$D\Psi_1$ and $D\Psi_2$ cannot be onto. Hence any point of the image is a critical value.
$\Psi_1(M \times M \times \mathbb{R})$ is equal to the critical value of Ψ_1
and $\Psi_2(T_x M)$ is equal to the critical value of Ψ_2.

Any point in $\mathbb{R}^N$ not contained in the images of Ψ_1 and Ψ_2 are regular values for both maps. Hence, by Sard's theorem both Ψ_1

and ψ_2 are not onto. Let $v \in \mathbb{R}^N$ be a regular value (i.e., not in the image of ψ_1 or ψ_2).

Let Π_v be the hyperplane in $\mathbb{R}^N$ perpendicular to the vector v.



Let $\pi : \mathbb{R}^N \longrightarrow \Pi_v \simeq \mathbb{R}^{N-1}$ be the orthogonal projection.

Consider the map $\pi \circ F : M \longrightarrow \Pi_v \simeq \mathbb{R}^{N-1}$.

$\pi \circ F$ is still a 1-1 immersion.
Repeating this process in case $2n+1 < N-1$ we'll finally obtain an embedding into $\mathbb{R}^{n+1}$!
This finishes the proof. ∎

Remark: Note that this process may not give an embedding into $\mathbb{R}^{2n+1}$.

$n=1$, $M = S^1$
$2n+1 = 3$

No projection of the Trefoil onto a plane is an embedding.

Video 1P

Differential Forms:

Let $V \subseteq \mathbb{R}^n$ open subset, $x_1, \ldots, x_n$ coordinates on $\mathbb{R}^n$

$T_x V = V \times \mathbb{R}^n$ $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_{i=1}^n$ sections of $T_x V$.

Any vector field on V has the form

$$F = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i}\Big|_p, \text{ for some smooth functions } a_i : V \to \mathbb{R}.$$

Similarly, $T^*V = V \times \mathbb{R}$ $\{ dx_i|_p \}_{i=1}^n$ sections of T^*V.

Any 1-form ω on V has the form

$$\omega = \sum_{i=1}^n a_i(p)\, dx_i|_p, \text{ for some smooth functions } a_i : V \to \mathbb{R}.$$

To define k-forms and wedge products we need some algebraic preliminaries:

V real vector space of dimension n.
$V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ dual vector space.

If $\mathcal{B} = \{e_1, \ldots, e_n\}$ is a basis for V then $\mathcal{B}^* = \{e_1^*, \ldots, e_n^*\}$ is the dual basis for V^* so that

$$e_i^*(e_j) = \delta_{ij}, \quad i, j = 1, \ldots, n.$$

For any vector spaces V and W their tensor product $V \otimes W$ is defined as follows:

$$V \otimes W = V \times W \Big/ \begin{array}{l} (v_1+v_2, w) - (v_1, w) - (v_2, w) \\ (v, w_1+w_2) - (v, w_1) - (v, w_2) \\ \lambda(v,w) - (\lambda v, w),\ \lambda(v,w) - (v, \lambda w) \\ \text{for any } v_1, v_2 \in V,\ w_1, w_2 \in W \\ \lambda \in \mathbb{R} \end{array}$$

The equivalence class of (v, w) in the quotient vector space is denoted as $v \otimes w$.

Hence, $(v_1+v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$ or

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

$$\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w).$$

Similarly, one can define the tensor products $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ for vector spaces $V_1, \ldots, V_n$.

<u>Symmetric tensor</u>: Let V be a vector space, V^* be its dual and consider

$$(V^*)^{\otimes n} = V^* \otimes \cdots \otimes V^*.$$

If $f \in (V^*)^{\otimes n}$ then f defines a map

$f : V \times \cdots \times V \longrightarrow \mathbb{R}$, as follows:

$$f = \sum_I a_I \, e^*_{i_1} \otimes \cdots \otimes e^*_{i_n} \qquad I = (i_1, \ldots, i_n) \quad a_I \in \mathbb{R}.$$

$(v_1, \ldots, v_n) \in V \times \cdots \times V$, then

$$f(v_1, \ldots, v_n) = \sum_I a_I \, e^*_{i_1}(v_1) \cdots e^*_{i_n}(v_n)$$

<u>Ex</u> $V = \mathbb{R}^3$, $V^* = \langle e_1^*, e_2^*, e_3^* \rangle$

$f = 2e_1^* \otimes e_2^* - 3e_2^* \otimes e_3^*$ 2-tensor

$f((1,0,3),(2,-6,5)) = 2\cdot 1\cdot(-2) - 3\cdot 0\cdot 5 = -4$

A tensor $f \in V^{\otimes n}$ is called symmetric if

$f(v_1, \dots, v_n) = f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) =: (f\circ\sigma)(v_1, \dots, v_n)$ for any

permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$.

A tensor $f \in V^{\otimes n}$ is called alternating if

$f \circ \sigma = \text{sign}(\sigma) f$

The determinant function $\det : \mathbb{R}^n \times \dots \times \mathbb{R}^n \to \mathbb{R}$ is an alternating dual tensor:

$\det : \mathbb{R}^n \times \dots \times \mathbb{R}^n \longrightarrow \mathbb{R}$ $(v_1, \dots, v_n) \longmapsto \det \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

From linear algebra we know that any alternating n-tensor on $\mathbb{R}^n$ is a scalar multiple of the determinant function.

<u>Notation:</u> $e_1, \dots, e_n$ basis for V

$e_1^*, \dots, e_n^*$ dual basis for V^*

$e_1^* \wedge \dots \wedge e_n^*$ will denote the determinant tensor.

$e_1^* \wedge \dots \wedge e_n^*(v_1, \dots, v_n) = \det(a_{ij})$, where

$v_i = \sum_{j=1}^{n} a_{ij} e_j$. $\quad\Big|\quad$ $= \sum_{\sigma \in S_n} \text{sgn}(\sigma)\, e_1^*(v_{\sigma(1)}) \cdots e_n^*(v_{\sigma(n)})$

$V = T_p U$, $U \subseteq \mathbb{R}^n$ open.

$$\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}$$

$V^* = T_p^* U$, $\{ dx_i|_p \}$

An alternating k-tensor on V^* has the form

$$\omega = \sum_{|I| = k} a_I \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Consider these as functions of p we obtain so called differential k-forms on U

$$\omega(p) = \sum_{|I| = k} a_I(p) \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$a_I : U \longrightarrow \mathbb{R}$ smooth function.

__Ex:__ A differential 2-form on $\mathbb{R}^3$ is as follows: x, y, z coordinates on $\mathbb{R}^3$

$$\omega = a(x,y,z)\, dx \wedge dy + b(x,y,z)\, dy \wedge dz + c(x,y,z)\, dz \wedge dx$$

__Remark:__ $dx_i \wedge dx_j = - dx_j \wedge dx_i$ for any i, j. thus, $dx_i \wedge dx_i = 0$, $\forall i$.

Wedge product of a k-form and a l-form.

$$(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \wedge (dx_{j_1} \wedge \cdots \wedge dx_{j_l})$$

$$\doteq dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

Note that if ω_1 is a k-form and ω_2 is an l-form then

$$\omega_1 \wedge \omega_2 = (-1)^{kl} \omega_2 \wedge \omega_1.$$

<u>Exterior Derivative:</u> If $\omega = \sum_I a_I dx_I$,

$I = (i_1, \ldots, i_k)$ multi indices, $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$,

is a differential k-form on $U \subseteq \mathbb{R}^n$ then its exterior derivative is defined to be the differential $k+1$-form on U given as

$$d\omega = \sum_I (da_I) \wedge dx_I, \text{ where}$$

$$da_I = \sum_{i=1}^{n} \frac{\partial a_I}{\partial x_i} dx_i, \text{ is the differential of } f.$$

<u>Proposition:</u> 1) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ if ω is a k-form.

2) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$

3) $d(\lambda \omega) = \lambda \, d\omega$, $\lambda \in \mathbb{R}$.

4) $d^2 = 0$

The vector space of all differential k-forms on U is denoted as $\Omega^k(U)$. So d is a linear map

$$d : \Omega^k(U) \to \Omega^{k+1}(U).$$

Hence, the identity $d^2 = 0$ means that the composition

$$\Omega^k(U) \xrightarrow{d} \Omega^{k+1}(\Omega) \xrightarrow{d} \Omega^{k+2}(U) \text{ is zero.}$$

Video 19

Proof 1) ω k-form and η l-form.

$$d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k\,\omega\wedge d\eta$$

$$\omega = \sum_{|I|=k} a_I\,dx_I\ ,\quad \eta = \sum_{|J|=l} b_J\,dx_J$$

$I=(i_1,\dots,i_k)$ $1\le i_1<i_2<\dots<i_k\le n$, $J=(j_1,\dots,j_l)$
$dx_I = dx_{i_1}\wedge\dots\wedge dx_{i_k}$

$$\omega\wedge\eta = \sum_{\substack{|I|=k\\|J|=l}} a_I\cdot b_J\,dx_I\wedge dx_J$$

It will be enough to prove this for $\omega = a\,dx_I$ and $\eta = b\,dx_J$.

$$d(ab) = \sum \frac{\partial(ab)}{\partial x_i}\,dx_i$$

$$\begin{aligned} d(\omega\wedge\eta) &= d(ab\,dx_I\wedge dx_J)\\ &= d(ab)\wedge dx_I\wedge dx_J\\ &= (b\,da + a\,db)\wedge dx_I\wedge dx_J\\ &= b\,da\wedge dx_I\wedge dx_J + a\,db\wedge dx_I\wedge dx_J\\ &= (da\wedge dx_I)\wedge(b\,dx_J) + (-1)^k\,a\,dx_I\wedge(db\wedge dx_J)\\ &= d\omega\wedge\eta + (-1)^k\,\omega\wedge d\eta \end{aligned}$$

In general

$$\begin{aligned} d(\omega\wedge\eta) &= d\Big(\sum_{I,J} a_I b_J\,dx_I\wedge dx_J\Big)\\ &= \sum_{I,J} d(a_I b_J\,dx_I\wedge dx_J)\\ &= \sum_{I,J} (da_I\,dx_I)\wedge(b_J\,dx_J) + (-1)^k (a_I\,dx_I)\wedge(db_J\wedge dx_J)\\ &= \sum_J d\omega\wedge(b_J\,dx_J) + (-1)^k\sum_J \omega\wedge(db_J\wedge dx_J)\\ &= d\omega\wedge\eta + (-1)^k\,\omega\wedge d\eta. \end{aligned}$$

4) $\omega = \sum_I a_I \, dx_I$ k-form

$$d\omega = \sum_I da_I \wedge dx_I$$
$$= \sum_I \sum_{j=1}^{n} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I$$

$$d^2\omega = \sum_I \sum_{j=1}^{n} d\left(\frac{\partial a_I}{\partial x_j}\right) \wedge dx_j \wedge dx_I$$
$$= \sum_I \sum_{j,l=1}^{n} \frac{\partial^2 a_I}{\partial x_l \partial x_j} dx_l \wedge dx_j \wedge dx_I$$
$$= \sum_I \Big(\sum_{j>l} \frac{\partial^2 a_I}{\partial x_l \partial x_j} dx_l \wedge dx_j$$
$$+ \sum_{j<l} \frac{\partial^2 a_I}{\partial x_l \partial x_j} dx_l \wedge dx_j \Big) \wedge dx_I$$
$$= \sum_I \Big(\sum_{j>l} \frac{\partial^2 a_I}{\partial x_l \partial x_j} dx_l \wedge dx_j - \sum_{j<l} \frac{\partial^2 a_I}{\partial x_l \partial x_j} dx_j \wedge dx_l \Big) \wedge dx_I$$
$$= \sum_I \Big(\sum_{j>l} \frac{\partial^2 a_I}{\partial x_l \partial x_j} dx_l \wedge dx_j - \sum_{l<j} \frac{\partial^2 a_I}{\partial x_j \partial x_l} dx_l \wedge dx_j \Big) \wedge dx_I$$
$$= \sum_I \Big(\sum_{j>l} \underbrace{\Big(\frac{\partial^2 a_I}{\partial x_l \partial x_j} - \frac{\partial^2 a_I}{\partial x_j \partial x_l} \Big)}_{=0} dx_l \wedge dx_j \Big) \wedge dx_I$$
$$= 0. \quad \blacksquare$$

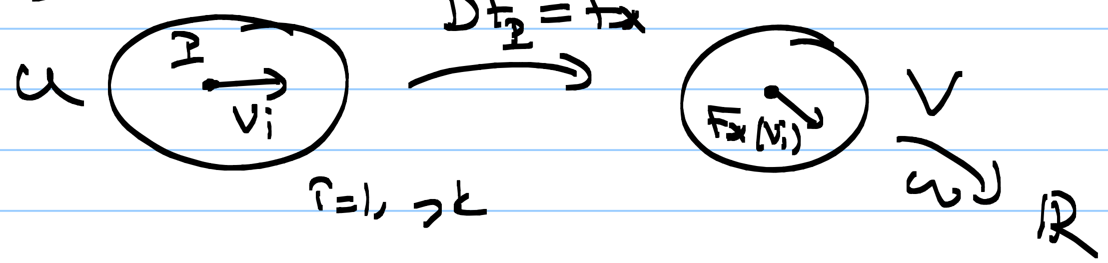
<u>Pull back of differential Forms:</u>

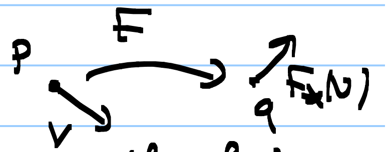
$F: U \to V$, $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ open sub.

Let $\omega \in \Omega^k(V)$ be a k-form on V.

The pull back of ω via F is an k-form on U defined as

$$\underline{\underline{F^*\omega}}(v_1,\dots,v_k) = \omega(F_*(v_1),\dots,F_*(v_k))$$





<u>Ex</u> $\omega = a(x)\, dx_i$

$$F^*(\omega)(v) = \omega(F_*(v))$$

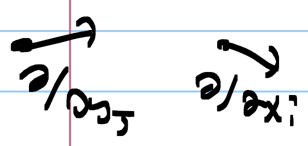
$$= \omega_q(DF_p(v))$$

$F = (f_1,\dots,f_m)$, $F: U \to V$, U: n, V: m; $y_1 \dots y_n$, $x_1 \dots x_n$

$v = \sum v_j \frac{\partial}{\partial y_j}$

$$= a(q)\, dx_i (Df_p(v))$$

$$= a(F(p))\, dx_i \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(p) & \dots & \frac{\partial f_1}{\partial y_n}(p) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial y_1}(p) & \dots & \frac{\partial f_m}{\partial y_n}(p) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$



$$= a(F(p))\, dx_i \left[\sum_{k=1}^{m} \sum_{j=1}^{n} \frac{\partial f_k}{\partial y_j} v_j \frac{\partial}{\partial x_k} \right]$$

$v = \sum v_j \frac{\partial}{\partial x_j}$

$$= a(F(p)) \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j} v_j$$

$$= a(F(p)) \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j} dx_j(v)$$

$$= a(F(p))\, df_i(v)$$

$F = (f_1,\dots,f_m)$

$$F^*(a\, dx_i) = a(F)\, df_i$$

<u>Example:</u> $F: \mathbb{R}^2 \to \mathbb{R}^3$, $F(x_1,x_2) = (x_1+x_2, x_1x_2, x_1^3 + 3x_1x_2+1)$

x_1, x_2 $\quad y_1, y_2, y_3$

Video 20

$$F^*(dy_1) = d(x_1+x_2) = dx_1 + dx_2$$
$$F^*(dy_2) = d(x_1 x_2) = x_2\,dx_1 + x_1\,dx_2$$
$$F^*(dy_3) = d(x_1^3+3x_1x_2+1) = 3x_1^2\,dx_1 + 3x_2\,dx_1 + 3x_1\,dx_2$$

Fact: $F^*(\omega\wedge\eta) = F^*\omega \wedge F^*\eta$. (Exercise!)

Example (Continued)

$$F^*(dy_1\wedge dy_2) = F^*(dy_1)\wedge F^*(dy_2)$$
$$= (dx_1+dx_2)\wedge(x_2\,dx_1 + x_1\,dx_2)$$
$$= (x_1 - x_2)\,dx_1\wedge dx_2$$

Example: $(x,y) = (\underbrace{x(u,v)}_{f_1}, \underbrace{y(u,v)}_{f_2}) = f(u,v)$

$x(u,v) = u/v, \quad y(u,v) = u+v$

$$F^*(dx\wedge dy) = F^*dx\wedge F^*dy$$
$$= d(u/v)\wedge d(u+v)$$
$$= \left(\frac{du}{v} - \frac{u\,dv}{v^2}\right)\wedge(du+dv)$$
$$= \left(\frac{1}{v} + \frac{u}{v^2}\right)du\wedge dv$$

Proposition: 1) $d(F^*\omega) = F^*(d\omega)$ (Exercise!)
2) $F^*(\omega\wedge\eta) = F^*(\omega)\wedge F^*(\eta)$

Remark: $F : \underset{y_1\dots y_n}{\mathbb{R}^n} \longrightarrow \underset{x_1\dots x_n}{\mathbb{R}^n}$ smooth map (coordinate change function). Then

$$F^*(dx_1\wedge\dots\wedge dx_n) = F^*(dx_1)\wedge\dots\wedge F^*(dx_n)$$

$F = (f_1, \dots, f_n)$

$\Rightarrow F^*(dx_1 \wedge \dots \wedge dx_n) = df_1 \wedge \dots \wedge df_n$

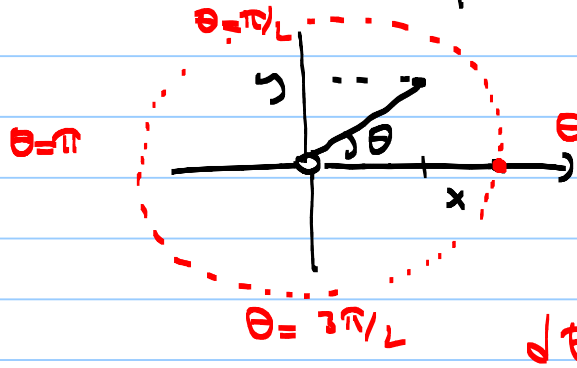
$$= \left(\sum_j \frac{\partial f_1}{\partial y_j} dy_j\right) \wedge \dots \wedge \left(\sum_j \frac{\partial f_n}{\partial y_j} dy_j\right)$$

$$= \det\left(\frac{\partial f_i}{\partial y_j}\right) dy_1 \wedge \dots \wedge dy_n$$

<u>Examples of Forms:</u> 1) $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\})$,

$$\omega = \frac{x\,dy - y\,dx}{x^2+y^2} = d\theta, \quad \theta = \tan^{-1}\frac{y}{x} \in \Omega^0(\mathbb{R}^2 \setminus y\text{-axis})$$

θ is a multivalued function on $\mathbb{R}^2 \setminus \{0,0\}$.

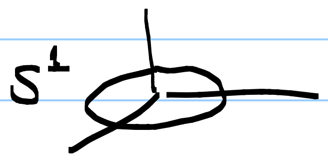


$\theta = 0,\ \theta = 2\pi$

$\theta = \tan^{-1}\frac{y}{x} + \underline{n 2\pi}$

$d\theta = d(\tan^{-1} y/x) + 0$ is now single valued.

2) $\mathbb{R}^3 \setminus \{(x,y,z) \mid x^2+y^2-1 = 0,\ z = 0\}$



$$\omega = \frac{(x^2+y^2-1)\,dz - z\,d(x^2+y^2-1)}{(x^2+y^2-1)^2 + z^2}$$

$\omega \in \Omega^1(\mathbb{R}^3 \setminus S^1)$

<u>Forms on Manifolds:</u>

A k-form on a smooth manifold is a smooth section of $\Lambda^k(T^*M) \longrightarrow M$, where

$\Lambda^k(T^*M)$ can be seen as follows:

$$M = \bigcup_i U_i, \quad U_i \simeq \overset{x_1 \dots x_n}{V_i} \subseteq \mathbb{R}^n \text{ open.}$$

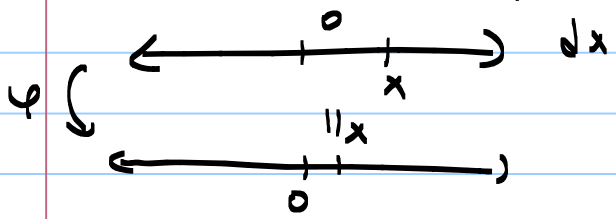
$$T_x M = \bigcup_i T_x U_i, \quad T^*M = \bigcup T^* U_i$$

$$(T^*M)^{\otimes k} = \underset{dx_{i_1}}{T^*M} \underset{\otimes}{\otimes} \cdots \underset{\otimes\, dx_{i_k}}{\otimes T^*M}$$

$$\mathrm{Alt}\left((T^*M)^{\otimes k}\right) \hookrightarrow dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Ex: $S^1 = \mathbb{R} \dot\cup \mathbb{R} / t \sim 1/t \qquad t \neq 0$

Let $\omega = dx$ 1-form on $\mathbb{R}$.



$$\varphi(x) = 1/x$$
$$\varphi^*(dx) = -\frac{1}{x^2} dx$$

$$y = \frac{1}{x} \qquad dy = -\frac{1}{x^2} dx$$

Example: $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy \in \Omega^2(\mathbb{R}^3).$$

$T_p S^2 \subseteq T_p \mathbb{R}^3$, we may identify ω with its restriction to S^2.

