METU

Northern Cyprus Campus

Math 219 Differential Equations I. Exam 05.11.2009		
Last Name: KEY Name : Student No:	Dept./Sec. : Time : 17: 40 Duration : 120 *minutes*	Signature
6 QUESTIONS ON 5 PAGES		TOTAL 100 POINTS

1	2	3	4	5	6			

Question 1 (20 pts.)

Consider the linear homogenous differential equation $ty'' + 2y' + ty = 0$, whose one solution is supposed to be of the form $y_1(t) = t^m \sin t$.

(a) Determine the constant m. For the derivatives of $y_1(t)$ we have

$$y_1'(t) = m t^{m-1} \sin(t) + t^m \cos(t)$$

$$y_1''(t) = \left(m(m-1) t^{m-2} - t^m\right) \sin(t) + 2m t^{m-1} \cos(t)$$

It follows that $t y_1'' + 2y_1' + t y_1 =$

$$= m(m+1) t^{m-1} \sin(t) + 2(m+1) t^m \cos(t)$$

Since the functions $\{t^{m-1}\sin(t), t^m \cos(t)\}$ form a linearly independent set of functions, the only possible choice for m is -1.

(b) The second solution of the considered differential equation is given as $y_2(t) = t^m \cos t$ with the same m found in the item a). Show that these solutions are linearly independent.

So, $y_2(t) = t^{-1}\cos(t)$. Let's calculate the Wronskian of the family $\{y_1, y_2\}$:

$$W(t) = \begin{vmatrix} t^{-1}\sin(t) & t^{-1}\cos(t) \\ -t^{-2}\sin(t) + t^{-1}\cos(t) & -t^{-2}\cos(t) - t^{-1}\sin(t) \end{vmatrix} =$$

$$= -t^{-3}\sin(t)\cos(t) - t^{-2}\sin^2(t) + t^{-3}\sin(t)\cos(t) - t^{-2}\cos^2(t)$$

$$= -t^{-2} \neq 0 \quad (t \neq 0).$$

Since we deal with solutions of the hom. differential equation, we conclude that $\{y_1, y_2\}$ is a linearly independent solution set.

(c) Find the largest interval in which the solutions of the differential equation above with the initial values $y(-1) = 1$, $y'(-1) = 0$ exist and unique. Explain your answer.

Note that $y'' + p(t)y' + q(t) = 0$ with $p(t) = \frac{2}{t}$, $q(t) = 1$. Both functions are continuous on $\mathbb{R}\setminus\{0\}$. But $-1 < 0$.

So, the largest interval where the IVP has the solution is $(-\infty, 0)$ thanks to the Exist-Uniq. Theorem.

Question 2 (20 pts.)

Find the solutions of the following initial value problem

$$\begin{cases} y'' + a^2 y = \cos(2t), \\ y(0) = 1, \quad y'(0) = 1, \end{cases}$$

for the following values of the constant a:

(i) $a = 1$: We can assume that $a \geq 0$ and

$y = C_1 \cos(at) + C_2 \sin(at)$ is a general solution of the relevant hom. equation. We need to find a special solution $Y(t)$ of the non hom. eq.

Let's use the Method of Undet. Coeff:

$Y(t) = A\cos(2t) + B\sin(2t)$ whenever $a \neq 2$, for we don't have duplications. So,

$Y'(t) = -2A\sin(2t) + 2B\cos(2t)$,

$Y''(t) = -4A\cos(2t) - 4B\sin(2t)$ and

$(a^2-4)A\cos(2t) + (a^2-4)B\sin(2t) = \cos(2t)$

Since $\{\cos(2t), \sin(2t)\}$ are lin. independent, we have

$B = 0$ $(a \neq 2)$ and $A = \frac{1}{a^2-4}$. Thus

$$y = C_1\cos(at) + C_2\sin(at) + \frac{1}{a^2-4}\cos(2t)$$

is the general sol. whenever $a \neq 2$. Using the initial conditions, we derive that

$$C_1 = \frac{a^2-5}{a^2-4}, \quad C_2 = \frac{1}{a}.$$

If $a = 1$ then $C_1 = \frac{4}{3}$, $C_2 = 1$.

(ii) $a = 0$: In this case

$$y = C_1 + C_2 t + \left(-\frac{1}{4}\cos(2t)\right)$$

But $y(0) = C_1 - \frac{1}{4} = 1$, $y'(0) = C_2 = 1$, that is,

$C_1 = \frac{5}{4}$, $C_2 = 1$ and

$$y = \frac{5}{4} + t - \frac{1}{4}\cos(2t)$$

is the sought solution.

