Northern Cyprus Campus

A	Differential Equations II. Midterm	
Code : *Math 219* Acad.Year : *2009-2010* Semester : *Fall* Date : *10.12.2009* Time : *17:40* Duration : *120 minutes*	Last Name : Name : KEY Department: Signature:	Student No.: Section:
	6 QUESTIONS ON 5 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6		

Please show your work in all questions.

1. (10+10=20 points) The homogeneous differential equation

$$y^{(8)} - 3y^{(7)} + 4y^{(6)} - 4y^{(4)} + 4y^{(3)} = 0$$

has characteristic equation $r^3(r+1)(r^2-2r+2)^2 = 0$.

(a) Write the general solution to the differential equation.

Note that $r^2-2r+2=(r-1)^2+1$. So, we have the following roots $r=0^{③}, -1^{①}, (1\pm i)^{②}$ of the characteristic equation. Hence

$$y = C_1 + C_2 t + C_3 t^2 + C_4 e^{-t} + C_5 e^t\cos(t) + C_6 e^t \sin(t) + C_7 te^t\cos(t) + C_8 te^t\sin(t)$$ – general solution.

(b) Write the **form** of the particular solution (used in the method of undetermined coefficients) for the nonhomogeneous differential equation

$$y^{(8)} - 3y^{(7)} + 4y^{(6)} - 4y^{(4)} + 4y^{(3)} = te^t\sin t + e^t + 1.$$

We have

$$y_s(t) = t^2\left((At+B)e^t\cos(t) + (Ct+D)e^t\sin(t)\right) + Ee^t + t^3F,$$

where A, B, C, D, E, F are undetermined coefficients.

2. (3+3+3+3+3+3+2=20 points) Mark each of the graphs below as "forced" or "free" and "undamped", "damped", or "overdamped". Also match them with their formula.

(A) $u'' + 2u = 0$

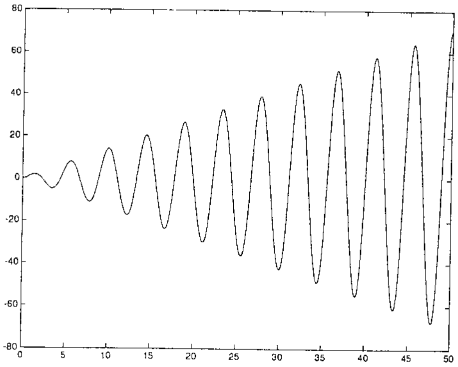
(B) $u'' + 2u = 4\cos(2t)$

(C) $u'' + 2u = 4\cos\left(\sqrt{2}\,t\right)$

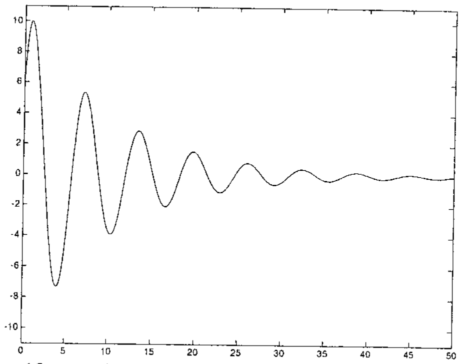
(D) $u'' + 5u' + 6u = 0$

(E) $u'' + \frac{1}{4}u = 0$

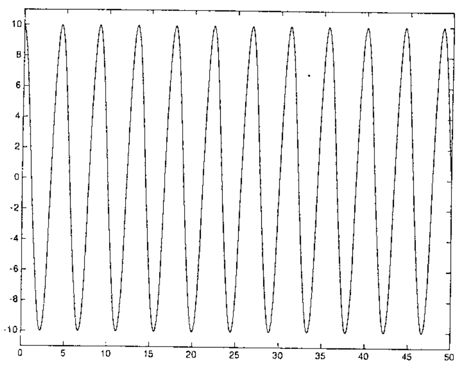
(F) $u'' + 20u' + 101u = 0$



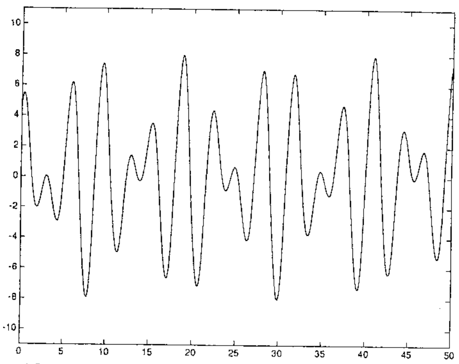
(forced/free)
(undamped/damped/overdamped) $\boxed{C}$



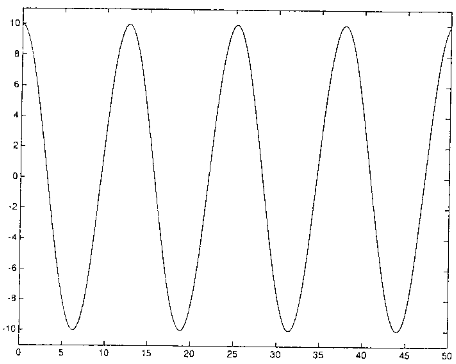
(forced/free)
(undamped/damped/overdamped) $\boxed{F}$