$i: S^2 \longrightarrow \mathbb{R}^3$, $\quad i^*(\omega)$: restriction of ω to S^2.

$$\varphi: \mathbb{R}^2 \longrightarrow S^2, \quad \varphi(s,t) = \Big(\underset{\overset{\parallel}{x}}{\frac{2s}{1+s^2+t^2}}, \underset{\overset{\parallel}{y}}{\frac{2t}{1+s^2+t^2}}, \underset{\overset{\parallel}{z}}{\frac{1-s^2-t^2}{1+s^2+t^2}}\Big)$$

$\varphi^*(\omega) = \varphi^*(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy)$

$dx = d\left(\frac{2s}{1+s^2+t^2}\right)$, $dy = d\left(\frac{2t}{1+s^2+t^2}\right)$, $dz = d\left(\frac{1-s^2-t^2}{1+s^2+t^2}\right)$

$$\Rightarrow \varphi^*\omega = \frac{4\, ds \wedge dt}{(1+s^2+t^2)^2} \in \Omega^2(\mathbb{R}^2).$$

~ ∪ ~

All the operations on forms on $\mathbb{R}^n$ can be done on forms on manifolds, such as exterior derivative, pull back, etc.

$$\cdots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \Omega^{k+2}(M) \xrightarrow{d} \cdots$$

$d^2 = 0$

<u>Definition</u>: A k-form $\omega \in \Omega^k(M)$ on a smooth manifold is called closed if $d\omega = 0$.
A k-form $\omega \in \Omega^k(M)$ is called exact if $\omega = d\eta$, for some $k-1$-form $\eta \in \Omega^{k-1}(M)$.

<u>Remark</u>: 1) Since $d^2 = 0$ on M, if $\omega = d\eta$ is exact then $d\omega = d^2\eta = 0$ so that ω is also closed. Hence, any exact form on M is closed.

2) $\omega = \frac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\})$ is closed ($d\omega = 0$) but it is <u>not</u> exact.

$\omega = d\theta$, $\theta = \tan^{-1} y/x$ is <u>not</u> a 0-form on $\mathbb{R}^2 \setminus \{(0,0)\}$.

<u>Definition</u>: For any smooth manifold M the quotient vector space of closed forms by the

Video 21

vector space exact forms is called the kth de Rham cohomology vector space of M.

$$H^k_{DR}(M) = \frac{\text{closed } k\text{-forms on } M}{\text{exact } k\text{-forms on } M} = \frac{Z^k(M)}{B^k(M)}$$

Orientation:

Orientation of Vector spaces / $\mathbb{R}$.

Let V be a finite dim'l real vector space. We define an equivalence relation on the collection of all ordered bases of V as follows:

If $\mathcal{B} = (v_1, \ldots, v_n)$ and $\mathcal{B}' = (w_1, \ldots, w_n)$ are two ordered bases let

$$(a_{ij}) = A = [I]_{\mathcal{B}}^{\mathcal{B}'}$$ be the base change matrix from the base $\mathcal{B}$ to $\mathcal{B}'$. Note that

$$v_i = \sum_{j=1}^{n} a_{ij} w_j \qquad I : V \to V.$$
$$v_{\mathcal{B}} \longmapsto v_{\mathcal{B}'}$$

Then we say that $\mathcal{B}$ and $\mathcal{B}'$ are related if $\det A > 0$. If $\mathcal{B}, \mathcal{B}'$ and $\mathcal{B}''$ are bases for V then note that

$$[I]_{\mathcal{B}}^{\mathcal{B}''} = [I]_{\mathcal{B}'}^{\mathcal{B}''}[I]_{\mathcal{B}}^{\mathcal{B}'}$$ and hence we see

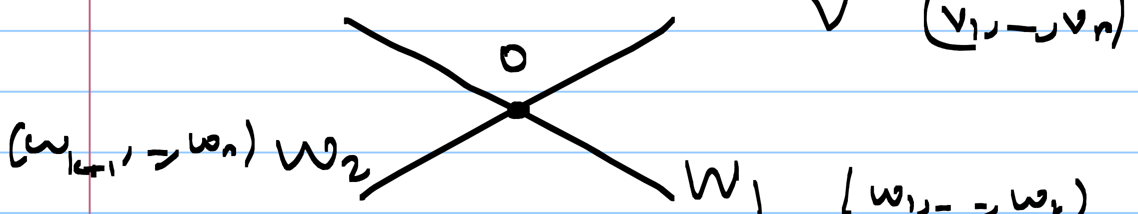
that if $\mathcal{B}$ and $\mathcal{B}'$ are not related any other base $\mathcal{B}''$ is either related to $\mathcal{B}$ or $\mathcal{B}'$. So there are only two equivalence classes. A choice of an equivalence class is called an orientation of V. Hence any real vector space has exactly two orientations.

If $V=(0)$ is the trivial vector space $/\mathbb{R}$ an orientation on V is such a choice of sign $\pm$.

<u>Exercise</u>: Show that $\sim$ is indeed an equivalence relation on the collection of all ordered bases of V.

<u>Orientation of the intersection of oriented subspaces</u>:

Let V be an oriented real vector space. Let W_1 and W_2 be two subspaces of V which are both oriented and $\dim V = \dim W_1 + \dim W_2$ and $W_1 \cap W_2 = \{0\}$.



$\dim W_1 = k$, $\dim W_2 = n-k$.

The orientation of the intersection $\{0\} = W_1 \cap W_2$ is either $+$ or $-$ depending on whether the bases $(v_1,\dots,v_n)$ and $(w_1,\dots,w_k, w_{k+1},\dots,w_n)$ have the same orientation or not.

<u>Example</u>: $V=\mathbb{R}^4 = (e_1,e_2,e_3,e_4)$

$W_1 = \text{span}(e_1,-e_2)$, $W_2 = (e_3,e_4)$, $W_1 \cap W_2 = \{0\}$.

$(e_1,-e_2,e_3,e_4) \overset{?}{\sim} (e_1,e_2,-e_3,e_4)$

No, they are not related?

Here, $W_1 \cap W_2$ has orientation "$-$".

Remark: The orientations of $W_1 \cap W_2$ and $W_2 \cap W_1$ may <u>not</u> be the same.

Indeed, $W_1 \cap W_2 = (-1)^{kl}\, W_2 \cap W_1$

$k = \dim W_1$, $l = \dim W_2$, $\dim V = n = k + l$.

Canonical Orientation of Complex Vector Spaces:

A finite dimensional complex vector space V is a $2n$-dimensional real vector space ($n = \dim_{\mathbb{C}} V$, $2n = \dim_{\mathbb{R}} V$) together with a complex structure, a linear map J of V

$$J : V \to V \quad \text{satisfying } J^2 = -\mathrm{Id}_V.$$

Example: $V = \mathbb{C} = \mathbb{R}^2$ $\quad J : \mathbb{R}^2 \to \mathbb{R}^2$, $J^2 = -\mathrm{Id}$.

$\mathcal{B} = \{(1,0),(0,1)\}$ $\quad J(1,0) = (0,1)$, $J(0,1) = -(1,0)$.

$$[J]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad J^2 \Leftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Theorem: Any finite dimensional complex vector space has a canonical orientation (as considered as a real vector space).

Let V be a complex vector space of dimension n.

$\dim_{\mathbb{R}} V = 2n$, $\quad J : V \to V$, $\quad J^2 = -\mathrm{Id}$.

We'll construct a real ordered basis of the form

$(v_1, Jv_1, v_2, Jv_2, \ldots, v_n, Jv_n)$

Let $v_1 \in V$, $v_1 \neq 0$ (assuming $n > 0$). Let $v_2 = J v_1$.

Claim: v_1 and v_2 are linearly independent.

Proof Let $a_1 v_1 + a_2 v_2 = 0$ be given for some $a_1, a_2 \in \mathbb{R}$. Then

$$J(a_1 v_1 + a_2 v_2) = J(0) = 0$$

$$\Rightarrow a_1 J v_1 + a_2 J v_2 = 0$$

$$\Rightarrow a_1 v_2 + a_2 \underbrace{J(J v_1)}_{-v_1} = 0 \Rightarrow a_1 v_2 - a_2 v_1 = 0.$$

Then we have the system

$$\begin{aligned} a_1 v_1 + a_2 v_2 &= 0 \\ -a_2 v_1 + a_1 v_2 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $v_i \neq 0$, $\det \begin{bmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{bmatrix} = 0 \Rightarrow a_1^2 + a_2^2 = 0$

So $a_1 = a_2 = 0$.

If $\dim V \geq 3$ then there is a vector $v_3 \in V$ with $v_3 \notin \operatorname{span}\{v_1, v_2\} = \operatorname{span}\{v_1, J v_1\}$.

Define v_4 as $v_4 = J v_3$.

Claim: The set $\{v_1, v_2, v_3, v_4\}$ is linearly independent.

Proof Suppose that $a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0$ for some real numbers $a_1, \dots, a_4$.

$$\begin{array}{rl} a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0 & \Big/ \; a_3 \\ -a_2 v_1 + a_1 v_2 - a_4 v_3 + a_3 v_4 = 0 & \Big| \; -a_4 \\ \hline + & \end{array}$$

$$(a_1 a_3 + a_2 a_4) v_1 + (a_2 a_3 - a_1 a_4) v_2 + (a_3^2 + a_4^2) v_3 = 0.$$

Since v_1, v_2, v_3 are linearly independent we see

Video 22

that $a_3^2 + a_4^2 = 0 \Rightarrow a_3 = a_4 = 0$. This implies

$a_1 v_1 + a_2 v_2 = 0$ and hence $a_1 = a_2 = 0$.

Iterating this process n times gives the required basis. $\blacksquare$

This gives an oriented basis for (V, J) of the form

$(v_1, Jv_1 = v_2, v_3, Jv_3 = v_4, \ldots, v_{2n-1}, Jv_{2n-1} = v_{2n})$.

Claim: If $(u_1, Ju_1 = u_2, \ldots, u_{2n-1}, Ju_{2n-1} = u_{2n})$ is another such basis then the bases induce the same orientation

Proof: We know that $\beta = \{v_1, v_3, \ldots, v_{2n-1}\}$ and $\beta' = \{u_1, u_3, \ldots, u_{2n-1}\}$ are two complex bases for the complex vector space V.

Let $A = [I]_{\beta}^{\beta'} \in \mathbb{C}^{n \times n}$ be the base change matrix. If $A = (a_{ij})$ set

$$\alpha_{ij} = \begin{pmatrix} Re(a_{ij}) & -Im(a_{ij}) \\ Im(a_{ij}) & Re(a_{ij}) \end{pmatrix}.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ & & a_{ij} & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix} \longleftrightarrow \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & & \alpha_{1n} \\ \vdots & & & \\ & & \alpha_{ij} & \\ \alpha_{n1} & \alpha_{n2} & & \alpha_{nn} \end{pmatrix}$$

$$\parallel$$

$$\alpha = [I]_{\{v_1, v_2, \ldots, v_{2n}\}}^{\{u_1, u_2, \ldots, u_{2n}\}}$$ must show: $\det(\alpha) > 0$.

Ex $A = (a) = (\lambda + i\mu)$ $\quad \alpha = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$ $\det \alpha = |a|^2 > 0$

Note that interchanging two rows of A corresponds to interchanging two rows of α with another two rows of α so that this operation does not change the sign of $\det \alpha$.

Also multiplying a row of A with a complex scaler $\xi = \lambda + i\mu$ and adding to another row corresponds to multiplying an array of 2 rows of α by the 2×2-matrix $\begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$ and adding to another array of 2-rows of α.

This way A will transform to a matrix of the form and α will transform to a matrix of the form

$$A \sim \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \; d_n \end{pmatrix} \qquad d_j = \epsilon_j + \theta_j i \qquad \alpha \sim \begin{pmatrix} \ddots & & \\ & \begin{matrix} \epsilon_j & -\theta_j \\ \theta_j & \epsilon_j \end{matrix} & \\ & & \ddots \end{pmatrix}$$

$$\det \alpha = \prod_{j=1}^{n} |d_j|^2 = \prod_{j=1}^{n} (\epsilon_j^2 + \theta_j^2) > 0.$$

Hence, any complex vector space has a canonical orientation.

Orientation of Manifolds:

A manifold M is called orientable if

each tangent space can be oriented coherently in the following sense.

For any $p \in M$ there is a coordinate chart $\varphi : U \to V \subseteq \mathbb{R}^n_{x_1,\dots,x_n}$ so that

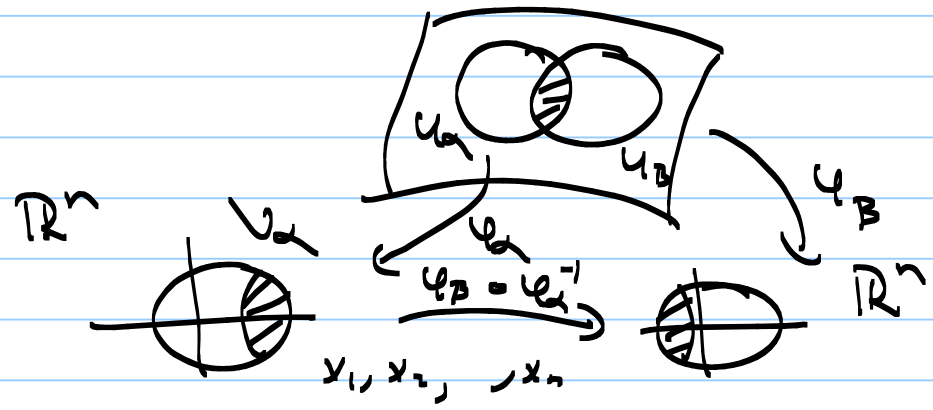
$\varphi^{-1}_* : T_{\varphi(p)} V \to T_p M$ the ordered basis

$$\left\{ (\varphi^{-1})_*\left(\frac{\partial}{\partial x_1}\right), \dots, (\varphi^{-1})_*\left(\frac{\partial}{\partial x_n}\right) \right\}$$ is the

orientation chosen at $T_p M$.

Proposition: Note that M is orientable if and only if there is an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ of M so that

$\det(\varphi_\beta \circ \varphi_\alpha^{-1})_* > 0$, when each V_α is considered as an open set in a fixed $\mathbb{R}^n$ with coordinates $x_1, \dots, x_n$.



$\det((\varphi_\beta \circ \varphi_\alpha^{-1})_*) > 0$ at all points.

Corollary: Any connected smooth manifold has either two orientations or it is not orientable.

Proposition: A smooth connected n-manifold M is orientable if and only if there is an n-form ν on M so that $\nu(p) \neq 0$ at any point.

Example: $\mathbb{R}^n$, $\nu = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$

Proof: Suppose that there is a nowhere zero n-form ν on M. Then we orient each coordinate chart as follows: Take any coordinate chart

$$\varphi : U \longrightarrow V \subseteq \mathbb{R}^n \quad \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$$

$\left(\varphi_*^{-1} \frac{\partial}{\partial x_1}, \ldots, \varphi_*^{-1} \frac{\partial}{\partial x_n}\right)$ ordered basis for T_pM.

$\nu\left(\varphi_*^{-1} \frac{\partial}{\partial x_1}, \ldots, \varphi_*^{-1} \frac{\partial}{\partial x_n}\right) \neq 0$ and hence either

>0 or <0 at all points of U.

If ">0" then we keep $\varphi : U \to V$. If "<0" then replace $\varphi : U \to V$ by the composition

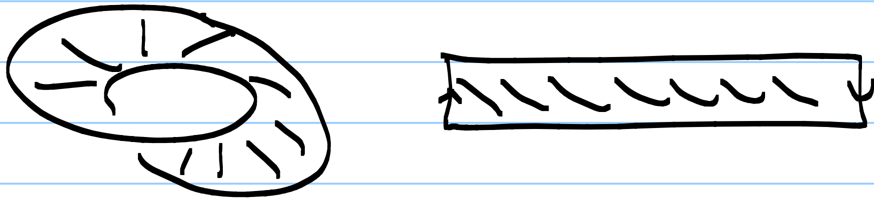
$$U \xrightarrow{\varphi} \underset{\mathbb{R}^n}{\underset{\cap}{V}} \xrightarrow{(x_1, \ldots, x_n) \longmapsto (x_1, \ldots, -x_n)} \underset{\mathbb{R}^n}{V'}$$, call φ'.

Now for the new atlas all values $\nu(\varphi_*^{-1}(\partial/\partial x_1), \ldots, \varphi_*^{-1}(\partial/\partial x_n)) > 0$, for each

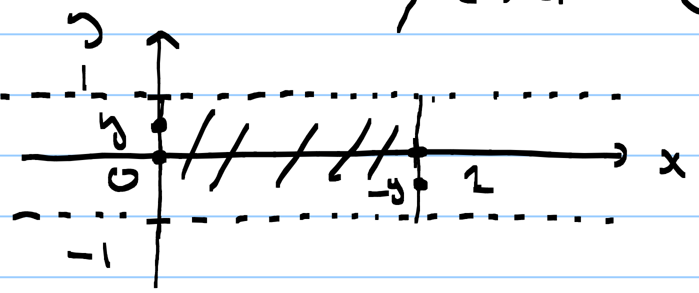
coordinate chart.

Conversely, if M is orientable then we may construct a nowhere zero n-form ν as using so called partition of unity, which we postpone for the time being. ∎

<u>Example:</u> Möbius Band is <u>not</u> orientable.

MB 

$$MB = \mathbb{R} \times (-1,1) \Big/ (x,y) \sim (x+1,-y)$$



Assume MB is orientable. Then there is a nowhere zero two form μ on MB.

$$P : \mathbb{R} \times (-1,1) \longrightarrow \underline{MB}.$$

Let $\omega = P^* \mu$, which is also nowhere zero since P is locally a diffeomorphism.

So, $\omega = P^*\mu = f(x,y)\, dx \wedge dy$ for some smooth function $f : \mathbb{R} \times (-1,1) \longrightarrow \mathbb{R} \setminus \{0\}$.

Consider the map $\sigma : \mathbb{R} \times (-1,1) \longrightarrow \mathbb{R} \times (-1,1)$ given by $\sigma(x,y) = (x+1,-y)$.

Then $P = P \circ \sigma$. Thus $P^* \mu = (P \circ \sigma)^* \mu$

Video 23

$$\omega = \sigma^*(\rho^* p) = \sigma^* \omega$$

$\sigma(x,y) = (x+1, -y)$

$\sigma^*(dx) = dx, \quad \sigma^*(dy) = -dy$

$$f(x,y)\, dx \wedge dy = \omega = \sigma^*\omega = (f\circ\sigma)(\sigma^* dx \wedge \sigma^* dy)$$

$$\Rightarrow f(x,y)\, \underline{dx \wedge dy} = f(x+1,-y)\, dx \wedge (-dy)$$
$$= -f(x+1,-y)\, dx \wedge dy.$$

$(\partial/\partial x, \partial/\partial y)$

$f(x,y) = -f(x+1,-y)$ for all $(x,y) \in \mathbb{R}\times(-1,1)$

$f(1,1) = -f(2,-1)$ is a contradiction to the fact $f(x,y) > 0$ $\forall (x,y)$ or $f(x,y) < 0$ for all (x,y).

Special Forms on Complex Manifolds:

$\mathbb{C}^n$: $\quad \mathbb{C}^n = \mathbb{R}^{2n}$ $\quad z_k = x_k + i y_k, \quad k = 1, \ldots, n.$

$$\omega = \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k, \quad \begin{array}{l} dz_k = dx_k + i\, dy_k \\ d\bar{z}_k = dx_k - i\, dy_k \end{array}$$

$$dz_k \wedge d\bar{z}_k = (dx_k + i\, dy_k) \wedge (dx_k - i\, dy_k)$$
$$= -2i\, dx_k \wedge dy_k$$

$$\frac{i}{2}\, dz_k \wedge d\bar{z}_k = dx_k \wedge dy_k.$$

$$\Rightarrow \omega = \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k = \sum_{k=1}^{n} dx_k \wedge dy_k.$$

Note that $\omega = d\left(\sum_{i=1}^{n} x_k dy_k\right)$, since

$d(x_k dy_k) = dx_k \wedge dy_k$. Hence, $d\omega = d\left(d\left(\sum_{k=1}^{n} x_k dy_k\right)\right) = 0$,

so that ω is a closed two form.

ω is a non-degenerate two form: If $u \neq 0$ vector in $\mathbb{R}^{2n}$ then there is another $v \in \mathbb{R}^{2n}$ so that $\omega(u,v) \neq 0$. For example, if $u_1 \neq 0$ then choose $v = (\underset{=}{0}, \underset{=}{1}, 0, \ldots, 0)$. Then

$$\omega(u,v) = \left(\sum_{k=1}^{n} dx_k \wedge dy_k\right)(u,v) = u_1 \neq 0$$

Note that $\omega^n = \omega \wedge \omega \wedge \ldots \wedge \omega$

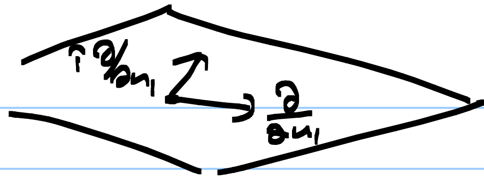
$$= \left(\sum_{k=1}^{n} dx_k \wedge dy_k\right) \wedge \ldots \wedge \left(\sum_{k=1}^{n} dx_k \wedge dy_k\right)$$

$$= n!\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n$$

$\omega^n > 0$.

<u>Definition</u>: Let M^{2n} be a smooth manifold of dimension $2n$. If $\omega \in \Omega^2(M^{2n})$ is a closed non degenerate form on M^{2n} then ω is called a symplectic two form on M and the pair (M, ω) is called a symplectic manifold.

<u>Lemma</u>: Assume $\mathbb{C}^n$ and ω are as above. Let $V \subseteq \mathbb{C}^n$ be a complex l-dim'l subspace. Then considered as a real $2l$-dim'l subspace of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with its preferred complex orientation $\omega^l(\partial/\partial u_1, i\,\partial/\partial u_1, \ldots, \partial/\partial u_l, i\,\partial/\partial u_l) > 0$ on V.