(iii) $a = 2$: We have duplication. Therefore we put

$Y(t) = t(A\cos(2t) + B\sin(2t))$. Then

$$Y'(t) = A\cos(2t) + B\sin(2t) + t(-2A\sin(2t) + 2B\cos(2t))$$

$$Y''(t) = -2A\sin(2t) + 2B\cos(2t) - 2A\sin(2t) + 2B\cos(2t) +$$
$$+ t(-4A\cos(2t) - 4B\sin(2t))$$

Hence $Y''(t) + 4Y(t) = -4A\sin(2t) + 4B\cos(2t) = \cos(2t)$,

that is, $A = 0$, $B = \frac{1}{4}$ ($\{\sin(2t), \cos(2t)\}$ is a lin. ind. set)

Thus $y = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{4}t\sin(2t)$ is the general solution. Finally, let's use the initial values:

$$y(0) = C_1 = 1, \quad y'(0) = 2C_2 = 1, \text{ that is,}$$

$$y(t) = \cos(2t) + \frac{1}{2}\sin(2t) + \frac{1}{4}t\sin(2t)$$

is the sought solution of the IVP.

Question 3 (15 pts.)

Solve the differential equation $ydx+(2x-ye^y)dy=0$ given that the integrating factor $\mu(x,y)=y$.

The dif equation $y^2dx+y(2x-ye^y)dy=0$ is exact by the assumption. So, $\exists\,\Psi(x,y)$ such that

$$\frac{\partial\Psi}{\partial x}=y^2,\quad \frac{\partial\Psi}{\partial y}=2xy-y^2e^y$$ From the first one, we derive that

$$\Psi(x,y)=xy^2+C(y)\Rightarrow\frac{\partial\Psi}{\partial y}=2xy+C'(y)=2xy-y^2e^y$$

$\Rightarrow C'(y)=-y^2e^y$. Using the Integ. by Parts trick, we derive that

$$C(y)=-\int y^2e^y\,dy=-y^2e^y+2\int ye^y\,dy=$$

$$=-y^2e^y+2(ye^y-e^y)=-y^2e^y+2ye^y-2e^y+C$$

Hence

$$xy^2-y^2e^y+2ye^y-2e^y=C$$

is an implicitly defined solution set.

Question 4 (15 pts.)

Verify that $y_1(t) = e^t$ is a solution of $(t-1)y'' - ty' + y = 0, \quad t > 1$. Find the second linearly independent solution by the method of reduction of order.

$y_1(t) = e^t$ is a solution: $(t-1)e^t - te^t + e^t = 0$.

Put $y_2(t) = v(t)e^t$. Then $y_2' = v'e^t + ve^t$,

$y_2'' = v''e^t + v'e^t + v'e^t + ve^t = v''e^t + 2v'e^t + ve^t$;

and

$(t-1)v''e^t + 2(t-1)v'e^t + (t-1)ve^t - tv'e^t - tve^t + ve^t =$

$= (t-1)v''e^t + (t-2)v'e^t = 0$. So,

$$(t-1)v'' + (t-2)v' = 0 \quad \text{or}$$

$\dfrac{dv'}{v'} = \dfrac{2-t}{t-1} \; (t>1) \Rightarrow \ln|v'| = -t + \ln(t-1) + C$

$\Rightarrow v' = c(t-1)e^{-t} \Rightarrow v = cte^{-t} + k \Rightarrow$

$y_2(t) = cte^{-t}e^t + ke^t = ct + ke^t$.

One can assume that $k=0$, for we need lin. independent solutions. Thus $y_2(t) = t$.

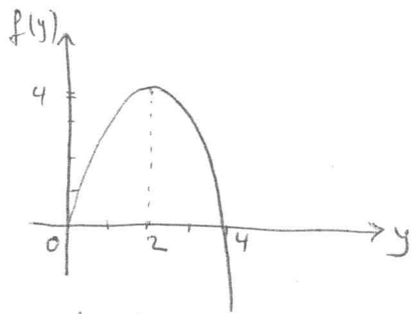
Check:

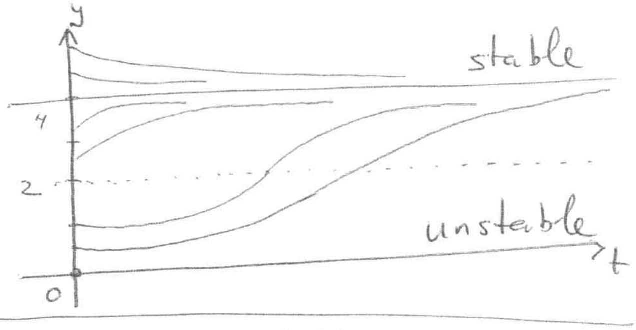
$$W(t) = \begin{vmatrix} e^t & t \\ e^t & 1 \end{vmatrix} = e^t - te^t = (1-t)e^t \neq 0 \; (t>1).$$

Question 5 (15 pts.)

