M E T U
Northern Cyprus Campus

Mat-219 Differential Equations Final Exam 05.06.2009		
Last Name : Name : Student No:	Dept./Sec. : Time : 16: 30 Duration : 120 *minutes*	Signature
5 QUESTIONS ON 5 PAGES		TOTAL 100 POINTS

1	2	3	4	5			

EACH PROBLEM - 20 POINTS.

Question 1. Using the substitution $v = \dfrac{1}{y^2}$, solve the first order nonlinear differential equation $y' + \dfrac{2}{x}y = \dfrac{y^3}{x^2}$, $x > 0$ *(Hint: the indicated substitution will convert the nonlinear equation into a linear differential equation that you know how to solve).*

Question 2. Using the Variation of Parameters Method, solve the nonhomogeneous system $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}(t)$ with $A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$ and $\mathbf{g}(t) = e^{-t} \begin{bmatrix} 1 \\ \tan^2(t) \end{bmatrix}$, $t \in (-\pi/2, \pi/2)$.

Question 3. Find the general solution of the system $\mathbf{x}'(t) = \begin{bmatrix} -3 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -3 \end{bmatrix} \mathbf{x}(t)$.

Question 4. **(a)** Consider the function $f(x) = e^x$ over the interval $[0, \pi]$. Extend this function to the interval $[-\pi, \pi]$ as an odd function and find its Fourier series expansion $S(x) = \frac{a_0}{2} + \sum_n a_n \cos(nx) + b_n \sin(nx)$.

(b) Is it true that $S(x) = f(x)$ for all x? What does Fourier Convergence Theorem say in that concern? Explain your answer and sketch the graphs of $f(x)$ and $S(x)$ approximately.

(c) Using the Separation of Variables Method and items (a), (b), solve the following heat conduction problem

$$\begin{cases} u_{xx} = u_t, & 0 < x < \pi, \quad t > 0, \\ u\left(x, 0\right) = f\left(x\right) = e^x, & 0 \leq x \leq \pi, \\ u\left(0, t\right) = u\left(0, \pi\right) = 0, & t > 0. \end{cases} .$$

Demonstrate all steps of the method. Do not write an exact formula for the solution.

Question 5. Let $A \in M_3$ be a matrix with its Jordan matrix $J = T^{-1}AT$.

(a) Show that $\mathrm{tr}(J) = \mathrm{tr}(A)$ and $\mathrm{tr}(J)$ is the sum of all eigenvalues from $\sigma(A)$, where tr indicates the trace of a matrix. (*Hint: use the definition of the Jordan matrix J and the linear algebra formula* $\mathrm{tr}\,(XY) = \mathrm{tr}\,(YX)$ *for all matrices* $X, Y \in M_3$).

(b) Using the formula $\Psi\,(t) = Te^{Jt}$ for the fundamental matrix of the system $\mathbf{x}'\,(t) = A\mathbf{x}\,(t)$, prove Abel's formula for the Wronskian

$$W\,(t) = \det \Psi\,(t) = Ce^{\mathrm{tr}(A)t}.$$

Explain why $C \neq 0$? (*Hint: use the linear algebra formula* $\det\,(XY) = \det\,(X)\det\,(Y)$ *and item (a), where* det *indicates the determinant of a matrix).*