$\mathbb{C}^n$  $V \cong \mathbb{C}^\ell \subseteq \mathbb{C}^n$

Choose complex coordinates $w_1, \dots, w_\ell$ on V. Then
If $z_1, \dots, z_n$ are as above (coor. on $\mathbb{C}^n$)
we have

$$(z_1, \dots, z_j, \dots, z_n) = L(w_1, \dots, w_\ell)$$
$$= (a_{11}w_1 + \cdots + a_{\ell 1}w_1, \dots, a_{1n}w_1 + \cdots + a_{\ell n}w_n)$$

$$\bar z_j = \bar a_{1j}\bar w_1 + \cdots + \bar a_{\ell j}\bar w_\ell \qquad L : V \cong \mathbb{C}^\ell \longrightarrow \mathbb{C}^n$$

$$L^*(dz_j) = \sum a_{kj}\, dw_k \qquad \omega \in \Omega^2(\mathbb{C}^n)$$

$$L^*(d\bar z_j) = \sum \bar a_{kj}\, d\bar w_k$$

$$L^*(\omega) = L^*\Big(\frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar z_j\Big)$$

$$L^*(\omega^\ell) = ?$$

$$L^*(dz_{k_1}\wedge d\bar z_{k_1}\wedge \cdots \wedge dz_{k_\ell}\wedge d\bar z_{k_\ell})$$
$$= \overline{\det(A_{k_1 \dots k_\ell})}\,(\det(A_{k_1 \dots k_\ell}))\, dw_1 \wedge d\bar w_1 \wedge \cdots \wedge dw_\ell \wedge d\bar w_\ell$$
$$= |\det(A_{k_1 \dots k_\ell})|^2\, dw_1 \wedge d\bar w_1 \wedge \cdots \wedge dw_\ell \wedge d\bar w_\ell$$

where $A_{k_1 \dots k_\ell}$ is the $\ell \times \ell$ matrix consisting of the column $k_1, \dots, k_\ell$ of A.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{\ell 1} & a_{\ell 2} & & a_{\ell n} \end{pmatrix}_{\ell \times n}$$

$$\Rightarrow L^*(\omega^\ell)\Big(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial u_\ell}, \frac{\partial}{\partial v_\ell}\Big) > 0.$$

complex oriented real basis for V

Video 24

Integration on Manifolds:

$\omega \in \Omega^n_c(\mathbb{R}^n)$ compactly supported n-form,

say $\omega = f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$, $x_1, \ldots, x_n$ chosen coordinate system on $\mathbb{R}^n$. Then the integral of ω over $\mathbb{R}^n$ is defined as the Riemann integral

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

$\operatorname{supp}(\omega) = \overline{\{x \in \mathbb{R} \mid \omega(x) \neq 0\}}$, ω is called compactly supported if $\operatorname{supp}(\omega)$ is compact in $\mathbb{R}^n$.

Remark: If $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ diffeomorphism say

$$\varphi(y_1, \ldots, y_n) = (g_1(y_1, \ldots, y_n), \ldots, g_n(y_1, \ldots, y_n)) = (x_1, \ldots, x_n).$$

Then $\varphi^*\omega = \varphi^*(f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n)$

$$= \varphi^*(f)\,(y_1, \ldots, y_n)\ \varphi^*(dx_1 \wedge \ldots \wedge dx_n)$$

$$= (f \circ \varphi)\, d(\varphi^*(x_1)) \wedge \cdots \wedge d(\varphi^*(x_n))$$

$$= (f \circ \varphi)\, dg_1 \wedge \cdots \wedge dg_n$$

where $dg_i = \sum_j \frac{\partial g_i}{\partial y_j} dy_j$,

$$= (f \circ \varphi) \det\left(\frac{\partial(g_1, \ldots, g_n)}{\partial(y_1, \ldots, y_n)}\right) dy_1 \wedge \ldots \wedge dy_n.$$

So $$\int_{\mathbb{R}^n} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = \int_{\mathbb{R}^n} (f \circ \varphi)(y_1, \ldots, y_n) \det\left(\frac{\partial(g_1, \ldots, g_n)}{\partial(y_1, \ldots, y_n)}\right) dy_1 \cdots dy_n$$

provided that $\det\left(\frac{\partial(s_1,\dots,s_n)}{\partial(y_1,\dots,y_n)}\right) > 0$ at all points.

So $\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} \varphi^* \omega$, when $x_1,\dots,x_n$ and $y_1,\dots,y_n$ induce orien-
$x_1 \dots x_n$ $\quad y_1 \dots y_n$ tations on both $\mathbb{R}^n$'s.
$\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)\left(\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_n}\right)$

To define integration on manifolds we need so called a partition of unity subordinate to an open cover of a manifold.

Definition: Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in A}$ be an open cover for M. The a family of C^∞-funtions $\{f_\lambda : M \to [0,1]\}_{\lambda\in\Lambda}$ satisfying the conditions below is called a partition of unity subordinate to the open cover $\{U_\alpha\}$:

1) For any $\lambda \in \Lambda$ there is some $\alpha \in A$ so that $\{p\in M \mid f_\lambda(p) \neq 0\} \subseteq U_\alpha$.

2) For any $p \in M$ there is some open subset $p \in V$ in M so that the number of λ's with $V \cap \{p \in M \mid f_\lambda(p) \neq 0\} \neq \emptyset$ is finite.

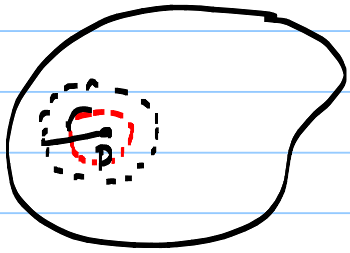
3) For any $p\in M$, $\sum_{\lambda\in\Lambda} f_\lambda(p) = 1$.

Theorem: If M is a smooth manifold and $\{U_\alpha\}$ is an open cover for M then there is a partition of unity subordinate to this covering.

Sketch of proof for the compact case:

$\rho : \mathbb{R} \to [0,1]$

$U \subseteq \mathbb{R}^n$ open, $p \in U$. Choose a ball $B(p,r) \subseteq U$.



$$\rho\left(2\frac{|x-p|}{r}\right) = \begin{cases} 1 & \text{on } B(p,\frac{r}{2}) \\ 0 & \text{on } U \setminus B(p,r) \end{cases}$$

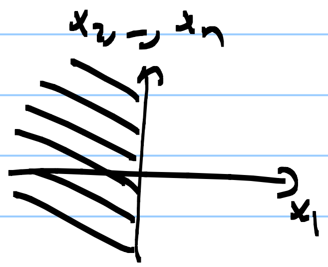
Now if M is a compact manifold for any $p \in M$ choose a coordinate system $\varphi_p : M \to V_p$ and $\mathring{\rho}_p : M \to [0,1]$ so that $\mathring{\rho}_p \equiv 1$ on an open subset $\tilde{U}_p$ in U_p containing p and $\mathring{\rho}_p \equiv 0$ on $M \setminus U_p$.

Since M is compact and the collection $\{\tilde{U}_p\}_{p \in M}$ is an open cover for M there are $p_1, \dots, p_N$ so that $M = \tilde{U}_{p_1} \cup \dots \cup \tilde{U}_{p_N}$.

Now define $\rho_{p_j} = \dfrac{\mathring{\rho}_{p_j}}{\mathring{\rho}_{p_1} + \dots + \mathring{\rho}_{p_N}}$. Then $\{\rho_{p_j}\}_{j=1}^{N}$ is the required family.

Manifolds with Boundary

$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \le 0\}$



A second countable Hausdorff space M is a topological manifold with boundary if for any $p \in M$ there are open sets $p \in U \subseteq M$ and $V \subseteq \mathbb{H}^n$ and a homeomorphism $\varphi : U \to V$.

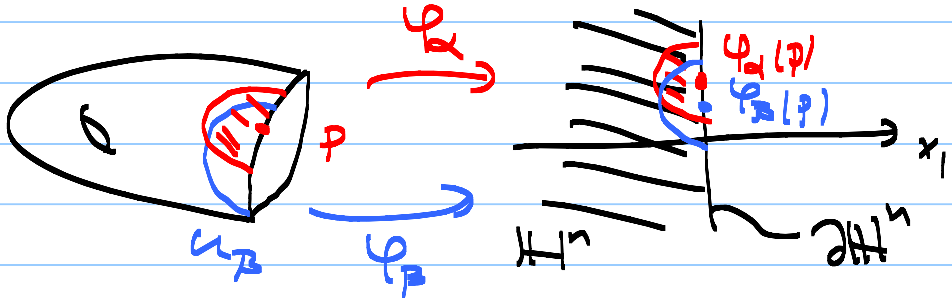
Video 25

Let $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ be an atlas for M so that for any α, β with $U_\alpha \cap U_\beta \neq \emptyset$ the composition

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta) \text{ has an}$$

extension to open subsets of $\mathbb{R}^n$ which is C^∞.

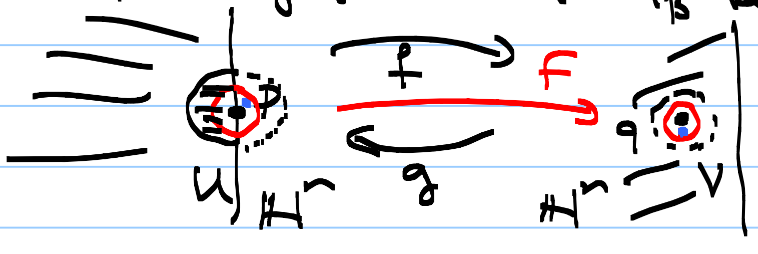
A point in M will be called a boundary point if it is sent under some φ_α to a point in $\partial \mathbb{H} = \{(x_1, \dots, x_n) \in \mathbb{H}^n \mid x_1 = 0\}$.



Proposition: If a point $p \in M$ is mapped to a boundary point of $\mathbb{H}^n$ under some coordinate chart then it is mapped to boundary points under any coordinate chart.

The set of boundary points of M is denoted as ∂M and ∂M is a C^∞-manifold of dimension $\dim M - 1$. ∂M has no boundary.

Proof: Assume on the contrary that a point of M is mapped to a boundary point of $\mathbb{H}^n$ under some φ_α and it is mapped also to an interior point of $\mathbb{H}^n$ under some φ_β. Then we have the following picture: $f = \varphi_\beta \circ \varphi_\alpha^{-1}$, $g = \varphi_\alpha \circ \varphi_\beta^{-1}$



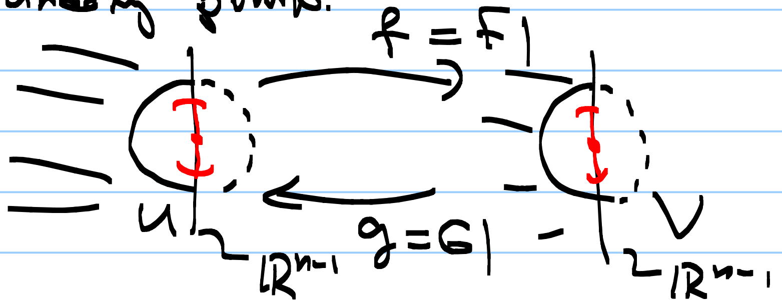
Let F and G be the C^∞-extensions of f and g to some open subsets U, V of $\mathbb{R}^n$.

$$f = F|_{U\cap H^n}, \quad g = f^{-1} = G|_{V\cap H^n}$$

$$p = (0, x_2, \ldots, x_n), \; f(p) = q = (y_1, \ldots, y_n) \notin V \cap \partial H^n$$
$$y_1 < 0.$$

On $U \cap H^n$, $G\circ f = Id$ and both are differentiable and thus on $\mathrm{Int}(U\cap H^n)$ $D(G\circ F) = Id$. Being equal to Id is a closed condition and thus $D(G\circ F) = Id$ on $U\cap \partial H$. In particular, $D(G\circ F)(p) = Id$. Hence, $DG(F(p))\circ DF(p) = Id$ so that $DF(p)$ is an isom. So, by the Inverse Function Thm F is a diffeomorphism near p. Hence $g = f^{-1}$ maps some points whose first coordinate is negative to points whose first coordinate is positive. This is a contradiction.

We may that ∂M is an $n-1$ dimensional manifold as follows: F and G are differentiable functions and they map boundary points to boundary points.



$$f = \varphi_\beta \circ \varphi_\alpha^{-1}$$
$$g = \varphi_\alpha \circ \varphi_\beta^{-1}$$

Also $f(U\cap\partial H^n) = F(U\cap \partial H^n) \subset V \cap \partial H^n$.

Hence, the restriction of φ_α's to ∂M provide a C^∞-manifold structure of ∂M. As no point in ∂M has a neighborhood homeomorphic to an open set in $\mathbb{R}^{n-1}$ and thus ∂M has no boundary: $\partial(\partial M) = \phi$.

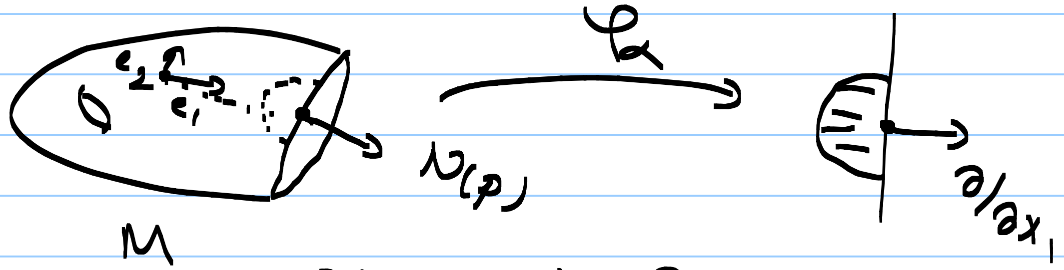
STOKES' THEOREM:

Definition of the integral of a differential form $\omega \in \Omega^n_c(M)$ on M, $n = \dim M$, where the subscript c on $\Omega^n_c(M)$ represents forms with compact support.

$$\omega \in \Omega^k_c(M), \quad supp(\omega) = \overline{\{p \in M \mid \omega(p) \neq 0\}}.$$

Let $\{\varphi_\alpha : U_\alpha \longrightarrow V_\alpha \subseteq \mathbb{H}^n\}_{\alpha \in \Delta}$ be an oriented atlas for M.

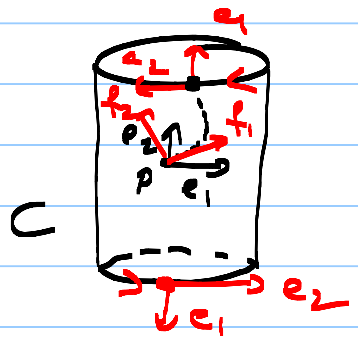
Orientation of ∂M: Assume that M is oriented.



$$D\varphi_\alpha(N(p)) = \frac{\partial}{\partial x_1}$$

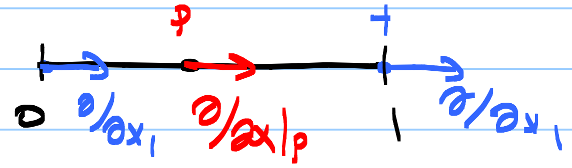
A basis $\beta = \{v_1, \dots, v_{n-1}\}$ of $T_p \partial M$ is positively oriented if $\beta' = \{N(p), v_1, v_2, \dots, v_{n-1}\}$ is a positively oriented basis for $T_p M$.

Ex



$$T_p C = span\{e_1, e_2\}$$

Ex: $M = [0,1] \quad T_pM = \{ \frac{\partial}{\partial x}|_p \}$



$\partial M = \{ 0^-, 1^+ \}$

$$\int_0^1 f(x)\,dx = F(1) - F(0) \qquad F' = f$$

$$\int_{[0,1]} dF = \int_{\partial[0,1]} F \doteq -F(0) + F(1) \qquad \partial[0,1] = \{0^-, 1^+\}$$

$$\int_M d\omega = \int_{\partial M} \omega \qquad \text{(STOKES' THM)}$$

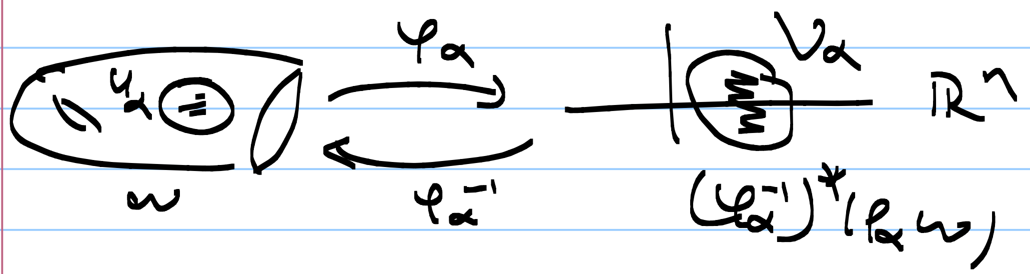
$\omega \in \Omega^{n-1}(M)$, $n = \dim M$.

Let $\{\rho_\alpha : M \to [0,1]\}$ be a partition of unity subordinate the oriented cover $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\Lambda}$ of M.

$$\sum_\alpha \rho_\alpha(p) = 1.$$

Now the integral $\int_M \omega$ is defined to be the

sum $$\int_M \omega \doteq \sum_{\alpha\in\Lambda} \int_{V_\alpha} (\varphi_\alpha^{-1})^* (\underline{\underline{\rho_\alpha \omega}})$$



One can check that if $\{\tilde{\rho}_\alpha : M \to [0,1]\}$ is another partition of unity subordinate the same $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ then

$$\sum_\alpha \int_{V_\alpha} (\varphi_\alpha^{-1})^*(\rho_\alpha \omega) = \sum_\alpha \int_{V_\alpha} (\varphi_\alpha^{-1})^*(\tilde{\rho}_\alpha \omega)$$

Indeed we have

$$\sum_{\alpha \in \Lambda} \int_{V_\alpha} (\varphi_\alpha^{-1})^*(\rho_\alpha(p)\,\omega) = \sum_\alpha \int_{V_\alpha} (\varphi_\alpha^{-1})^*\Big(\overbrace{\sum_\beta \tilde{\rho}_\beta(p)}^{1 \,=}\, \rho_\alpha(p)\,\omega\Big)$$

$$= \sum_\alpha \int_{V_\alpha} \sum_\beta (\varphi_\alpha^{-1})^*(\tilde{\rho}_\beta(p)\,\rho_\alpha(p)\,\omega)$$

$$= \sum_{\alpha,\beta} \int_{V_\alpha} \underbrace{(\varphi_\alpha^{-1})^*(\tilde{\rho}_\beta(p)\,\rho_\alpha(p)\,\omega)}_{\text{''}\,\eta}$$

$$\overset{*}{=} \sum_{\alpha,\beta} \int_{V_\beta} \underbrace{(\varphi_\beta^{-1})^*(\tilde{\rho}_\beta(p)\,\rho_\alpha(p)\,\omega)}_{(\varphi_\alpha \circ \varphi_\beta^{-1})^*(\eta)}$$

$$\vdots$$

$$= \sum_\beta \int_{V_\beta} (\varphi_\beta^{-1})^*(\tilde{\rho}_\beta(p)\,\omega),$$

where the equality $*$ is obtained as replacing the differential form by its pull back under

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \underset{\substack{\cap \\ V_\beta}}{\varphi_\beta(U_\alpha \cap U_\beta)} \to \underset{\substack{\cap \\ V_\alpha}}{\varphi_\alpha(U_\alpha \cap U_\beta)}.$$

This finishes the proof. ∎

Integration of a 0-form, i.e., a function f on an oriented 0-manifold
$$M = \{ p_n^{\epsilon_n} \mid \epsilon_n = \pm 1 \}$$

We need $f \in \Omega_c^0(M)$ so that $f(p_n) = 0$ for all but finitely many n.
$$\int_M f \doteq \sum_n \epsilon_n f(p_n).$$

<u>Theorem</u> (Stokes) If $\omega \in \Omega_c^{n-1}(M)$, where M is an oriented manifold with boundary then
$$\int_M d\omega = \int_{\partial M} \omega \quad , \text{ where } n = \dim M$$

and ∂M is oriented by the orientation of M.

<u>Proof</u>: Assume the above set up.
$$\int_M d\omega = \sum_\alpha \int_{V_\alpha} (\varphi_\alpha^{-1})^* (\rho_\alpha(p)\omega)$$

Let $\omega_\alpha \doteq \rho_\alpha \omega \in \Omega_c^{n-1}(M)$. Then $\omega = \sum_\alpha \omega_\alpha$.