Draw the direction fields, determine the equilibrium (or critical) points and describe their stabilities of the following autonomous differential equations:

(a) $y' = -y^2 + 4y$, $f(y) = y(4-y)$

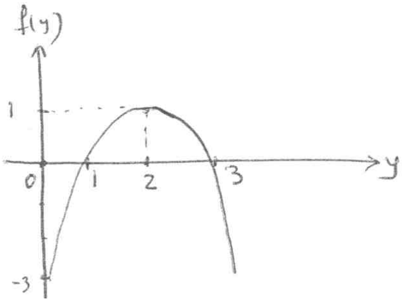


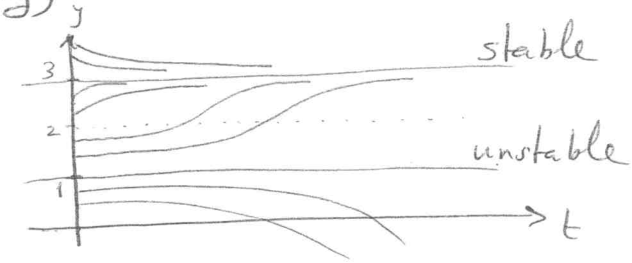


$y' = f(y)$
$y'' = f'(y)\,f(y)$

y	$0 \le y \le 2$	$2 \le y \le 4$	$4 \le y$
Sign $f(y)$	+	+	−
Sign $f'(y)$	+	−	−
Solutions	increasing conc. up	increasing conc. down	decreasing conc. up

(b) $y' = -y^2 + 4y - 3$, $f(y) = (y-1)(3-y)$

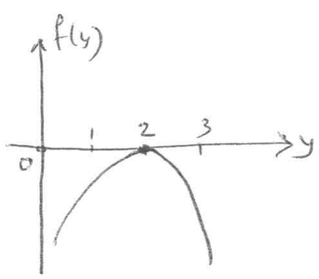


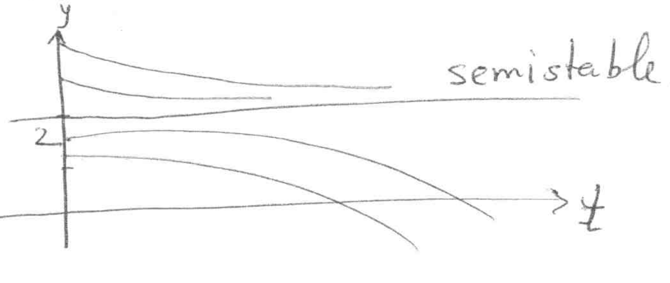


$y' = f(y)$
$y'' = f'(y)\,f(y)$

y	$0 \le y \le 1$	$1 \le y \le 2$	$2 \le y \le 3$	$3 \le y$
Sign $f(y)$	−	+	+	−
Sign $f'(y)$	+	+	−	−
Solutions	decreasing conc. down	increasing conc. up	increasing conc. down	decreasing conc. up

(c) $y' = -y^2 + 4y - 4$. $f(y) = -(y-2)^2$





$y' = f(y)$
$y'' = f'(y)\,f(y)$

y	$0 \le y \le 2$	$2 \le y$
Sign $f(y)$	−	−
Sign $f'(y)$	+	−
Solutions	decreasing conc. down	decreasing conc. up

Question 6 (20 pts.) A pond containing 1000 tons of water initially has $500kg$ of pollutant in it. Water from the pond flows at a rate of 10 tons per hour to a sewage system which cleans 90 percent of the pollutant it receives, and pumps the water (without any loss) back to the pond with the same rate. Assume that the pollutant is homogeneously distributed and dissolved in water so that it does not effect its volume.

(a) Write a differential equation for the amount of pollutant $Q(t)$ (in kg) in the pond at time t.

The concentration of pollutant in the pond at time t is $\frac{Q(t)}{1000}$. So, the rate at which pollutant leaves the pond is $10 \cdot \frac{Q(t)}{1000}$.
The 90 percent of $Q(t)$ is $\frac{9}{10} Q(t)$. So, the concentration of pollutant in the cleaned water is $\frac{1}{10} \frac{Q(t)}{1000}$ $\Rightarrow$ the rate at which pollutant enters is $\frac{Q(t)}{1000}$. Hence $\frac{dQ}{dt}$ = rate in − rate out

$$= \frac{Q(t)}{1000} - 10 \frac{Q(t)}{1000} \quad \text{or} \quad \frac{dQ}{dt} = -\frac{9}{1000} Q(t)$$

(b) Solve this differential equation to obtain an expression for $Q(t)$.

It is a separable dif. equation: $\frac{dQ}{Q} = -\frac{9}{1000} dt$

$\Rightarrow Q(t) = C e^{-\frac{9}{1000}t}$ Since $Q(0) = 500$, it follows that

$$Q(t) = 500 e^{-\frac{9}{1000}t}$$

(c) Determine how much time it takes to reduce $Q(t)$ to half of its initial value.

Let's find t when $Q(t) = 250$. So,

$$-\frac{9}{1000} t = \ln\left(\frac{1}{2}\right) \Rightarrow t = \frac{1000}{9} \ln(2) \simeq 77 \text{ h}.$$