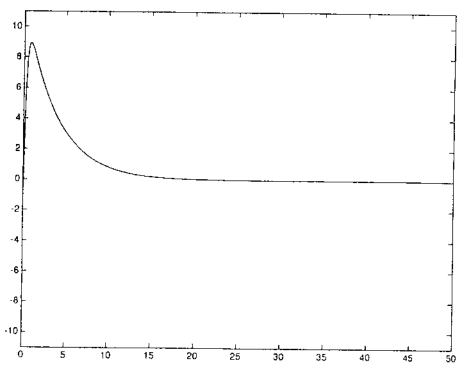
(forced/free)
(undamped/damped/overdamped) $\boxed{A}$



(forced/free)
(undamped/damped/overdamped) $\boxed{B}$



(forced/free)
(undamped/damped/overdamped) $\boxed{E}$



(forced/free)
(undamped/damped/overdamped) $\boxed{D}$

3. (15+5=20 points) Consider the initial value problem

$$y'' + 4y = -\delta(t-\pi) + \delta(t-2\pi) - \delta(t-3\pi), \quad y(0)=0,\ y'(0)=1$$

where $\delta(t-c)$ is the unit impulse function at c.

(a) Solve the initial value problem using Laplace transforms.

$$s^2\mathcal{L}\{y\} - 0 - y'(0) + 4\mathcal{L}\{y\} = -e^{\pi s} + e^{-2\pi s} - e^{-3\pi s}$$

or

$$(s^2+4)Y(s) = 1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s}.\quad \text{Thus}$$

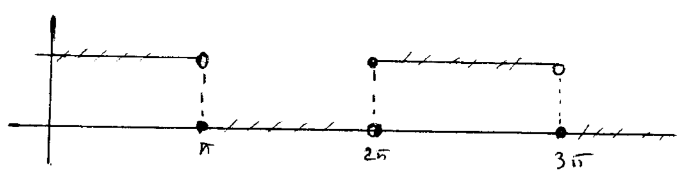
$$Y(s) = \frac{1}{s^2+4} - e^{-\pi s}\frac{1}{s^2+4} + e^{-2\pi s}\frac{1}{s^2+4} - e^{-3\pi s}\frac{1}{s^2+4}.$$

It follows that

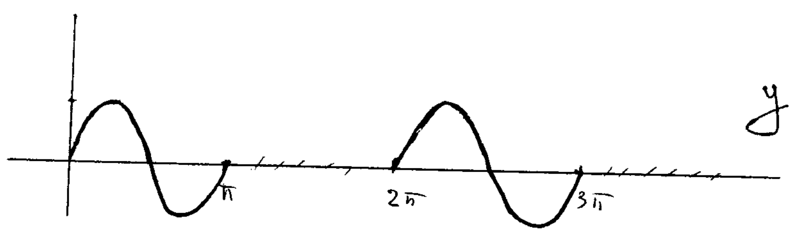
$$y(t) = \frac{1}{2}\sin(2t) - u_\pi(t)\frac{1}{2}\sin(2(t-\pi)) + \frac{1}{2}u_{2\pi}(t)\sin(2(t-2\pi))$$
$$- \frac{1}{2}u_{3\pi}(t)\sin(2(t-3\pi)).$$

(b) Graph your solution. Let's simplify the expression for $y(t)$:

$$y(t) = \frac{1}{2}\left(1 - u_\pi(t) + u_{2\pi}(t) - u_{3\pi}(t)\right)\sin(2t).$$



$1 - u_\pi + u_{2\pi} - u_{3\pi}$



$y(t)$

4. (10 points) Compute the inverse Laplace transform of $F(s) = \dfrac{s\,e^{-3s}}{(s+4)^2+4}$.

First note that

$$F(s) = e^{-3s}\frac{s+4}{(s+4)^2+4} - 2e^{-3s}\frac{2}{(s+4)^2+4} = e^{-3s}\mathcal{L}\{e^{-4t}\cos(2t)\}$$

$$-2e^{-3s}\mathcal{L}\{e^{-4t}\sin(2t)\}.$$ Then

$$f(t) = u_3(t)\left[e^{-4(t-3)}\cos 2(t-3) - 2e^{-4(t-3)}\sin 2(t-3)\right]$$

5. (10 points) Compute the convolution of step functions $u_2 * u_3$ (you must show work for credit).

Let's use the Laplace transform.

$$\mathcal{L}\{u_2 * u_3\} = \mathcal{L}\{u_2\}\,\mathcal{L}\{u_3\} = \frac{e^{-2s}}{s}\,\frac{e^{-3s}}{s} = \frac{e^{-5s}}{s^2}$$

Then

$$u_2 * u_3 = \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^2}\right\} = u_5(t)\,(t-5).$$

6. (5+15=20 points) Consider the Chebyshev equation (with $\alpha = 1$):

$$(1 - x^2)y'' - xy' + y = 0.$$

(a) Find the radius of convergence of the series solution to the Chebyshev equation about $x_0 = 0$.

Since $p(x) = \frac{-x}{1-x^2}$, $q(x) = \frac{1}{1-x^2}$ are analytic functions about $x_0 = 0$, whose radii convergence are 1, we conclude that $\rho = 1$.

(b) Compute the first **five** **<u>nonzero</u>** terms of the series solution to the Chebyshev equation about $x_0 = 0$.

Put $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

So,

$$2a_2 + a_0 = 0$$

$$6a_3 - a_1 + a_1 = 0$$

$$a_{n+2} = \frac{n-1}{n+2} a_n, \quad n \geq 2$$

Thus

$$y = a_0 + a_1 x - \frac{a_0}{2} x^2 - \frac{a_0}{6} x^4 - \frac{a_0}{10} x^6 - \frac{5a_0}{70} x^8 + \cdots$$