Hence, $d\omega = \sum_\alpha d\omega_\alpha$. Integration is linear

and hence it is enough to show that
$$\int_M d\omega_\alpha = \int_{\partial M} \omega_\alpha .$$
$V_\alpha \subseteq \mathbb{R}^n, \mathbb{H}^n$ open $\quad x_1, \ldots, x_n$

$$(\varphi_\alpha^{-1})^* \omega_\alpha = \sum_{i=1}^{n} a_i \; dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$(\varphi_\alpha^{-1})^* (d\omega_\alpha) = d\left((\varphi_\alpha^{-1})^* (\omega_\alpha) \right)$$

$$= d\left(\sum_{i=1}^{n} a_i \; dx_1 \wedge - \wedge \widehat{dx_i} \wedge - \wedge dx_n \right)$$

$$= \sum_{i=1}^{n} da_i \wedge dx_1 \wedge - - \wedge \widehat{dx_i} \wedge - - \wedge dx_n$$

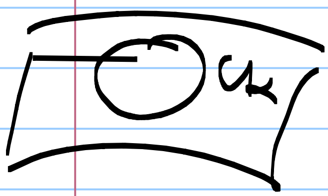
$$= \sum_{i=1}^{n} \left(\sum_{j} \frac{\partial a_i}{\partial x_j} dx_j \right) \wedge dx_1 \wedge - - \wedge \widehat{dx_i} \wedge - \wedge dx_n$$

$$= \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} dx_i \wedge dx_1 \wedge - - \wedge \widehat{dx_i} \wedge - - \wedge dx_n$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial a_i}{\partial x_i} dx_1 \wedge - - \wedge dx_i \wedge - - \wedge dx_n .$$

<u>Case 1</u> $\varphi_\alpha(U_\alpha) = V_\alpha = \mathbb{R}^n$

$$\int_M d\omega_\alpha = \int_{U_\alpha} d\omega_\alpha = \int_{V_\alpha} (\varphi_\alpha^{-1})^* (d\omega_\alpha)$$



$$= \int_{\mathbb{R}^n} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial a_i}{\partial x_i} dx_1 \wedge - - \wedge dx_i \wedge - - dx_n$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^n} \frac{\partial a_i}{\partial x_i} dx_1 \cdots dx_n$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i} dx_i \right) dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

$$= \sum_{i=1}^{n} \int_{\mathbb{R}^{n-1}} \left(\lim_{x_i \to \infty} a_i(x) - \lim_{x_i \to -\infty} a_i(x) \right) dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

$x = (x_1, \ldots, x_i, \ldots, x_n)$

Video 27

Note that both limits are equal zero since ω_α and hence $a_i(x)$'s are compactly supported.

Hence, $\int_M d\omega = 0$.

Case 2) $\varphi_\alpha(U_\alpha) = V_\alpha = \mathbb{H}^n$.

$$\int_M d\omega_\alpha = \int_{U_\alpha} d\omega_\alpha = \int_{\mathbb{H}^n} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial a_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{H}^n} \frac{\partial a_i}{\partial x_i} dx_1 \cdots dx_n$$

$$= \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{0} \frac{\partial a_1}{\partial x_1} dx_1 \right) dx_2 \cdots dx_n$$

$$= \int_{\mathbb{R}^{n-1}} \left(a_1(0, x_2, \dots, x_n) - \lim_{x_1 \to -\infty} a_1(x_1, x_2, \dots, x_n) \right) dx_2 \cdots dx_n$$

$$= \int_{\mathbb{R}^{n-1}} \left(a_1(0, x_2, \dots, x_n) - 0 \right) dx_2 \cdots dx_n$$

$$= \int_{\partial\mathbb{H}^n} a_1(0, x_2, \dots, x_n)\, dx_2 \cdots dx_n$$

$$= \int_{\partial\mathbb{H}^n} a_1(x)\, dx_2 \wedge \cdots \wedge dx_n$$

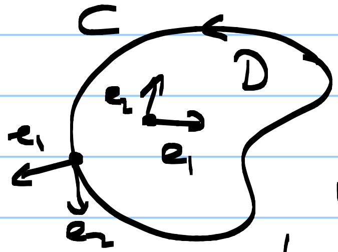
$$= \int_{\partial\mathbb{H}^n} \sum_{i=1}^{n} a_i(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$= \int_{\partial\mathbb{H}^n} \omega$$

$$= \int_{\partial M} \omega.$$

Remark Special Cases of Stokes' Theorem.

Green's Thm



$\omega \in \Omega^1(D)$

$\omega = f(x,y)\,dx + g(x,y)\,dy$

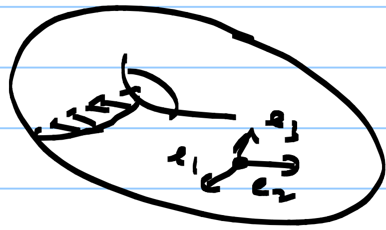
$d\omega = (g_x - f_y)\,dx \wedge dy$

$$\int_C \omega = \int_{\partial D} \omega = \int_D d\omega$$

$$\Rightarrow \int_C f(x,y)\,dx + g(x,y)\,dy = \int_D (g_x - f_y)\,dx \wedge dy$$

$$= \int_D (g_x - f_y)\,dx\,dy.$$

Gauss's Thm



$D \subseteq \mathbb{R}^3$ manifold with boundary

$\partial D = S$ surface in $\mathbb{R}^3$

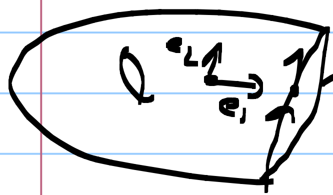
Let $\omega \in \Omega^2(D)$, $\omega = f\,dx \wedge dy + g\,dy \wedge dz + h\,dz \wedge dx$

$$d\omega = (f_z + g_x + h_y)(dx \wedge dy \wedge dz)$$
$$= \operatorname{div} F \; dx \wedge dy \wedge dz, \qquad F = (f, g, h) \text{ vector field on } D.$$

Stokes' Thm
$$\int_S F \cdot \vec{N}\,dS = \int_{S = \partial D} \omega = \int_D d\omega = \int_D \operatorname{div} F \; dx\,dy\,dz$$

Green's Thm for Surfaces in $\mathbb{R}^3$: Classical Stokes

$$\Sigma \subseteq D \subseteq \mathbb{R}^3$$

 $\sim \partial\Sigma = C$

$$\omega \in \Omega^1(D), \quad \omega = f\,dx + g\,dy + h\,dz$$

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy + dh \wedge dz \\ &= (g_x - f_y)\,dx\wedge dy + (f_z - h_x)\,dz \wedge dx \\ &\qquad + (h_y - g_z)\,dy \wedge dz \end{aligned}$$

Stokes' Thm $\displaystyle\int_C \omega = \int_{\partial\Sigma} \omega = \int_\Sigma d\omega$

$$\int_C f\,dx + g\,dy + h\,dz = \int_\Sigma \operatorname{curl}(F)\cdot n\,ds$$

$$F = (f, g, h)$$

<u>Example:</u> Let $M \subseteq \mathbb{C}^n$ be a compact complex submanifold (connected). Then $\dim M = 0$.

<u>Proof</u>: Assume on the contrary that $\dim M = l > 0$.

$$\omega = \sum_{j=1}^n dx_j \wedge dy_j = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$$

$$\iota^*(\omega^l) = c\, l! \left(\frac{i}{2}\right)^l dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_l \wedge d\bar{w}_l,$$

where $\iota : V \longrightarrow \mathbb{C}^n$ inclusion map of an l-dimensional complex vector subspace. Here $c > 0$ is a positive real number.

$w_1, \ldots, w_\ell$

$M \subseteq \mathbb{C}^n$ cx. submanifold. Take V as T_pM as a complex subspace. $(\partial M = \emptyset)$

$$T_pM = \operatorname{span}\left\{ \frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_i} \right\} = \left\{ \frac{\partial}{\partial u_1}, i\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_\ell}, i\frac{\partial}{\partial u_\ell} \right\}$$

$$w_j = u_j + i v_j, \quad \partial/\partial v_j = i\, \partial/\partial u_j$$

Hence, the restriction of ω^ℓ to T_pM is

$$c\left(\frac{i}{2}\right)^\ell \ell!\; dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_\ell \wedge d\bar{w}_\ell$$

$$= c\,\ell!\,(du_1 \wedge dv_1 \wedge du_2 \wedge dv_2 \wedge \cdots \wedge du_\ell \wedge dv_\ell)$$

The orientation (coming from complex manifold structure) on T_pM is

$$\left\{ \frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1} = i\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_\ell}, \frac{\partial}{\partial v_\ell} = i\frac{\partial}{\partial u_\ell} \right\}$$

Hence $\int_M \omega > 0$. On the other hand,

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j = d\left(\sum_{j=1}^{n} x_j\, dy_j \right) = d\eta,$$

where $\eta = \sum_{j=1}^{n} x_j\, dy_j$.

Thus, $\omega^\ell = d(\eta) \wedge \omega^{\ell-1} = d(\eta \wedge \omega^{\ell-1})$,

because $d(\omega^{\ell-1}) = 0$ since ω is closed.

In particular, ω^ℓ is exact. Now,

$$0 < \int_M \omega^\ell = \int_M d(\eta \wedge \omega^{\ell-1}) = \int_{\partial M = \emptyset} \eta \wedge \omega^{\ell-1} = 0,$$

a contradiction!

Example: $\mathbb{C}^2 \setminus \{(0,0)\} \big/ (z_1, z_2) \sim (2z_1, 2z_2)$

is

$S^3 \times S^1$

So $S^3 \times S^1$ has a complex manifold structure.

As a real 6-dimensional manifold it sits in $\mathbb{R}^4 \times \mathbb{R}^2 = \mathbb{R}^6$.

However, we've just seen that it does not embed in any $\mathbb{C}^n$ as a complex submanifold.

Question: $\mathbb{CP}^n$ a compact complex manifold.

Does $S^3 \times S^1$ sit in $\mathbb{CP}^n$ as a complex submanifold for some n?

Answer: No! since $S^3 \times S^1$ has no second cohomology.

Video 28

Volumes of the disc and the Sphere in $\mathbb{R}^n$:

$D^n(r) = \{x \in \mathbb{R}^n \mid \|x\| \le r\}$ the n-disc in $\mathbb{R}^n$.

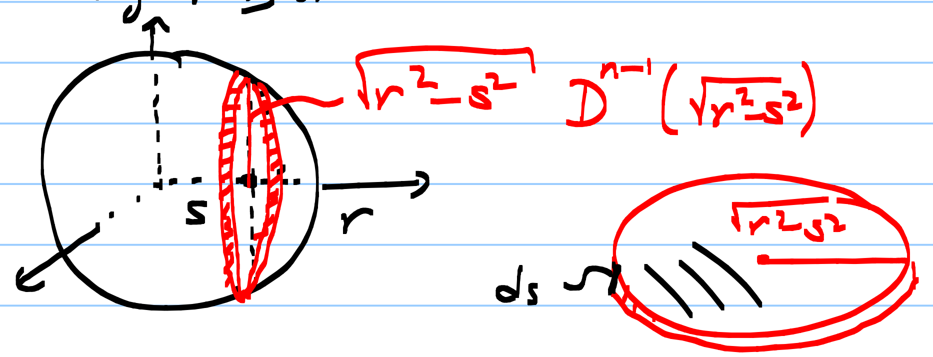
$$Vol(D^n(r)) \doteq \int_{D^n(r)} dx_1 \wedge \dots \wedge dx_n$$

$$= \int_{D^n(r)} dx_1 \dots dx_n$$

Claim: $Vol(D^n(r)) = A_n r^n$ for some constant A_n.

Proof: $Vol(D^1(r)) = Vol([-r, r]) = 2r$ so that $A_1 = 2$.

Assume that A_{n-1} exists so that $Vol(D^{n-1}(r)) = A_{n-1} r^{n-1}$ for any $r \ge 0$.



$$Vol(D^n(r)) = 2 \int_0^r Vol(D^{n-1}(\sqrt{r^2 - s^2}))\, ds$$

$$= 2 A_{n-1} \int_0^r (\sqrt{r^2 - s^2})^{n-1}\, ds$$ by the Induction hypothesis

Let $s = r \sin\theta$, $\theta \in [0, \pi/2]$.

$$(r^2 - s^2)^{n-1/2}\, ds = (r^2 - r^2 \sin^2\theta)^{n-1/2}\, r \cos\theta\, d\theta$$

$$\Rightarrow Vol(D^n(r)) = 2 A_{n-1} \int_0^{\pi/2} r^n \cos^n\theta\, d\theta.$$

$$\mathrm{Vol}(D^n(r)) = 2A_{n-1}\, r^n \overbrace{\int_0^{\pi/2} \cos^n\theta \, d\theta}^{B_n}$$

Let $B_n = \displaystyle\int_0^{\pi/2} \cos^n\theta \, d\theta = \underbrace{\sin\theta(\cos\theta)^{n-1}\Big|_0^{\pi/2}}_{=0} + \int_0^{\pi/2} \sin^2\theta\,(\cos\theta)^{n-2}(n-1)\,d\theta$

$v=(\cos\theta)^{n-1}$, $du = \cos\theta\, d\theta$
$dv = (n-1)(-\sin\theta)(\cos\theta)^{n-2}$
$u = \sin\theta$

$$\Rightarrow B_n = \int_0^{\pi/2} \sin^2\theta(\cos\theta)^{n-2}(n-1)\,d\theta$$

$$= \int_0^{\pi/2}\left[(\cos\theta)^{n-2} - (\cos\theta)^n\right] d\theta$$

$$= (n-1)B_{n-2} - (n-1)B_n$$

$$nB_n = (n-1)B_{n-2} \Rightarrow B_n = \frac{n-1}{n} B_{n-2}.$$

These imply that

$$B_{2n+1} = \frac{(2^n n!)^2}{(2n+1)!} \quad \text{and} \quad B_{2n} = \frac{(2n)!}{(2^n n!)^2}\,\frac{\pi}{2}.$$

Since we have $A_n = 2A_{n-1}B_n$ we get

$$A_{2n+1} = \frac{2^{n+1}\pi^n}{1\cdot 3\cdots(2n+1)} \quad \text{and} \quad A_{2n} = \frac{\pi^n}{n!}$$

For example, $\mathrm{Vol}(D^4(r)) = A_4 r^4 = \dfrac{\pi^2}{2!} r^4$.

Note that $S^{n-1}(r) = \partial D^n(r)$ and thus
$\mathrm{Vol}(S^{n-1}(r)) = \frac{d}{dr}(D^n(r)) = \frac{d}{dr}(A_n r^n) = nA_n r^{n-1}.$

For example, $Vol(S^3(r)) = \frac{d}{dr} Vol(D^4) = 2\pi^2 r^3$.

De Rham Cohomology:

M smooth manifold of dimension n.

$\Omega^k(M)$: the vector space of (smooth) k-forms on M. Then we have a, so called, chain complex:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \cdots$$

$d^2 = 0$ and hence $Im(d : \Omega^{k-1}(M) \to \Omega^k(M))$ is contained in $Ker(d : \Omega^k(M) \to \Omega^{k+1}(M))$.

$$H^k_{dR}(M) \doteq \frac{Ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{Im(d : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

$H^k_{dR}(M)$ is called the k^{th} de Rham cohomology of M.

Example: If M is a smooth manifold then

$H^0(M) \simeq \mathbb{R}^{b_0}$, where b_0 is the "number" of connected components of M.

$$H^0(M) = \frac{\text{closed 0-forms on } M}{\text{Exact 0-forms on } M} \cong \frac{\mathbb{R}^{b_0}}{(0)} \cong \mathbb{R}^{b_0}.$$

A zero form is just a function $f: M \to \mathbb{R}$. Since it must be closed we have $df = 0$.

So, $0 = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \Rightarrow \frac{\partial f}{\partial x_i} = 0$ on M for all i.

Hence, f is locally constant on M. In other words it is a constant on each connected component of M.

$f \in \mathbb{R}^{b_0}$

Ex $M = \mathbb{Z}$ $\quad \cdot \quad \underset{n-1}{\cdot} \quad \underset{n}{\cdot} \quad \underset{n+1}{\cdot} \quad \cdots \quad \cdot$

$f(n) = a_n, \quad a_n \in \mathbb{R}$

Ex M any connected manifold, then $H^0_{dR}(M) \cong \mathbb{R}$.

Proposition: If $f: M \to N$ is a smooth map, then the pull back operation $f^*: \Omega^k(N) \to \Omega^k(M)$, $\omega \longmapsto f^*(\omega)$

induces an vector space homomorphism
$$f^*: H^k_{dR}(N) \to H^k_{dR}(M).$$

Proof: $f: M \to N$, $f^*: \Omega^k(N) \to \Omega^k(M)$

$$H^k_{dR}(N) = \frac{Z^k(N) \leftarrow \text{closed } k\text{-forms on } N}{B^k(N) \leftarrow \text{Exact } k\text{-forms on } N}$$

If $\omega \in Z^k(N)$ then $d(f^*(\omega)) = f^*(\underset{0}{d\omega}) = 0$

so that $f^*(\omega) \in Z^k(M)$. So we have a homomorphism
$$f^*: Z^k(N) \to Z^k(M).$$

Consider the composition

Video 29

$$\begin{array}{ccccc} \omega \in \overline{Z^k(N)} & \xrightarrow{f^*} & Z^k(M) & \longrightarrow & \dfrac{Z^k(M)}{B^k(M)} \doteq H^k_{dR}(M). \\ \downarrow & & & & \\ [\omega] \in \dfrac{Z^k(N)}{B^k(N)} \doteq H^k_{dR}(N) & & \dashrightarrow ?\, f^* & & \end{array}$$

<u>must check</u>: If $[\omega_1] = [\omega_2]$ then $[f^*(\omega_1)] = [f^*(\omega_2)]$.

Since $[\omega_1] = [\omega_2] \Rightarrow \omega_1 - \omega_2 \in B^k(N)$

$\Rightarrow \omega_1 - \omega_2 = d\eta$ for some $\eta \in B^{k-1}(N)$.

Then, $f^*(\omega_1) - f^*(\omega_2) = f^*(d\eta) = d(f^*(\eta))$

so that it is exact. Hence,

$[f^*(\omega_1)] = [f^*(\omega_2)]$ in $H^k_{dR}(M)$. ■

<u>Proposition</u>: Wedge product induces a product on cohomology:

$$H^k_{dR}(M) \times H^\ell_{dR}(M) \longrightarrow H^{k+\ell}_{dR}(M), \ ([\omega],[\eta]) \mapsto [\omega \wedge \eta].$$

<u>Proof</u>: Let $[\omega_1] = [\omega_2]$ in $H^k_{dR}(M)$. Then

$\omega_1 - \omega_2 = d\nu$, for some $\nu \in \Omega^{k-1}(M)$.

So $(\omega_1 - \omega_2) \wedge \eta = (d\nu) \wedge \eta = d(\nu \wedge \eta)$
$= d\nu \wedge \eta + (-1)^{k-1} \nu \wedge \underset{0}{d\eta}$

$\Rightarrow \omega_1 \wedge \eta - \omega_2 \wedge \eta = d(\nu \wedge \eta)$

$\Rightarrow [\omega_1 \wedge \eta] = [\omega_2 \wedge \eta]$ in $H^{k+l}_{dR}(M)$. ∎

This implies that $H^*_{dR} = \bigoplus_k H^k_{dR}(M)$ is a graded $\mathbb{R}$-algebra.

<u>Proposition</u> The map $I: H^1_{dR}(S^1) \longrightarrow \mathbb{R}$ given by $I([\omega]) = \int_{S^1} \omega$ is an $\mathbb{R}$-vector space isomorphism

<u>Proof:</u> Consider the 1-form $\omega = \frac{1}{2\pi}(x\,dy - y\,dx)$ on $S^1 \subseteq \mathbb{R}^2$ ($i: S^1 \to \mathbb{R}^2$ inclusion map
$\omega = i^*(1/2\pi \; x\,dy - y\,dx)$)

Since S^1 is 1-dimensional then $\nexists$ no nonzero 2-form on S^1 and thus $d\omega = 0$, i.e., ω is closed.
Let $P: \mathbb{R} \to S^1$ given by $P(t) = (\cos t, \sin t)$.
Note that $P|_{[0,2\pi]} : [0, 2\pi] \longrightarrow S^1$ is a coordinate path except the end points.

$$\int_{S^1} \omega = \int_{S^1 \setminus \{(1,0)\}} \omega = \int_{(0,2\pi)} P^*\omega = \int_{[0,2\pi]} P^*\omega$$

$P(t) = (\cos t, \sin t) = (x, y)$, $x = \cos t$, $y = \sin t$
$dx = -\sin t\, dt$, $dy = \cos t\, dt$
$\Rightarrow P^*\omega = P^*(x\,dy - y\,dx)\frac{1}{2\pi}$

$$\Rightarrow P^*\omega = \frac{1}{2\pi}\left(\cos t\,(\cos t)\,dt - \sin t\,(-\sin t)\,dt\right)$$

$$= \frac{1}{2\pi}\,dt.$$

So $\displaystyle\int_{S^1} \omega = \int_{[0,2\pi)} (1/2\pi)\,dt = 1.$

Also note that $I : H^1_{dR}(S^1) \to \mathbb{R}$ is linear:

$$I(a[\omega] + b[\eta]) = \int_{S^1} [a\omega + b\eta]$$

$$= \int_{S^1} a\omega + b\eta$$

$$= a\int_{S^1} \omega + b\int_{S^1} \eta$$

$$= a\,I([\omega]) + b\,I([\eta]).$$

Note that we must that I is well defined:

$I : H_{dR}(S^1) \longrightarrow \mathbb{R}$, $I([\omega]) = \displaystyle\int_{S^1} \omega$

If $[\omega_1] = [\omega_2]$ then $\displaystyle\int_{S^1} \omega_1 = \int_{S^1} \omega_2$. (?)

Since $[\omega_1] = [\omega_2]$ we have $\omega_1 - \omega_2 = d\nu$ for some $\nu \in \Omega^0(S^1)$.

Stokes' Thm.

$$\text{Then } \int_{S^1} \omega_1 - \int_{S^1} \omega_2 = \int_{S^1} \omega_1 - \omega_2 = \int_{S^1} d\nu \overset{\downarrow}{=} \int_{\partial S^1 = \phi} \nu = 0.$$

So $\mathcal{I}: H^1_{dR}(S^1) \longrightarrow \mathbb{R}$ is an onto linear map.

To finish the proof we must show $\ker \mathcal{I} = (0)$.

Let $[\eta] \in H^1_{dR}(S^1)$ so that $\mathcal{I}([\eta]) = \int_{S^1} \omega = 0$.

must show: η is exact on S^1.

$$0 = \int_{S^1} \eta = \int_0^{2\pi} p^*\eta , \qquad p^*\eta \in \Omega^1(\mathbb{R})$$

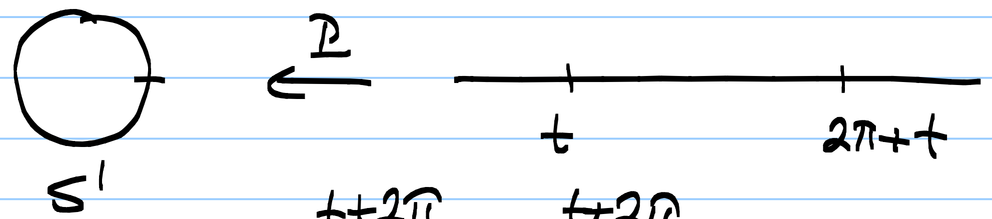
$$\Rightarrow p^*\eta = f(t)\,dt$$

So we have $0 = \int_0^{2\pi} f(t)\,dt.$

Let $F(t) = \int_0^t f(s)\,ds$. Then

$$F(t+2\pi) - F(t) = \int_0^{t+2\pi} f(s)\,ds - \int_0^t f(s)\,ds$$

$$= \int_t^{t+2\pi} f(s)\,ds = 0$$



$$0 = \int_{S^1} \eta = \int_t^{t+2\pi} p^*\eta = \int_t^{t+2\pi} f(s)\,ds$$

So $F: \mathbb{R} \longrightarrow \mathbb{R}$ is periodic: $F(t+2\pi) = F(t)$.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d, "P"'] & \mathbb{R} \\ S^1 \arrow[ur] & \end{tikzcd}$$

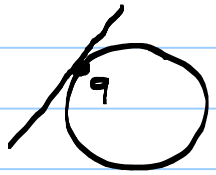
$\tilde{F}(\cos t, \sin t) \doteq F(t)$

$$F = \tilde{F} \circ P.$$

$$P^* \eta = f(t)dt = dF.$$

Now $P^*(d\tilde{F}) = d(P^*\tilde{F}) = d(\tilde{F} \circ P) = dF.$

$$\Rightarrow P^*(d\tilde{F}) = P^*\eta.$$



must show $d\tilde{F} = \eta$ on $T_q S^1$.

Let $v \in T_q S^1$, then since $P_* : T_{t_0}\mathbb{R} \to T_q S^1$ when $P(t_0) = q$, is an isomorphism because P is a local diffeomorphism. So $v = P_*(u)$ for some $u \in T_{t_0} V$.

Now $P^*(d\tilde{F})(u) = P^*(\eta)(u)$

$$d\tilde{F}(P_*(u)) = \eta(P_*(u))$$

$$d\tilde{F}(v) = \eta(v), \text{ for all } v \in T_q S^1.$$

So, $d\tilde{F} = \eta$ so that η is exact! This finishes the proof.

<u>Remark</u>: $H^1_{DR}(S^1) \simeq \mathbb{R}$, $H^0_{DR}(S^1) \simeq \mathbb{R}$

$\dim S^1 = 1$ any k-form, when $k \geq 2$, is zero on S^1 and $H^k_{DR}(S^1) \simeq 0$.

Video 30

Poincaré Lemma: Let $I \subseteq \mathbb{R}$ be an interval. Then for any smooth manifold M we have

$$H^k_{DR}(M \times I) \simeq H^k_{DR}(M).$$

Proof: Consider the projection map $Pr: M \times I \to M$, $Pr(x,t) = x$, $\forall (x,t) \in M \times I$. Let $a \in I$ and define the inclusion map $i_a : M \to M \times I$ given by $i_a(x) = (x, a)$. So, $i_a(M) = M \times \{a\}$.

Note that the composition $Pr \circ i_a : M \to M$ is the identity map Id_M. This induces an isomorphism on cohomology:

$$H^k_{DR}(M) \xrightarrow{Pr^*} H^k_{DR}(M \times I) \xrightarrow{i_a^*} H^k_{DR}(M)$$

$$i_a^* \circ Pr^* = (Pr \circ i_a)^* = (Id_M)^* = Id_{H^k_{DR}(M)}.$$

Hence, Pr^* is injective and i_a^* is surjective.

Let $U \subseteq M$ be a coordinate chart. Locally $M \times I$ can be seen as $U \times I$. Let $x_1, \dots, x_n$ be the coordinates on U. Also let t be the coordinate on I.

Let $I = (i_1, \dots, i_{k-1})$ and $J = (j_1, \dots, j_k)$, where $1 \leq i_1 < \dots < i_{k-1} \leq n = \dim M$, $1 \leq j_1 < \dots < j_k \leq n = \dim M$

Now any k-form on $U \times I$ is given as

$$\sum_J g_J(x,t)\, dx_J + \sum_I f_I(x,t)\, dx_I \wedge dt, \text{ for some}$$

functions on $U \times I$.

Consider the map $P: \Omega^k(M\times I) \to \Omega^{k-1}(M)$ given by

$$P(f(x,t)\,dx_I \wedge dt) \doteq (-1)^{k-1}\left(\int_a^t f(x,s)\,ds\right) dx_I$$

and $P(g(x,t)\,dx_J) \doteq 0$.

<u>Claim:</u> 1) $(d\circ P + P\circ d)(f(x,t)\,dx_I \wedge dt) = f(x,t)\,dx_I \wedge dt$

and 2) $(d\circ P + P\circ d)(g(x,t)\,dx_J) = (g(x,t) - g(x,a))\,dx_J$.

Let's first see that the claim finishes the proof. Note for the composition
$T_a \circ Pr : M\times I \to M\times I$ we have

$(Pr^* \circ T_a^*)(f(x,t)\,dx_I \wedge dt) = 0$ and

$(Pr^* \circ T_a^*)(g(x,t)\,dx_J) = g(x,a)\,dx_J$.

Hence, for any $\omega \in \Omega^k(M\times I)$, we have

$(d\circ P + P\circ d)(\omega) = \omega - (Pr^* \circ T_a^*)\,\omega$.

If ω is closed, then

$$\underbrace{d(P(\omega))}_{\text{exact}} - P(\underset{\overset{\shortparallel}{0}}{d\omega}) = \omega - (Pr^* \circ T_a^*)\,\omega$$

and hence, $[\omega] = (Pr^* \circ T_a^*)([\omega])$
$= (T_a \circ Pr)^*([\omega])$, so that

$(T_a \circ Pr)^*$ is identity on $H^k_{DR}(M)$. Thus,
$T_a^*: H^k_{DR}(M\times I) \to H^k_{DR}(M)$ and
$Pr^*: H^k_{DR}(M) \to H^k_{DR}(M\times I)$ are inverses of each other and thus they are both isomorphisms.

Video 31

Proof of Claim: 1) $(d\circ P + P\circ d)(f(x,t)\,dx_I \wedge dt) = f(x,t)\,dx_I \wedge dt$.

$P : \Omega^k(M\times\mathbb{R}) \to \Omega^{k-1}(M\times\mathbb{R})$,

$$P(f(x,t)\,dx_I \wedge dt) = \left((-1)^{k-1}\int_a^t f(x,s)\,ds\right) dx_I \quad \text{and}$$

$$P(g(x,t)\,dx_J) = 0. \qquad I = (i_1, \dots, i_{k-1}) \quad \in \Omega^{k+1}(M)$$

$$(P\circ d)(f(x,t)\,dx_I \wedge dt) = P\left(\sum_l \frac{\partial f}{\partial x_l}\,dx_l \wedge dx_I \wedge dt\right)$$

$$= (-1)^k \sum_l \left(\int_a^t \frac{\partial f}{\partial x_l}(x,s)\,ds\right) dx_l \wedge dx_I$$

$$(d\circ P)(f(x,t)\,dx_I \wedge dt) = d\left[(-1)^{k-1}\left(\int_a^t f(x,s)\,ds\right) dx_I\right]$$

$$= (-1)^{k-1}\left(f(x,t)\,dt \wedge dx_I + \sum_l \left(\int_a^t \frac{\partial f}{\partial x_l}(x,s)\,ds\right) dx_l \wedge dx_I\right)$$

So $(P\circ d + d\circ P)(f(x,t)\,dx_I \wedge dt) = (-1)^{k-1} f(x,t)\,dt \wedge dx_I$
$= f(x,t)\,dx_I \wedge dt$.

This finishes the proof of the Poincaré lemma. ∎

Corollary: For any $k \geq 0$ integer and differentiable manifold M we have

$$H^i_{DR}(M \times \mathbb{R}^k) \simeq H^i_{DR}(M), \text{ for any } i.$$

Corollary $H^i_{DR}(\mathbb{R}^k) = H^i_{DR}(\{pt\} \times \mathbb{R}^k) \simeq H^i_{DR}(\{pt\}) = 0$ if $i \geq 1$ and $H^0_{DR}(\mathbb{R}^k) \simeq \mathbb{R}$.

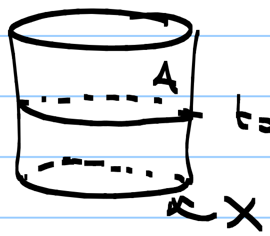
Corollary: The isomorphism $i_a^*: H^k_{DR}(M \times I) \to H^k_{DR}(M)$ is independent of the point $a \in I$.

Proof Just note that $i_a^* = (pr^*)^{-1}$. ∎

Definition: A map $r: X \to A$, where $A \subseteq X$ a subspace of a topological space X, is called a **retraction** if $r(a) = a$, for all $a \in A$.

Example: For any space X, let $Y = X \times I$ and $A = X \times \{t_0\}$, $t_0 \in I$. Then the map

$r: Y = X \times I \longrightarrow X \times \{t_0\} = A$, $r(x,t) = (x, t_0)$ is a retraction.

$Y = X \times I$ 

Proposition: If $r: M \to L$ is a retraction of smooth manifolds then $r^*: H^k_{DR}(L) \to H^k_{DR}(M)$ is injective for all k.

Proof Consider the composition $r \circ i: L \to L$, where $(r \circ i)(x) = r(i(x)) = i(x) = x$.

Thus, $(r \circ i)^* = id$ on $H^k_{DR}(L)$.

$$H^k_{DR}(L) \xrightarrow{r^*} H^k_{DR}(M) \xrightarrow{i^*} H^k_{DR}(L)$$

$= id$

Here, r^* is injective and i^* is surjective. ∎

<u>Corollary</u>: There is no retraction of D^2 onto its boundary $\partial D^2 = S^1$.

<u>Proof</u>.

$r : D^2 \to \partial D^2 = S^1$

Suppose on the contrary that there is a retraction $r : D^2 \to \partial D^2 = S^1$. Then by the above corollary $r^* : H^1_{DR}(S^1) \to H^1_{DR}(D^2)$ must be injective, which is a contradiction.

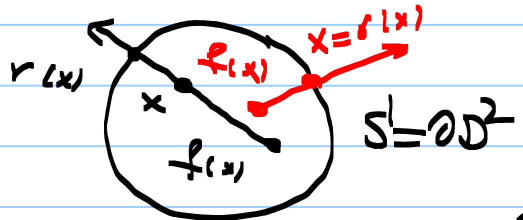
$$\begin{array}{ccc} H^1_{DR}(S^1) & \longrightarrow & H^1_{DR}(D^2) \\ \text{IS} & & \text{IS} \\ \mathbb{R} & \xrightarrow[\text{inj.}]{} & (0) \end{array}$$

∎

<u>Corollary</u>: Any smooth map $f : D^2 \to D^2$ has a fixed point.

<u>Proof</u>: Assume on the contrary that f has no fixed points. So, $f(x) \neq x$ for all $x \in D^2$. Define the map

$r : D^2 \to \partial D^2$, $x \mapsto r(x)$, the intersection of the ray $\overrightarrow{f(x)\,x}$ with the boundary circle ∂D^2.



$r(x)$ is a smooth map and $r(x) = x$ for any $x \in S^1 = \partial D^2$.

Hence, $r : D^2 \to S^1 = \partial D^2$ is a retraction. This is a contradiction and the proof finishes. ∎

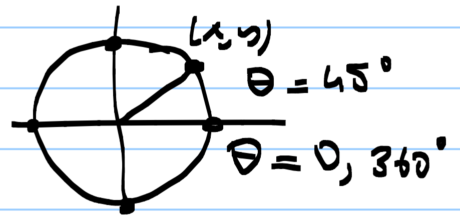
Winding, Rotation and Linking numbers:

By the Poincaré lemma, since $\mathbb{R}^2 \setminus \{(0,0)\} \cong S^1 \times \mathbb{R}$
$\left((x,y) \longmapsto \left(\frac{(x,y)}{\|(x,y)\|}, \ln \|(x,y)\|\right)\right)$ is a C^∞-map, we

have $H^k_{DR}(\mathbb{R}^2 \setminus \{(0,0)\}) \simeq H^k_{DR}(S^1 \times \mathbb{R}) \simeq H^k_{DR}(S^1)$.

In particular, $H^1_{DR}(\mathbb{R}^2 \setminus \{(0,0)\}) \simeq \mathbb{R}$.

Let $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2} = \underline{\underline{d\theta}}$, $\quad \theta : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$
$(x,y) \longmapsto \tan^{-1}(y/x)$, a multivalued function.



$\theta = \tan^{-1}(y/x) + 2k\pi$
$k \in \mathbb{Z}$

One can see that $d\omega = 0$ so that ω is closed.
Also its integral over S^1 is

$$\int_{S^1} \underline{\underline{\omega}} = \int_0^{2\pi} \gamma^* \omega = \int_0^{2\pi} dt = 2\pi \neq 0.$$

$t \longmapsto (\cos t, \sin t)$, $t \in [0, 2\pi]$, $\quad \gamma : [0,2\pi] \longrightarrow S^1$
$\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2} = dt \qquad t \longmapsto (\cos t, \sin t)$

If $\omega = d\eta$ were exact then we would have

$$2\pi = \int_{S^1} \omega = \int_{S^1} d\eta = \int_{\partial S^1} \eta = \int_{\emptyset} \eta = 0,$$ a contradiction.

Hence, ω is closed but not exact so that

$0 \neq [\omega] \in H^1_{DR}(\mathbb{R}^2 \setminus \{(0,0)\}) \simeq \mathbb{R}$.

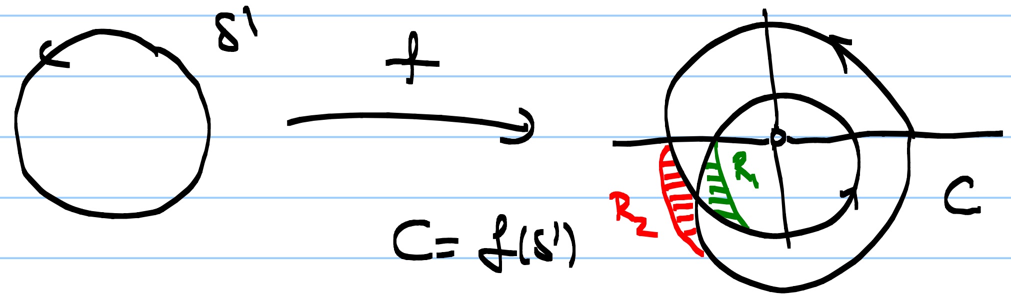
Video 32

Let's replace ω by $\frac{1}{2\pi}\omega'$ so that $\int_{S^1}\omega = 1$.

Winding number: If $f: S^1 \to \mathbb{R}^2\setminus\{(0,0)\}$ is any smooth map the number defined by

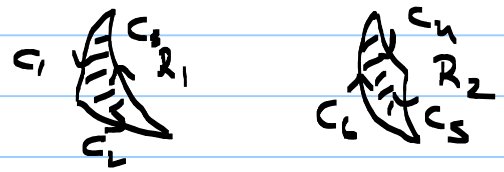
$$\int_{S^1} f^*(\omega) \text{ is an integer, called the}$$

winding number of $f: S^1 \to \mathbb{R}^2\setminus\{(0,0)\}$.



$$\omega = \frac{1}{2\pi}\,\frac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2\setminus\{(0,0)\}).$$

$$I_\omega(f) = \int_{S^1} f^*(\omega)$$

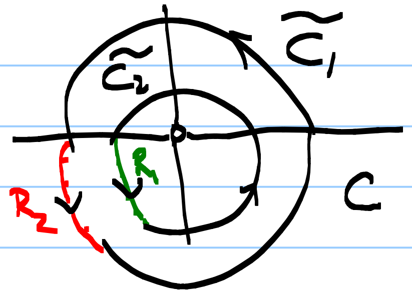


$\partial R_1 = C_1 + C_2 + C_3 \qquad \partial R_2 = C_4 + C_5 + C_6$

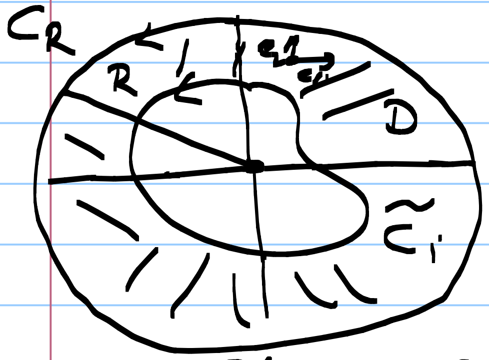
$$0 = \int_{R_1} \underset{0}{\underset{\shortparallel}{d\omega}} = \int_{\partial R_1}\omega = \int_{C_1+C_2+C_3}\omega \Rightarrow -\int_{C_3}\omega = \int_{C_1}\omega + \int_{C_2}\omega$$

Similarly, $0 = \int_{R_2} \underset{0}{\underset{\shortparallel}{d\omega}} = \int_{\partial R_2}\omega = \int_{C_4+C_5+C_6}\omega \Rightarrow -\int_{C_6}\omega = \int_{C_4}\omega + \int_{C_5}\omega.$

So, $\int_{-C_3}\omega = \int_{C_1}\omega + \int_{C_2}\omega$ and $\int_{-C_6}\omega = \int_{C_4}\omega + \int_{C_5}\omega.$



$$\int_C \omega = \int_{\tilde{C}_1} \omega + \int_{\tilde{C}_2} \omega$$



$$\partial D = -\tilde{C}_1 + C_R$$

$$0 = \int_D \underset{0}{\underset{\shortparallel}{d\omega}} = \int_{\partial D} \omega = \int_{-\tilde{C}_1 + C_R} \omega$$

$$\Rightarrow \int_{\tilde{C}_1} \omega = \int_{C_R} \omega$$

C_R: $x^2+y^2 = R^2$

$x = R\cos\theta,\ y = R\sin\theta$

$dx = -R\sin\theta\, d\theta$

$dy = R\cos\theta\, d\theta$

$$\omega = \frac{1}{2\pi} \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$= \frac{1}{2\pi} d\theta$$

$$\Rightarrow \int_{\tilde{C}_1} \omega = \int_{C_R} \omega = \int_0^{2\pi} \frac{1}{2\pi} d\theta = 1.$$

Hence, $\mathrm{Ind}(f) = \int_{S^1} f^*(\omega) = 1+1 = 2$

Theorem For any smooth map $f: S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$

$\mathrm{Ind}(f) = \int_{S^1} f^*(\omega)$ is always an integer.

Proof: $S^1 \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\} \xrightarrow{r} S^1$, where

$x \longmapsto f(x) \longmapsto \frac{f(x)}{\|f(x)\|}$

$r: \mathbb{R}^2 \setminus \{(0,0)\} \to S^1$ is a retraction.

$\mathbb{R}^2 \setminus \{(0,0)\} \xrightarrow{\simeq} S^1 \times \mathbb{R}$ diffeomorphism.
$x \longmapsto (r(x), \ln\|x\|)$

Poincaré lem

Moreover, $H^1_{DR}(\mathbb{R}^2 \setminus \{(0,0)\}) \simeq H^1_{DR}(S^1 \times \mathbb{R}) \overset{\downarrow}{\cong} H^1_{DR}(S^1) \simeq \mathbb{R}.$

$p_1 : S^1 \times \mathbb{R} \to S^1 \Rightarrow p_1^* : H^1_{DR}(S^1) \to H^1_{DR}(S^1 \times \mathbb{R}).$
$\eta \longmapsto p_1^*(\eta)$

$i : S^1 \to S^1 \times \mathbb{R}$
$x \longmapsto (x, 0)$

$i^* : H^1_{DR}(S^1 \times \mathbb{R}) \to H^1_{DR}(S^1)$

$i^* \circ p_1^* = (p_1 \circ i)^* = (\mathrm{id}_{S^1})^* = \mathrm{id}_{H^1_{DR}(S^1)}$

In particular, $H^1_{DR}(\mathbb{R}^2 \setminus \{(0,0)\}) \xrightarrow{\simeq} H^1_{DR}(S^1) \simeq \mathbb{R}.$
$[\omega] \longmapsto [\omega|_{S^1}]$

$I_\omega(f) = \int_{S^1} (r \circ f)^*(\omega_{S^1})$ $\qquad \omega_{S^1} = \dfrac{x\,dy - y\,dx}{2\pi}$

$S^1 \xrightarrow{f} \mathbb{R}^2 \setminus \{(0,0)\} \xrightarrow{r} S^1$

$\longleftarrow \omega \longrightarrow$

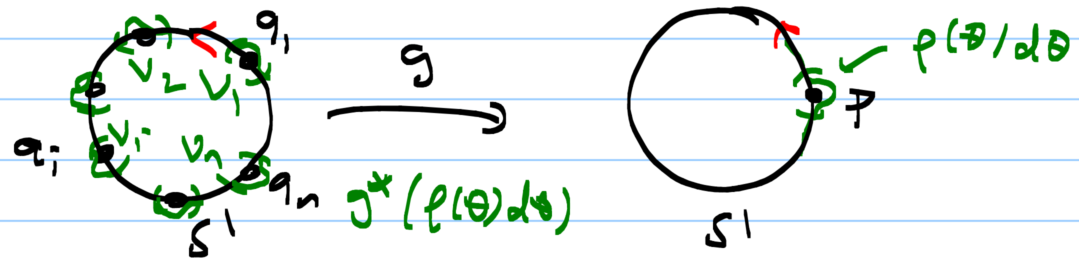
$\mathbb{R} \xrightarrow{\times\lambda} \mathbb{R}$

$S^1 \xrightarrow[g = r \circ f]{} S^1$

Claim $g^* : H^1_{DR}(S^1) \to H^1_{DR}(S^1)$ is given with multiplication by an integer $\lambda \in \mathbb{Z}$.

$g : S^1 \to S^1$ smooth. Let $p \in S^1$ be a regular value. $g^{-1}(p)$ is a 0 dim'l submanifold of S^1. Moreover, $g^{-1}(p)$ is a closed set. If $g^{-1}(p)$ is infinite then there is a sequence

(q_n) in $g^{-1}(p)$ so that $\lim q_n = q_0 \in g^{-1}(p)$.
This gives a contradiction to the fact that $g^{-1}(p)$ is a submanifold. Hence $g^{-1}(p)$ must be finite.



$g_*: T_{q_i}S^1 \longrightarrow T_p S^1$ is an isomorphism.

Define $\deg_{q_i}(g)$ the local degree of g at q_i as ± 1 depending on whether $g_*: T_{q_i}S^1 \to T_p S^1$ is orientation preserving or reversing.

Let $\deg g \doteq \sum_{i=1}^{n} \deg_{q_i}(g) \in \mathbb{Z}$

Now we claim that $\deg g = \int_{S^1} g^*(\omega)$

$H^1_{DR}(S^1) = \langle [\omega] \rangle \qquad \omega = \frac{1}{2\pi}(x\,dy - y\,dx) = \frac{1}{2\pi} d\theta$

Let $\eta = \rho(\theta)d\theta$, where $\int_{S^1} \rho(\theta)d\theta = 1$ and $\rho(\theta)$ is zero outside a small open set around p. Since g is a diffeomorphism around each q_i by choosing small enough open sets $V_1, \ldots, V_n$ around $q_1, \ldots, q_n$ and U around p so that $g: V_i \to U$ is a diffeomorphism we see that

$$\int_{S^1} g^*(\omega) \underset{*}{\overset{?}{=}} \int_{S^1} g^*(\rho(\theta)d\theta) = \int_{V_1 \cup \cdots \cup V_n} g^*(\rho(\theta))d\theta$$

$*\left[H^1_{DR}(S^1) \simeq \mathbb{R}, \quad I : H^1_{DR}(S^1) \to \mathbb{R} \right.$

$\eta \longmapsto \int_{S^1} \eta$ is an. $\left.\right]$

$\int_{S^1} \omega = 1 = \int_{S^1} \rho(\theta)\, d\theta$

$$\Rightarrow \int_{S^1} g^*(\omega) = \sum_{i=1}^{n} \int_{V_i} g^*(\rho(\theta)\, d\theta)$$

$$= \sum_{i=1}^{n} \deg_{q_i}(g) \underbrace{\int \rho(\theta)\, d\theta}_{1}, \text{ where}$$

$\deg(q_i)$ is the local degree of g at q_i.

$$= \sum_{i=1}^{n} \deg_{q_i}(g) = \deg(g).$$

■

<u>Remark:</u> If M is a compact oriented manifold (connected) then $H^n_{DR}(M) \simeq \mathbb{R}$. In this case we define the degree of any smooth map $f: M \to N$ between compact connected oriented smooth manifolds of the same dimension.

$$f: M \to N \Rightarrow f^*: \underset{\substack{\text{IS} \\ \mathbb{R}}}{H^n_{DR}(N)} \to \underset{\substack{\text{IS} \\ \mathbb{R}}}{H^n_{DR}(M)}$$

$\lambda = \deg(f)$ $\qquad \mathbb{R} \xrightarrow{\times \lambda} \mathbb{R}$

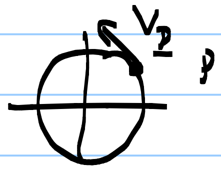
■

Video 33

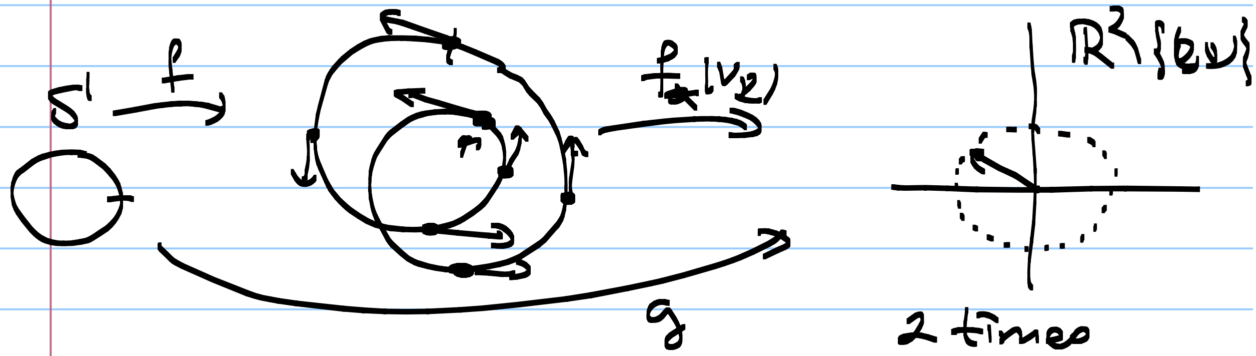
Rotation number of an immersion $f: S^1 \to \mathbb{R}^2$.

Let $g: S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ be the function defined by $g(p) = f_*(p)(v_p)$, where

$v_p \in T_pS^1$, $v_p = (-y, x)$.



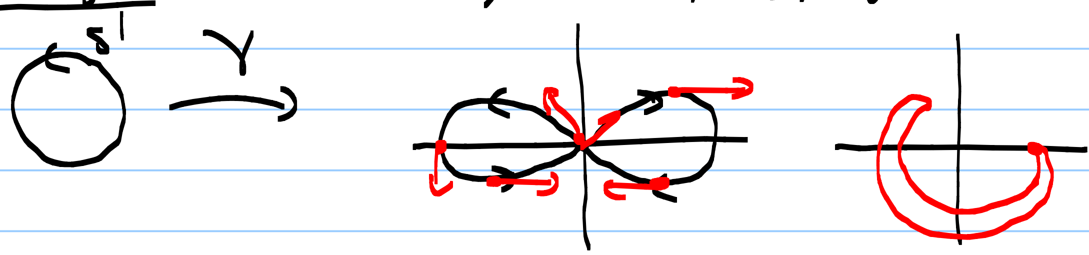
Note that since f is an immersion $g(p) \neq 0$, for any $p \in S^1$.



Rot(f) = winding number of g

It is the count of rotations of the tangent vector of the curve $f: S^1 \to \mathbb{R}^2$.

Example. $\gamma: S^1 \to \mathbb{R}^2$, $\gamma(x,y) = (y, xy)$



$$\text{Rot}(\gamma) = 0, \quad g(x,y) = \gamma_*(v_{(x,y)})$$

$$\text{Rot}(\gamma) = \int_{S^1} g^*(\omega), \quad \omega = \frac{1}{2\pi}\frac{x\,dy - y\,dx}{x^2+y^2}$$

$\gamma(x,y) = (y, xy) \qquad t \longmapsto (\cos t, \sin t)$

$\gamma(t) = (\sin t, \cos t \sin t)$

$g(t) = \dot{\gamma}(t) = (\cos t, \cos 2t) \qquad \cos t \sin t = \frac{1}{2}\sin 2t$

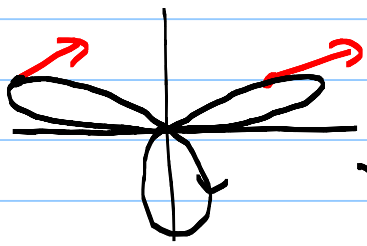
$$g^*(\omega) = \frac{1}{2\pi} \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{1}{2\pi} \frac{\cos 2\theta \sin\theta - 2\cos\theta \sin 2\theta}{(\cos^2\theta + \cos^2 2\theta)} d\theta$$

$$\text{Rot}(\gamma) = \int_{S^1} g^*(\omega) = \frac{1}{2\pi}\int_0^{2\pi} \qquad d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overbrace{\frac{\cos 2\theta \sin\theta - 2\cos\theta \sin 2\theta}{(\cos^2\theta + \cos^2 2\theta)}}^{\text{odd function}} d\theta$$

$$= 0.$$

<u>Example:</u> $r = \sin 3\theta$, $\theta \in [0, \pi]$.



$\text{Rot}(\gamma) = 2$

$\gamma(\theta) = (\sin 3\theta \cos\theta, \sin 3\theta \sin\theta)$

$g(\theta) = \dot{\gamma}(\theta) = (2\cos 4\theta + \cos 2\theta, 2\sin 4\theta - \sin 2\theta)$

$$g^*(\omega) = \frac{1}{2\pi} \frac{14 + 4\cos 6\theta}{5 + 4\cos 6\theta}, \quad \text{Rot}(\gamma) = \int_{S^1} g^*(\omega)$$

$$\text{Rot}(\gamma) = \frac{1}{2\pi}\int_0^{\pi} \frac{14 + 4\cos 6\theta}{5 + 4\cos 6\theta} d\theta \qquad \phi = 6\theta$$

$$= \frac{1}{2\pi}\int_0^{6\pi} \frac{14 + 4\cos\phi}{5 + 4\cos\phi} \frac{1}{6} d\phi$$

$$= \frac{1}{12\pi} \cdot 3 \int_0^{2\pi} \frac{14 + 4\cos\phi}{5 + 4\cos\phi} d\phi$$

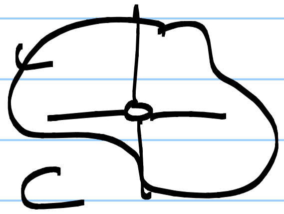
$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{14 + 4\cos\phi}{5 + 4\cos\phi}\, d\phi \qquad u = \tan\frac{\phi}{2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{18 + 10u^2}{(9+u^2)(1+u^2)}\, du$$

$$= 2.$$

<u>Linking Number</u>: $\mathbb{R}^2 \setminus \{(0,0)\}$

$S^1 \xrightarrow{f}$



$x=0,\ y=0$

$$\omega = \frac{1}{2\pi} \frac{x\,dy - y\,dx}{x^2+y^2}$$

$\int_{S^1} f^*(\omega)$ can be considered as linking of $(0,0)$ with the image of f.

$\int_C \omega$ if $f: S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ is an embedding.

In $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ take a curve C.
Its linking with z-axis can be defined as

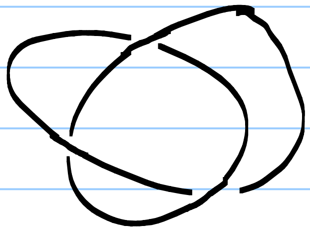
z-axis : $x=0, y=0$ in $\mathbb{R}^3$

The linking form of z-axis is $\dfrac{x\,dy - y\,dx}{2\pi(x^2+y^2)}$
$\in \Omega^1(\mathbb{R}^3 \setminus z\text{-axis})$

$$\mathrm{Link}(C, z\text{-axis}) = \int_C \omega$$

Example: Let K be a knot in $\mathbb{R}^3$ given by

$K: f = g = 0$, for some smooth functions $f, g: \mathbb{R}^3 \to \mathbb{R}$ with $\nabla f \times \nabla g \neq (0,0,0)$ at all points of K.



K linking form of K is

$$\omega_K = \frac{1}{2\pi} \frac{f\,dg - g\,df}{f^2 + g^2}$$

If L is any other knot in $\mathbb{R}^3 \setminus K$ then the linking $\mathrm{Link}(L,K)$ is defined as

$$\mathrm{Link}(L,K) = \int_L \omega_K.$$

One can define linking of submanifolds in $\mathbb{R}^n$:

$K \subseteq \mathbb{R}^{n+k}$ submanifold of dimension k. Assume it is as a complete intersection

$K: f_1 = 0, \ldots, f_n = 0$ $\qquad \frac{1}{2\pi} \frac{x\,dy - y\,dx}{x^2+y^2}$

$$\omega_K = \underbrace{\frac{1}{nA_n}}_{\text{volume of } S^{n-1}} \sum_{i=1}^{n} (-1)^{i-1} f_i \frac{df_1 \wedge \cdots \wedge df_{i-1} \wedge df_{i+1} \wedge \cdots \wedge df_n}{(f_1^2 + \cdots + f_n^2)^{n/2}}$$

$\left[\, n=2 \quad \omega_K = \frac{1}{2 \cdot A_2} \frac{f_1\,df_2 - f_2\,df_1}{f_1^2 + f_2^2} \,\right]$

$\omega_K \in \Omega^{n-1}(\mathbb{R}^{n+k} \setminus K)$ is a closed k-form

If L is a manifold in $\mathbb{R}^{n+k} \setminus K$ of dimension $\dim L + \dim K = n+k-1$, then the linking of L and K is defined as

$$\mathrm{Link}(L, K) = \int_L \omega_K \qquad (\dim L = n-1)$$

<u>Remark</u> $\frac{1}{nA_n} \sum_{i=1}^{n} (-1)^{i-1} x_i \frac{dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n}{(x_1^2 + \cdots + x_n^2)^{n/2}} = \omega_{S^n}$

is a closed $n-1$ form on $\mathbb{R}^n \setminus \{0\}$ with the property that

$$\int_{S^n} \omega_{S^n} = 1.$$

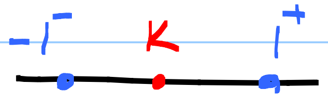
ω_{S^n} is closed and $H^{n-1}_{DR}(\underbrace{\mathbb{R}^n \setminus \{0\}}_{\mathbb{R} \times S^{n-1}}) \simeq \mathbb{R}$

<u>Special Case</u> 1) $K = \{0\} \subseteq \mathbb{R}$
$x = 0$

$$\omega_K = \frac{1}{1 \cdot A_1} \frac{x}{\sqrt{x^2}} = \frac{1}{2} \frac{x}{|x|} \quad \text{0-form on } \mathbb{R}\setminus\{0\}.$$

$L = \{1^+\}$ $\quad \mathrm{Link}(K, L) = \int_L \omega_K = \frac{1}{2}.$

$L = \{-1^-, 1^+\}$ $\quad \mathrm{Link}(K, L) = -\frac{1}{2}\left(\frac{-1}{|-1|}\right) + \frac{1}{2}\frac{1}{|1|}$
$= S^0$



$= 1.$

2) $K = S^0 = \{-1, 1\} \subseteq \mathbb{R}$

$f = x^2 - 1 = 0$

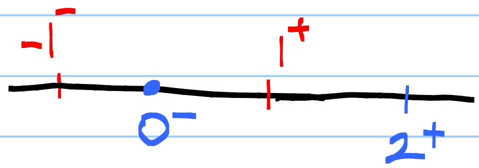
$2 = \mathrm{vol}(S^0) = \mathrm{vol}(1,-1)$

$$\omega_K = \frac{1}{2} \frac{f}{|f|} = \frac{x^2-1}{2|x^2-1|}.$$

$L = 0$, $\mathrm{Link}(L, K) = \dfrac{-1}{2}$

$L = \{0^-, 2^+\}$ $\mathrm{Link}(L, K) = \dfrac{-1}{2}\left(\dfrac{-1}{1}\right) + \dfrac{1}{2}\,\dfrac{3}{3}$

$= 1.$



Video 34

Cochain Complexes and Mayer-Vietoris's Exact Sequence for de Rham Cohomology

A cochain complex is a sequence of vector space homomorphisms of the form

(A_*, d_*)

$$\cdots \longrightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \xrightarrow{d_{n+1}} \cdots$$

A_n: vector space, d_n: vector space homomorphism

so that $d_n \circ d_{n-1} = 0$, for all n.

Hence, $\mathrm{Im}\, d_{n-1} \subseteq \ker(d_n)$.

The n^{th} cohomology of (A_*, d_*) is defined as the quotient vector space

$$H^n(A_*, d_*) \doteq \frac{\ker(d_n : A_n \to A_{n+1})}{\mathrm{Im}(d_{n-1} : A_{n-1} \to A_n)}.$$

Now consider homomorphisms of cochain complexes

$$\cdots \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to \cdots$$

$$\begin{tikzcd}
\arrow[r] & A_{n-1} \arrow[r, "d^A_{n-1}"] \arrow[d, "f_{n-1}"] & A_n \arrow[r, "d^A_n"] \arrow[d, "f_n"] & A_{n+1} \arrow[r, "d^A_{n+1}"] \arrow[d, "f_{n+1}"] & \cdots \\
\arrow[r] & B_{n-1} \arrow[r, "d^B_{n-1}"] \arrow[d, "g_{n-1}"] & B_n \arrow[r, "d^B_n"] \arrow[d, "g_n"] & B_{n+1} \arrow[r, "d^B_{n+1}"] \arrow[d, "g_{n+1}"] & \cdots \\
\arrow[r] & C_{n-1} \arrow[r, "d^C_{n-1}"'] & C_n \arrow[r, "d^C_n"'] & C_{n+1} \arrow[r, "d^C_{n+1}"'] & \cdots
\end{tikzcd}$$

We require that each rectangle is commutative.

$$\begin{array}{ccc} \bullet & \xrightarrow{d_n} & \bullet \\ f_n \downarrow & & \downarrow f_{n+1} \\ \bullet & \xrightarrow{d_n} & \bullet \end{array} \qquad f_{n+1} \circ d_n = d_n \circ f_n$$

A sequence of the form

$$\cdots \longrightarrow (A_*, d_*) \xrightarrow{f_*} (B_*, d_*) \xrightarrow{g_*} (C_*, d_*) \longrightarrow \cdots$$

is called exact at (B_*, d_*) if

$\operatorname{Im} f_* = \ker g_*$. In other words,

for all n, $\operatorname{Im}(f_n : A_n \to B_n) = \ker(g_n : B_n \to C_n)$.

A sequence of cochain complexes

$$0 \longrightarrow (A_*, d_*) \xrightarrow{f_*} (B_*, d_*) \xrightarrow{g_*} (C_*, d_*) \longrightarrow 0$$

is called short exact if it is exact at all vector spaces. Hence, it is short exact if and only if each

$0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$ is exact at all vector spaces. This is equivalent to

1) $\ker f_n = 0 \iff f_n$ is injective
2) $\operatorname{Im} f_n = \ker g_n$
3) $\operatorname{Im} g_n = C_n \iff g_n$ is onto.

Note that in this case, $C_n = \operatorname{Im} g_n = \dfrac{B_n}{\ker g_n} = \dfrac{B_n}{\operatorname{Im} f_n}$

$$\simeq \frac{B_n}{f(A_n)}$$

Or equivalently, $B_n \simeq A_n \oplus C_n$

<u>Theorem</u> (Algebraic Fact)

Suppose $0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$ is a short exact sequence of cochain complexes. Then there is a long exact sequence of chomology groups of these cochain complexes as follows:

$$\cdots \to H^{n-1}(C_*) \xrightarrow{\delta} H^n(A_*) \xrightarrow{f^*} H^n(B_*) \xrightarrow{g^*} H^n(C_*) \xrightarrow{\delta} H^{n+1}(A_*) \to$$

where δ is called the connecting homomorphism

<u>Proof:</u>

$$\begin{array}{ccccccc} \to A_{n-1} & \xrightarrow{d} & A_n & \xrightarrow{d} & A_{n+1} & \to & \\ \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_{n+1} & & \\ \to B_{n-1} & \xrightarrow[d]{} & B_n & \xrightarrow[d]{} & B_{n+1} & \to & \cdots \end{array}$$

$$f_n^* : H^n(A^*) \longrightarrow H^n(B^*)$$
$$\frac{\ker(d: A_n \to A_{n+1})}{\mathrm{Im}(d: A_{n-1} \to A_n)} \qquad \frac{\ker(d: B_n \to B_{n+1})}{\mathrm{Im}(d: B_{n-1} \to B_n)}$$

Let $a \in \ker(d: A_n \to A_{n+1})$. Define $f_n^*([a]) \doteq [f_n(a)]$.
<u>must show two things:</u>

1) $\underline{\underline{d(f_n(a))}} = 0$ so that $f_n(a)$ defines a cohomology class.

$df_n(a) = f_n(da) = f_n(0) = 0$ ✓

2) $[a] = [b] \overset{?}{\Rightarrow} [f_n(a)] = [f_n(b)]$

$[a]=[b] \Rightarrow [a-b]=0 \Rightarrow a-b = dc$, for some $c \in A_{n-1}$.

$$\Rightarrow f_n(a-b) = f_n(dc) = d\, f_n(c)$$

$$f_n(a) - f_n(b) = d\, f_n(c)$$

$$\Rightarrow [f_n(a)] - [f_n(b)] = [d\, f_n(c)] = 0.$$

$$\Rightarrow [f_n(a)] = [f_n(b)]$$

So, f^* and g^* are well defined.

How to define $\delta : H^{n-1}(C_*) \longrightarrow H^n(A)$?

$$\begin{array}{ccccccccc}
 & & \vdots & & u & \longmapsto & w & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_{n-1} & \xrightarrow{f} & B_{n-1} & \xrightarrow{g} & C_{n-1} & \to & 0 \\
 & & d\downarrow & & d\downarrow\ y & \longmapsto & d\downarrow\ z = z' & & \\
0 & \to & A_n & \xrightarrow{f} & B_n\ \ y-du & \xrightarrow{g} & C_n & \to & 0 \\
 & & d\downarrow\ x & \longmapsto & d\downarrow\ dy & \longmapsto & d\downarrow\ 0 & & \\
0 & \to & A_{n+1} & \xrightarrow{f} & B_{n+1} & \xrightarrow{g} & C_{n+1} & \to & 0 \\
 & & \downarrow\ dx & \longmapsto & \downarrow\ d^2y = 0 & & \downarrow & & \\
 & & \vdots\ \ 0 & & \vdots & & \vdots & &
\end{array}$$

$[z] = [z']$

$z - z' = dw$

$dx = 0$? L

$g(y) = z$

$f(x) = dy \quad g(u) = w$

$g(y) = z \quad g(du) = z - z'$

$g(y - du) = z'$

$f(x') = d(y - du) = dy$

$$\delta : H^{n-1}(C_*) \longrightarrow H^n(A_*)$$
$$[z] \in$$

Take any element $[z] \in H^{n-1}(C_*)$, where $z \in C_{n-1}$ such that $dz = 0$.

$f(x - x') = f(x) - f(x')$
$= dy - dy = 0$
$\Rightarrow x - x' = 0.$

We define $\delta([z])$ as $[x]$.

We must show that $[x]$ is uniquely determined by $[z]$. Done above!

One must also check that the sequence in the statement is exact.

We'll apply this tool to de Rham cohomology as follows:

Assume that a smooth manifold M is written as $M = U \cup V$ the union of two open subsets.

$$
\begin{tikzcd}
 & U \arrow[dr, "i_U"] & \\
U \cap V \arrow[ur, "j_U"] \arrow[dr, "j_V"'] & & U \cup V = M \\
 & V \arrow[ur, "i_V"'] &
\end{tikzcd}
$$

all are inclusion maps. This gives a diagram of homomorphisms:

$$
\begin{tikzcd}
 & \Omega^k(U) \arrow[dl, "j_U^*"'] & \\
\Omega^k(U \cap V) & & \Omega^k(\overset{M}{\overset{\shortparallel}{U \cup V}}) \arrow[ul, "i_U^*"'] \arrow[dl, "i_V^*"] \\
 & \Omega^k(V) \arrow[ul, "j_V^*"] &
\end{tikzcd}
$$

We have a short exact sequence of cochain complexes:

$$
0 \to \overset{d\downarrow}{\Omega^k(M)} \xrightarrow{(i_U^*,\, i_V^*)} \overset{d\downarrow}{\Omega^k(U)} \oplus \overset{d\downarrow}{\Omega^k(V)} \longrightarrow \overset{d\downarrow}{\Omega^k(U \cap V)} \to 0
$$

$$
\omega \longmapsto (i_U^*(\omega), i_V^*(\omega)) \longmapsto \omega|_{U\cap V} - \omega|_{U \cap V} = 0
$$

$$
(\eta, \theta) \longmapsto j_U^*(\eta) - j_V^*(\theta)
$$

one to one Im = ker onto

$\subseteq$

$\omega \in \Omega^k(M) = \Omega^k(U \cup V)$. Assume $i_U^*(\omega) = 0$ and $i_V^*(\omega) = 0$. So $\omega|_U = 0$, $\omega|_V = 0$.
$\Rightarrow \omega = 0$ on $U \cup V = M$. Hence, i_U^* is inj.

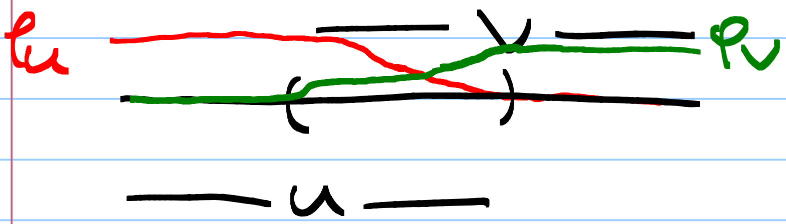
<u>must show:</u> $(\eta, \theta) \longmapsto j_U^*(\eta) - j_V^*(\theta)$ is onto.

Video 35

Let $\omega \in \Omega^k(U\cap V)$. Choose a partition of unity associated to the open cover $M = U\cup V$, say $\{\rho_U, \rho_V\}$:

$$\rho_U + \rho_V = 1$$

$$\overline{\mathrm{supp}(\rho_U)} \subseteq U, \quad \overline{\mathrm{supp}(\rho_V)} \subseteq V.$$



Now $\rho_U\cdot\omega$ has support in U. Thus we may extend $\rho_U\omega$ to $V\setminus(U\cap V)$ as zero.

Hence, we may regard $\rho_U\omega \in \Omega^k(V)$. Similarly, we may regard $\rho_V\omega \in \Omega^k(V)$.

Then

$$(\rho_V\omega, -\rho_U\omega) \longmapsto J_U^*(\rho_V\omega) - (J_V^*(-\rho_U\omega))$$
$$= \rho_U\omega + \rho_V\omega$$
$$= (\rho_U+\rho_V)\cdot\omega = 1\cdot\omega = \omega.$$

Hence, the last map is onto.

Finally, if $(\eta, \theta) \in \ker(J_U^* \oplus J_V^*)$ then

$\eta|_{U\cap V} = \theta|_{U\cap V}$. So define

$$\omega = \begin{cases} \eta & \text{on } U \\ \theta & \text{on } V \end{cases} \in \Omega^k(U\cup V) = \Omega^k(M)$$

then $J_U^*(\omega) = \eta$ and $J_V^*(\omega) = \theta$.

Now since the the sequence

$$0 \to \Omega^k(\underset{M}{\underbrace{U \cup V}}) \to \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V) \to 0$$

is exact for all k, we obtain a long exact sequence in cohomology

$$\cdots \to H^k_{dR}(M) \to H^k_{dR}(U) \oplus H^k_{dR}(V) \to H^k_{dR}(U \cap V) \xrightarrow{\delta} H^{k+1}_{dR}(M) \to \cdots$$

$\bullet$

<u>Example:</u> $H^k(S^n) = \begin{cases} \mathbb{R} & \text{if } k = 0, n \\ 0 & \text{otherwise} \end{cases}$, $n \geq 1$.

<u>Proof</u>: Induction on n.

$n = 1$, $H^k(S^1) = \begin{cases} \mathbb{R} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

Assume the result for n and let's consider S^{n+1}. Let $U = S^{n+1} \setminus \{N\}$, $V = S^{n+1} \setminus \{S\}$

$N = (0, -, 0, 1)$, $S = (0, -, 0, -1)$.

Then, clearly $U \cup V = S^{n+1}$; $U \simeq \mathbb{R}^{n+1} \simeq V$

$$U \cap V = S^{n+1} \setminus \{S, N\} \simeq \mathbb{R}^{n+1} \setminus \{0\} \simeq S^n \times \mathbb{R}$$

Now by the Mayer-Vietoris sequence we have

$$0 \to \underset{\underset{S^n \times \mathbb{R}}{\cong}}{H^n(U \cap V)} \xrightarrow{\delta} \underset{\underset{S^{n+1}}{\|}}{H^{n+1}(U \cup V)} \to \underset{\underset{\mathbb{R}^{n+1}}{\cong}}{\overset{\overset{0}{\|}}{H^{n+1}(U)}} \oplus \underset{\underset{\mathbb{R}^{n+1}}{\cong}}{\overset{\overset{0}{\|}}{H^{n+1}(V)}} \to \underset{\underset{S^n \times \mathbb{R}}{\|}}{\overset{\overset{0}{\|}}{H^{n+1}(U \cap V)}}$$

"Poincaré Lemma"
$H^n(S^n) \simeq \mathbb{R}$

$$\xrightarrow{\delta} H^{n+2}(\overset{\overset{0}{\shortparallel}}{U\cap V}) \to$$

$$0 \to \mathbb{R} \to H^{n+1}(S^{n+1}) \to 0$$

So, $H^{n+1}(S^{n+1}) \simeq \mathbb{R}$. $H^k(S^{n+1}) = 0$, $k \geq n+2$
and $H^0(S^{n+1}) \simeq \mathbb{R}$.

$H^k(S^{n+1}) = ?$ $1 \leq k \leq n$.

Exercise: Show that $H^k(S^{n+1}) = 0$ for $1 \leq k \leq n$.

In particular, we have $H^n_{dR}(S^n) \simeq \mathbb{R}$

and isomorphism is given by integration:

$$I : H^n_{dR}(S^n) \longrightarrow \mathbb{R}, \quad [\omega] \longmapsto \int_{S^n} \omega.$$

For $H^2_{dR}(S^2) \simeq \mathbb{R}$ and a generator is

given by $\omega = \dfrac{1}{4\pi} \dfrac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$

$\left[\omega \in \Omega^2(\mathbb{R}^3 \setminus \{0\})\right]$

Theorem (Gauss-Bonnet Thm).

If Σ_g is a genus g closed surface in $\mathbb{R}^3$ then

$$\int_{\Sigma_g} K\,dS = 4\pi(1-g) = 2\pi\underbrace{(2-2g)}_{= \chi(\Sigma_g)}$$

$\sigma^{-1}(\mathcal{P}) = \{q_1, q_2, \dots, q_{g+1}\}$

K = Gauss Curvature of Σ_g.

If Σ_g is given locally as the graph of a function $z = f(x,y)$ then

$$K\,dS = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2}\sqrt{1+f_x^2+f_y^2}\;dx\wedge dy$$

Gauss map: $\Sigma_g \subseteq \mathbb{R}^3$, $\sigma : \Sigma_g \longrightarrow S^2$
$p \longmapsto u(p)$,

$u(p)$ is a unit normal vector.

$$u(p) = (-f_x, -f_y, 1)\Big/\sqrt{1+f_x^2+f_y^2}$$

If $\omega = \dfrac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$, then

$$\sigma^*\omega = K\,dS$$

So, we must compute $\displaystyle\int_{\Sigma_g} K\,dS = \int_{\Sigma_g} \sigma^*\omega$

$[\omega] \in H^2_{dR}(S^2) \simeq \mathbb{R}$. Choose any 2-form η on S^2 so that

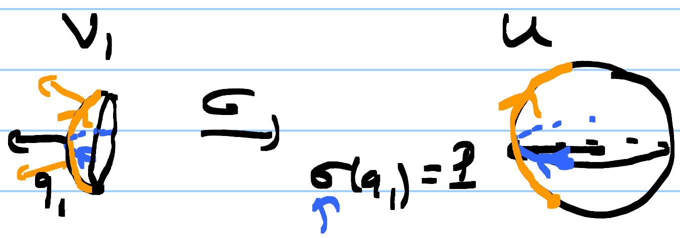
$$\int_{S^2}\eta = \int_{S^2}\omega = 4\pi$$

and η is supported in a small open set around the point $p \in S^2$.

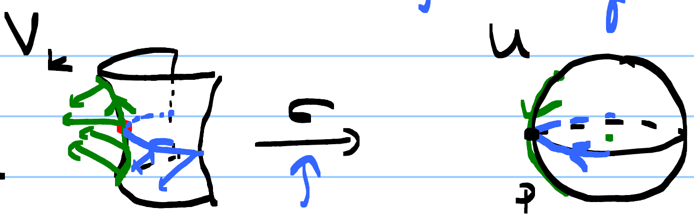
Video 36

Then $[\omega]=[\eta]$ so that
$$\int_{\Sigma_g}\sigma^*\omega=\int_{\Sigma_g}\sigma^*\eta$$

Note that σ is a diffeomorphism near each q_i.

q_1: 

orientation preserving

q_k: $k\geq 2$ 

orientation reversing

$$\int_{\Sigma_g}K\,dS=\int_{\Sigma_g}\sigma^*(\omega)=\int_{\Sigma_g}\sigma^*(\eta)=\int_{V_1\cup\cdots\cup V_k}\sigma^*(\eta)$$

$$=\sum_k\int_{V_k}\sigma^*(\eta)=\sum_{k=1}^{g+1}\pm\underbrace{\int_U\eta}_{4\pi}\qquad \sigma:V_k\to U \text{ diffeom.}$$

where $\pm$ sign is determined depending on σ is orientation preserving or reversing on V_k. We know that it is orientation preserving for $k=1$ and reversing for $k\geq 2$.

Hence, $\displaystyle\int_{\Sigma_g}K\,dS=(1-g)\int_U\eta=(1-g)4\pi.$

However, this is proved for a specially embedded surface Σ_g in $\mathbb{R}^3$.

What about an arbitrary embedded Σ_g in $\mathbb{R}^3$?

Let $i_0 : \Sigma_g \to \mathbb{R}^3$ be the special embedding and

$i_1 : \Sigma_g \to \mathbb{R}^3$ an arbitrary embedding.

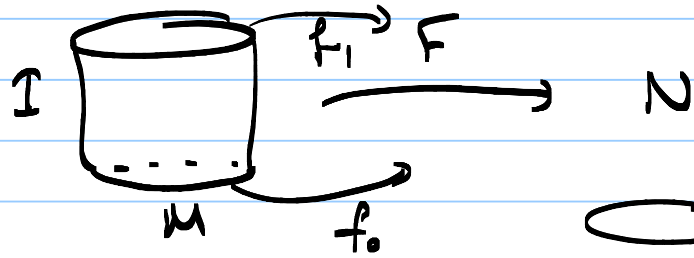
<u>must show:</u> $\displaystyle\int_{\Sigma_g = i_0(\Sigma_g)} K\,dS = \int_{i_1(\Sigma_g)} K\,dS$

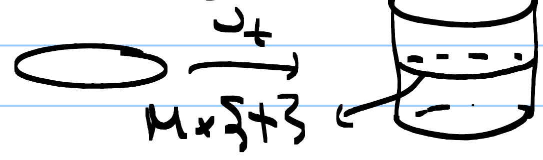
$$4\pi(1-g) = \int_{\Sigma_g} \sigma_{i_0}^*(\omega) \overset{?}{=} \int_{\Sigma_g} \sigma_{i_1}^*(\omega)$$

Yes

if i_0 and i_1 are homotopic!

<u>Definition:</u> Two maps f_0 and f_1 from M to N are called homotopic if there is a smooth map
$$F : M \times [0,1] \to N$$
so that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.





Note that if $J_t : M \to M \times I$ is the map $J_t(x) = (x,t)$ then

$f_0(x) = F(x,0) = F(J_0(x))$ and
$f_1(x) = F(x,1) = F(J_1(x))$.

Then $f_0^* = (F \circ J_0)^* = J_0^* \circ F^*$ and

$$\begin{aligned} &= J_t^* \circ F^* \\ &= J_1^* \circ F^* \\ &= (F \circ J_1)^* \\ &= f_1^* \end{aligned}$$

<u>Question:</u> Are f_0 and f_1 homotopic through immersions in $\mathbb{R}^3$?

These maps are homotopic through immersions in $\mathbb{R}^n$, $n \geq 7$.

Answer: Yes.

We must also replace S^2 by something else.

S^2: the space of oriented 2-dim'l subspaces of $\mathbb{R}^3$.

Let $\tilde{Gr}^+(2,n)$ be the space of all oriented 2-dim'l subspaces of $\mathbb{R}^n$. $Gr^+(2,n)$ is called the Grassmann manifold.

$$\begin{array}{ccccccc} S^2 = Gr^+(2,3) & \hookrightarrow & Gr^+(2,4) & \hookrightarrow & \cdots & \hookrightarrow & Gr^+(2,n) \\ \| & & & & & & \| \\ \mathbb{CP}^1 & & & & & & Q_{n-1} \end{array}$$

Q_{n-2} is the hypersurface in $\mathbb{CP}^{n-1}$ defined by the quadratic polynomial $z_1^2 + \cdots + z_n^2 = 0$

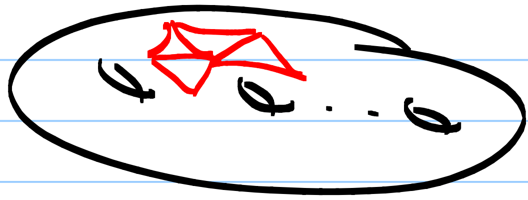
$$
\begin{tikzcd}
\overset{[\omega_1]}{S^2 = \mathbb{CP}^1} \arrow[r, "J"] \arrow[d, equal] & \overset{[\omega]}{\mathbb{CP}^{n-1}} \arrow[d, bend left, mapsto] \\
\mathbb{CP}^1 \arrow[r, hook] & Q_{n-2} \; [\omega_1] \arrow[u, hook] \\
& \Sigma_g \arrow[ul, "f_0"] \arrow[u, "f_1"']
\end{tikzcd}
$$

$$\int_{\Sigma_g} \sigma_{f_0}^*(\omega_1) = \int_{\Sigma_g} \sigma_{f_0}^*(J^*(\omega)) = \int_{\Sigma_g} (J \circ \sigma_{f_0})^*(\omega)$$

$$= \int_{\Sigma_g} \sigma_{f_1}^*(\omega)$$

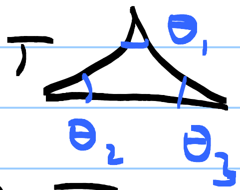
$\blacksquare$

A geometric proof: Divide the surface into triangles by geodesics.



Gauss' Thm

Geodesic triangle



$$\int_T K \, dS = (\theta_1 + \theta_2 + \theta_3) - \pi$$

Then the rest is a combinatorial computation.

Example 1) $\mathbb{R}^2$, $K = 0$

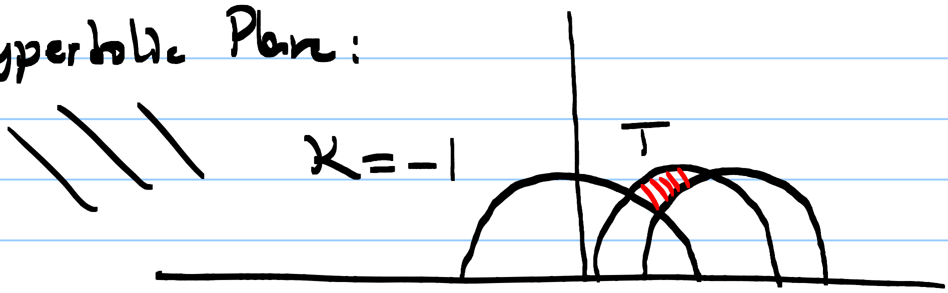
$$\int K \, dS = 0 = \theta_1 + \theta_2 + \theta_3 - \pi$$

2) S_r^2  $K(p) = \frac{1}{r^2}$

$$\int_{S_r^2} K(p)\, dS = \int_{S_r^2} \frac{1}{r^2}\, dS = \frac{1}{r^2}(4\pi r^2) = 4\pi$$

$$\theta_1 + \theta_2 + \theta_3 - \pi = \int_T K\, dS = \frac{1}{r^2}\int_T dS = \frac{\text{Area}(T)}{r^2}$$

3) Hyperbolic Plane:



$$\text{Area}(T) = -1 \cdot (\theta_1 + \theta_2 + \theta_3 - \pi)$$

Video 37

Integration of Vector Fields:

Theorem: Let M be a closed smooth manifold and X a smooth vector field in M. Then there is a unique "flow", a differentiable function

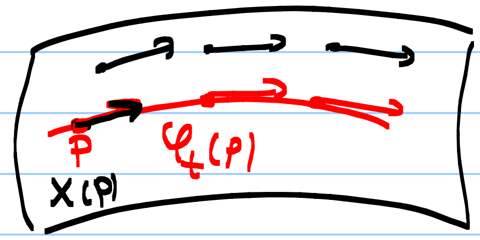
$$\varphi : \mathbb{R}\times M \longrightarrow M, \quad (t,p) \longmapsto \varphi(t,p) \quad \text{so that}$$

$$X(\varphi_t(p)) = \dot{\varphi}_t(p)$$

$\varphi_t(p) \doteq \varphi(t,p)$

$\dot{\varphi}_t(p) = \frac{d}{dt}\varphi_t(p)$

M



To prove the above theorem we need to recall the Existence-Uniqueness of O.D.E.'s and smooth dependence of the solutions with respect to the initial conditions.

Existence-Uniqueness of O.D.E.'s

B Banach Space, $B=(\mathbb{R}^n, \|\cdot\|)$, $U \subseteq B$ an open subset and $f: U \longrightarrow B$ a continuous function

We'll say that f is Lipschitz if there is some $\lambda > 0$ so that $\|f(x)-f(y)\| \leq \lambda \|x-y\|$, for all $x,y\in U$.

Theorem: Let B be a Banach space, $I \subseteq \mathbb{R}$ an open interval and $f: I\times B \longrightarrow B$ a continuous function, which is Lipschitz in the second coordinate. Then for any $t_0 \in I$, $y_0 \in B$, the

Initial Value Problem

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$ has a unique solution defined

on some interval $(t_0-\epsilon, t_0+\epsilon) \subseteq I$. Moreover, if $f \in C^k$ then $y \in C^{k+1}$.

<u>Proof.</u> We first convert the I.V.P. to an integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds.$$

Let $B_1 = C^0([t_0-\epsilon, t_0+\epsilon], B)$, where $\epsilon > 0$ is so that $\epsilon\lambda = \alpha < 1$, and λ is the above Lipschitz constant of f w.r.t. the second coordinate. Here B_1 is equipped with the supremum metric.

Define $\theta : B_1 \to B_1$ by

$$\theta(y)(t) = \underline{y_0} + \int_{t_0}^{t} f(s, y(s))\,ds.$$

<u>Claim.</u> θ is a contraction mapping

<u>Proof</u> If $y_1, y_2 \in B_1$ then

$$\|\theta(y_1)(t) - \theta(y_2)(t)\| = \left\| \int_{t_0}^{t} \left(f(s, y_1(s)) - f(s, y_2(s))\right) ds \right\|$$

$$\leq |t - t_0|\, \|f(s, y_1(s)) - f(s, y_2(s))\|$$

$$\leq \epsilon\, \lambda\, \|y_1(s) - y_2(s)\|$$

$$\leq \alpha\, \|y_1(t) - y_2(t)\|$$

So, θ is a contraction mapping. ∎

Then θ has a unique fixed point $y=\phi(t)$. So

$$\phi(t)=\theta(\phi(t))=y_0+\int_{t_0}^{t} f(s,\phi(s))\,ds,$$ and hence

$\phi(t)$ is the solution of the I.V.P.

The fact that $\phi(t)\in C^{k+1}$ follows from the identity
$$\phi(t)=y_0+\int_{t_0}^{t} f(s,\phi(s))\,ds.$$
∎

Continuous Dependence of the solution $\phi(t)$ on the initial conditions.

Choosing the interval I in the theorem smaller if necessary we may assume that for any $(t_1,y_1)\in I\times B$ the solution of the I.V.P. $y'=f(t,y(t))$, $y(t_1)=y_1$ will be in the

space $B_1=C^0([t_0-\epsilon,t_0+\epsilon],B)$.

Now for (t_0,y_0) and (t_0+h_1,y_0+h_2) let y^0 and y^h $(h=(h_1,h_2))$ denote the solutions corresponding to the I.C. (t_0,y_0) and (t_0+h_1,y_0+h_2), respectively.

First assume that f is bounded on U.

Consider the function $\Phi: I\times B\longrightarrow B_1$ mapping (t_0,y_0) to the unique solution of the I.V.P. $y'=f(t,y)$, $y(t_0)=y_0$.

Now,

$$\|\Phi(t_0+h_1, y_0+h_2) - \Phi(t_0, y_0)\| = \|y^h - y^0\|$$

$$= \|\theta(y^h) - \theta(y^0)\|$$

$$= \| \cancel{y_0} + h_2 + \int_{t_0+h_1}^{t} f(s, y^h(s))\,ds - \cancel{y_0} - \int_{t_0}^{t} f(s, y^0(s))\,ds \|$$

$$= \| h_2 - \int_{t_0}^{t_0+h_1} f(s, y^0(s))\,ds + \int_{t_0+h_1}^{t} \left(f(s, y^h(s)) - f(s, y^0(s))\right) ds \|$$

$$\leq \|h_2\| + \| \int_{t_0}^{t_0+h_1} f(s, y^0(s))\,ds \| + \| \int_{t_0+h_1}^{t} \left(f(s, \underline{y^h(s)}) - f(s, \underline{\underline{y^0(s)}})\right) ds \|$$

$$\leq \|h_2\| + |h_1|\,\|f\| + \underbrace{\lambda \|y^h(s) - y^0(s)\|\, \epsilon}_{= \alpha}$$

$$\leq \|h_2\| + |h_1|\,\|f\| + \alpha \|y^h - y^0\|$$

$$(1-\alpha)\|y^h - y^0\| \leq \|h_2\| + |h_1|\,\|f\|$$

$$\|y^h - y^0\| \leq \frac{\|h_2\| + |h_1|\,\|f\|}{1-\alpha}.$$

So Φ is continuous with respect to (t_0, y_0).

<u>Proof of the Theorem:</u> M compact smooth mfd.

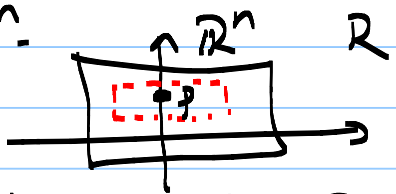
$X : M \to TM$ smooth vector field.

We must construct a function $\varphi : \mathbb{R} \times M \to M$ so that $X(\varphi_t(p)) = \dot{\varphi}_t(p)$ $\quad (\varphi_t(p) \doteq \varphi(t, p))$

Video 38

Let $p \in M$. Taking a coordinate neighborhood around p we may assume that $p \in \mathbb{R}^n$. So we'll solve equation
$$X(\varphi_t(p)) = \dot{\varphi}_t(p) \quad \text{on} \quad \mathbb{R}\times\mathbb{R}^n.$$

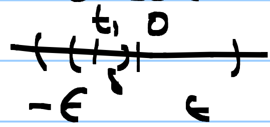
Take a closed rectangular region around $(0,p)$ in $\mathbb{R}\times\mathbb{R}^n$.



In this rectangle $X(\varphi)$ is Lipschitz w.r.t. p and hence taking R even smaller if necessary we can find a solution $\varphi : (-\epsilon_p, \epsilon_p)\times U_p \to M$ satisfying $\varphi(0,p) = p$. Since M is closed (compact connected) we can cover M with finitely U_p's so that taking $\epsilon = \min\{\epsilon_{p_1}, \ldots, \epsilon_{p_n}\} > 0$ we find a function
$$\varphi : (-\epsilon, \epsilon)\times M \to M$$
satisfying the equation $X(\varphi_t(p)) = \dot{\varphi}_t(p)$ and $\varphi_0 = \mathrm{Id}_M$.

We've seen that $\varphi_t(p)$ is continuous in (t,p).

Next we show that $\varphi_t(p)$ is defined for all $t \in \mathbb{R}$. For any $p \in M$ and $-\epsilon < t_1 < \epsilon$ choose an interval $(t_1 - \delta, t_1 + \delta) \subseteq (-\epsilon, \epsilon)$.



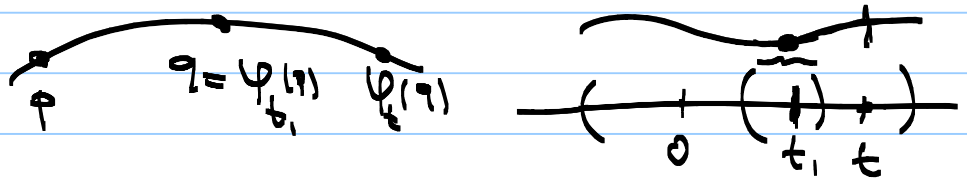
Now the function $f : (-\delta, \delta) \to M$, $f(t) = \varphi_{t_1+t}(p)$.
Also let $q = \varphi_{t_1}(p)$. Then the function

$g : (-\delta, \delta) \to M$ by $g(t) = \varphi_t(q)$ and $f(t)$ both solves the I.V.P.
$$\dot{y}(t) = X(y(t)) \text{ and } y(0) = q.$$

$$f'(t) = \dot{\varphi}_{t_1+t}(p) = X(\varphi_{t+t_1}(p)),\ \dot{g}(t) = \dot{\varphi}_t(q)$$

$$g(t) = \dot\varphi_t(\varphi_{t_1}(p)) = X(\varphi_t(\varphi_{t_1}(p))) = X(\varphi_{t+t_1}(p)) = \dot f(t)$$

and $f(0) = \varphi_{t_1}(p) = q = g(0)$



Hence, $f(t) = g(t) \Rightarrow \varphi_{t+t_1}(p) = \varphi_t(\varphi_{t_1}(p))$,

for all $p \in M$ and $t \in (t_1 - \delta, t_1 + \delta) \subseteq (-\epsilon, \epsilon)$.

So, for any $-\epsilon < s_i, t_j < \epsilon$ and $\sum_i s_i = \sum_j t_j$ we have

$$\varphi_{s_1} \circ \varphi_{s_2} \circ \cdots \circ \varphi_{s_k} = \varphi_{t_1} \circ \varphi_{t_2} \circ \cdots \circ \varphi_{t_\ell}$$

Any $t \in \mathbb{R}$ can be written as $t = t_1 + \cdots + t_\ell$ for finitely many $t_i \in (-\epsilon, \epsilon)$ so that φ extends to $\varphi : \mathbb{R} \times M \to M$.

Finally, we need to show that $\varphi : \mathbb{R} \times M \to M$ is smooth.

First note that $X(\varphi(t,p)) = \dot\varphi(t,p)$ can be written as

$$\varphi(p,t) = p + \int_0^t X(\varphi(s,p))\,ds.$$

Let $\varphi_0(t,p) = p$, $\varphi_0 = \mathrm{Id}_M$, and $\varphi_{n+1}(t,p) = p + \int_0^t X(\varphi_n(s,p))\,ds$

Now we'll use an idea of Arnold (his book on ODE Eqn. Thm 1.2.13): $\varphi_0 = \mathrm{Id}_M$ is of class C^∞.

The above recursion formula shows that each $\varphi_n : \mathbb{R}\times M \to M$ are smooth functions.

Let $\psi_n \doteq \partial\varphi_n/\partial p$. Then we have

$$\psi_0(t,p) = \mathrm{Id}, \quad \psi_{n+1}(t,p) = \mathrm{Id} + \int_0^t DX(\varphi_n(s,p))\,\psi_n(s,p)\,ds$$

$$\dot\varphi(t,p) = X(\varphi(t,p))$$

$$\partial\dot\varphi/\partial p = DX(\varphi(t,p))\cdot\partial\varphi/\partial p$$

ψ

By the same argument each ψ_n is smooth.

Hence, their limits $\varphi(t,p) = \lim_n \varphi_n(t,p)$ and $\psi(t,p) = \lim_n \psi_n(t,p)$ are both continuous.

Since $\psi_n = \partial\varphi_n/\partial p$ for all n we get $\psi = \partial\varphi/\partial p$ and hence ψ is continuous. So φ is continuously differentiable. Finally, φ satisfies

$$\psi = \frac{\partial\varphi}{\partial p} = \exp\int_0^t DX(\varphi(s,p))\,ds$$

and thus φ is infinitely many times differentiable.

$$\dot\varphi(t,p) = X(\varphi(t,p))$$

$$\frac{\partial\dot\varphi}{\partial p}(t,p) = DX(\varphi(t,p))\cdot\frac{\partial\varphi}{\partial p}(t,p)$$

$$\psi = \frac{\partial\varphi}{\partial p} \Rightarrow \dot\psi = DX(\varphi(t,p))\,\psi$$

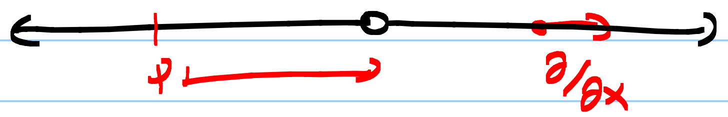
$$\Rightarrow \psi = \exp\int^t DX(\varphi(s,p))\,ds$$

Video 39

So, we have smooth map $\varphi : \mathbb{R}\times M \to M$ satisfying $(\varphi_t(p) = \varphi(t,p))$

$\varphi_0(p) = \mathrm{Id}_M$ and $\dot{\varphi}_t(p) = X(\varphi_t(p)), \forall t, p \in M.$

Remark: $M = \mathbb{R}\setminus\{0\}$, $X(p) = \dfrac{\partial}{\partial x}$



Lie Bracket: Let X, Y be vector fields on a smooth manifold M. Then $X\circ Y$ is not a vector field. However, $X\circ Y - Y\circ X$ is a vector field on M.

Take any $f : M \to \mathbb{R}$ smooth function. Assume that on a coordinate chart X and Y are given as

$$X(p) = \sum_{i=1}^{n} a_i(p)\, \partial/\partial x_i, \quad Y(p) = \sum_{j=1}^{n} b_j(p)\, \partial/\partial x_j.$$

Then
$$\begin{aligned}(X\circ Y)(f) &= X(Y(f)) \\ &= X\Big(\sum_j b_j \frac{\partial f}{\partial x_j}\Big) \\ &= \sum_i a_i \frac{\partial(b_j\, \partial f/\partial x_j)}{\partial x_i} \\ &= \sum_i a_i \left(\frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} + b_j \frac{\partial^2 f}{\partial x_i \partial x_j}\right)\end{aligned}$$

Similarly, $(Y\circ X)(f) = \sum b_j\left(\dfrac{\partial a_i}{\partial x_j}\dfrac{\partial f}{\partial x_i} + a_i \dfrac{\partial^2 f}{\partial x_j \partial x_i}\right)$

So, $(X\circ Y - Y\circ X)(f) = \sum_i a_i \dfrac{\partial b_j}{\partial x_i}\dfrac{\partial f}{\partial x_j} - \sum b_j \dfrac{\partial a_i}{\partial x_j}\dfrac{\partial f}{\partial x_i}$

$$\Rightarrow (X\circ Y - Y\circ X)(f) = \sum_{i,j} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right)\frac{\partial f}{\partial x_j}$$

So, $X\circ Y - Y\circ X$ is another vector field on M.

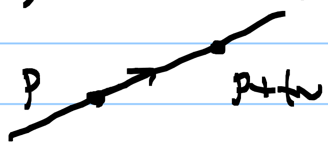
<u>Notation:</u> $[X,Y] \doteq X\circ Y - Y\circ X$

$[\,,\,]$ is a binary operation on the set of all vector fields on M.

<u>Lie Derivative:</u> Let $f:\mathbb{R}^n\to\mathbb{R}$ smooth function.

If $v\in\mathbb{R}^n$ any vector ($v\in T_p\mathbb{R}^n$) we have

$$D_v(f)(p) = \lim_{t\to 0}\frac{f(p+tv)-f(p)}{t}$$



If $X(p)=v$, $(t,p)\in\mathbb{R}\times\mathbb{R}^n$ is the constant vector field then $\varphi(t,p)=p+tv$ and hence

$$D_X(f)(p) = \lim_{t\to 0}\frac{f(\varphi(t,p))-f(p)}{t}.$$

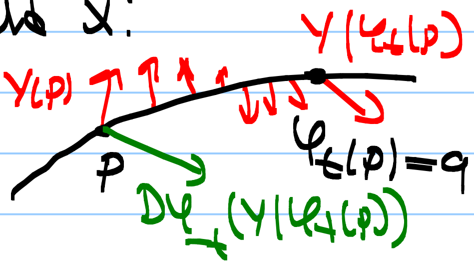
This enables us to define derivative of f along any vector field X using the flow of X.

$$X \rightsquigarrow \varphi_t, \quad D_v(f)(p) = \lim_{t\to 0}\frac{f(\varphi_t(p))-f(p)}{t}$$

$$\doteq L_X(f)(p)$$ the Lie derivative of f along X at p.

What about the derivative of a vector field Y along another vector field X?

$X \rightsquigarrow \varphi_t(p)$

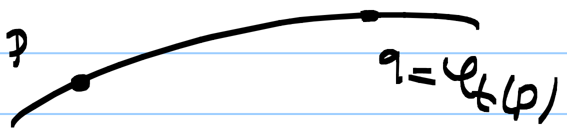


$$\mathcal{L}_X(Y) \doteq \lim_{t\to 0} \frac{1}{t}\left\{ D\varphi_{-t}\big(Y(\varphi_t(p)) - Y(p)\big)\right\}$$

$\varphi_t(q) = p, \; D\varphi_{-t} : T_qM \to T_pM$

The Lie derivative of Y along X.

What about Lie derivative of forms:



$\omega \in \Omega^k(M)$ $\qquad \varphi_t(p) = q$

$$\mathcal{L}_X(\omega)(p) \doteq \lim_{t\to 0} \frac{1}{t}\left\{ \varphi_t^*(\omega) - \omega\right\}(p)$$

Lie derivative of ω along X.

<u>Contraction of forms:</u> $\omega \in \Omega^k(M)$, X vector field on M.

$i_X \omega \in \Omega^{k-1}(M)$ defined by

$i_X\omega(Y_1, \dots, Y_{k-1}) \doteq \omega(X, Y_1, \dots, Y_{k-1})$, the contraction of ω by X.

Video 40

Proposition: $\mathcal{L}_X(i_Y\omega) = i_{\mathcal{L}_X(Y)}\,\omega + i_Y(\mathcal{L}_X\omega)$

(Theorem 3.1.5)

Theorem: 1) $\mathcal{L}_X(Y) = [X, Y]$

2) (Cartan Magic Formula)

$$\mathcal{L}_X\omega = i_X(d\omega) + d(i_X\omega).$$

Remark: If ω is closed, i.e., $d\omega = 0$ then

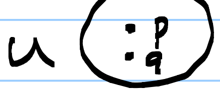
$$\mathcal{L}_X\omega = d(i_X\omega).$$

Proposition: Let X and Y be vector fields on M with flows φ and ψ, respectively. Then for any $t, s \in \mathbb{R}$ we have

$$\varphi_t \circ \psi_s = \psi_s \circ \varphi_t \text{ if and only if } [X, Y] = 0.$$

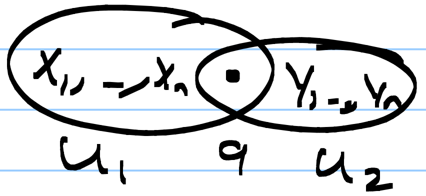
Frobenius Theorem:

Let M be a smooth manifold of dimension $m = n+k$ and assume that to each $p \in M$ is assigned an n-dimensional subspace Δ_p of T_pM. Suppose moreover that in a neighborhood U of $p \in M$ there are n linearly independent C^∞-vector fields $X_1, \ldots, X_n$, which form a basis of Δ_q for every $q \in U$.



$$\Delta_q = \text{span}\{X_1(q), \ldots, X_n(q)\}.$$

The collection Δ of Δ_p's $p \in M$ is called a <u>C^∞ n-plane distribution</u> of dimension n on M, and $X_1, \ldots, X_n$ is called a <u>local basis</u> of <u>Δ</u>.



$q \in U_1 \cap U_2$

$$\Delta_q = \text{span}\{X_i\} = \text{span}\{Y_j\}$$

$$[I]_{\{X_i\}}^{\{Y_j\}} = (a_{ij}) = A$$

$$y_j = a_{j1} X_1 + \cdots + a_{jn} X_n \qquad a_{ij} \in C^\infty(U_1 \cap U_2).$$

We say that Δ is <u>involutive</u> if there exists a local basis $X_1, \ldots, X_n$ in a neighborhood of each point such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k, \quad 1 \le i, j \le n$$

$$(c_{ij}^k \in C^\infty(U))$$

Finally, if Δ is a C^∞ distribution on M and N is a connected C^∞-submanifold of M such that for each $q \in N$ we have $T_qN \subseteq \Delta_q$, then we will say that N is an <u>integral submanifold</u> of the distribution Δ.

<u>Example:</u> $M = \mathbb{R}^{n+k}$, $x_1, \ldots, x_{n+k}$ coordinates on $\mathbb{R}^{n+k}$.

Let $X_i(p) = \frac{\partial}{\partial x_i}\Big|_p$, $i = 1, \ldots, n+k$. Then clearly,

$[X_i, X_j] = 0$, for all $i, j \in \{1, \ldots, n+k\}$.

Let $\Delta = \{\Delta_p\}_{p \in \mathbb{R}^{n+k}}$, $\Delta_p = \text{span}\{X_1(p), \ldots, X_n(p)\}$.

Thm Δ is an involutive n-plane distribution on $\mathbb{R}^{n+k}$ and $N = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}$ is an integral submanifold of Δ.

An n-dim'l distribution Δ on M is called completely integrable if each point $p \in M$ has a cubical coordinate neighborhood (U, φ) such that if $x_1, \ldots, x_m$ denote the local coordinates, the n vector fields

$$E_i = \varphi_*^{-1}\left(\partial/\partial x_i\right), \quad i = 1, \ldots, n, \text{ are a local}$$

basis on U for Δ. Note that in this case there is an n-dim'l integral submanifold N through each point $q \in U$ such that

$$T_q N = \Delta_q, \text{ that is } \dim N = n.$$

Note that the definition imply that a completely integrable n-plane distribution Δ is involutive.

Theorem (Frobenius)

A distribution Δ on a smooth manifold M is completely integrable if and only if it is involutive.

The lemma below is the main part of the proof:

Lemma: Let $X_1, \ldots, X_n$ be C^∞-vector fields defined on an open subset U of $\mathbb{R}^m$ such that

i) $\{X_1(q), \ldots, X_n(q)\}$ are linearly independent at each $q \in U$, and

ii) $[X_i, X_j] \equiv 0$ on U, $1 \le i, j \le n$.

Then there exists a coordinate neighborhood (V, ψ) around any $p \in U$ s.t.

$$X_i = E_i = \psi_*^{-1}\left(\frac{\partial}{\partial x_i}\right), \ i = 1, \dots, n, \text{ on } V.$$

Example: $M = \mathbb{R}^3$, let $X = \frac{\partial}{\partial x} + f_x(x,y)\frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial y} + f_y(x,y)\frac{\partial}{\partial z}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function ($\mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$).

Let's compute $[X, Y]$. Take any smooth function $g : \mathbb{R}^3 \to \mathbb{R}$.

Then $X(g) = \left(\frac{\partial}{\partial x} + f_x(x,y)\frac{\partial}{\partial z}\right)(g)$

$$= g_x + f_x(x,y)\, g_z.$$

$$Y(X(g)) = \left(\frac{\partial}{\partial y} + f_y(x,y)\frac{\partial}{\partial z}\right)(g_x + f_x(x,y)\, g_z)$$

$$= \cancel{g_{xy}} + \cancel{f_{xy}}\, g_z + \cancel{f_x\, g_{zy}} + \cancel{f_y\, g_{xz}}$$

$$+ f_y\,(\underset{\substack{\shortparallel \\ 0}}{f_{xz}}\, g_z + \cancel{f_x\, g_{zz}})$$

0 since $f = f(x,y)$

$$Y(g) = \left(\frac{\partial}{\partial y} + f_y(x,y)\frac{\partial}{\partial z}\right)(g) = g_y + f_y\, g_z.$$

$$X(Y(g)) = \left(\frac{\partial}{\partial x} + f_x(x,y)\frac{\partial}{\partial z}\right)(g_y + f_y g_z)$$

$$= \cancel{g_{xy}} + \cancel{f_{yx} g_z} + \cancel{f_y g_{zx}} + \cancel{f_x g_{yz}}$$

$$+ f_x(\underbrace{f_{yz}}_{=0} g_z + \cancel{f_y g_{zz}})$$

0 since $f = f(x,y)$.

So, $(XY - YX)(g) = 0$ for all g.

$\Rightarrow [X,Y] = 0$ and thus Δ is involutive.

Indeed, $z = f(x,y) + c - f(a,b)$ is the integral surface of the distribution passing through the point (a,b,c).