M E T U

Northern Cyprus Campus

Math 219 Differential Equations Final Exam 12.01.2010								
Last Name: Name : Student No:						Dept./Sec. : Time : 9: 00 Duration : 150 *minutes*	Signature	
6 QUESTIONS ON 5 PAGES							TOTAL 100 POINTS	
1	2	3	4	5	6			

Question 1 (20 pts.)

Solve the following initial value problem

$$x'(t) = Ax, \qquad A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since $\det(A-\lambda) = (1+\lambda)^2(1-\lambda)$, we conclude $\sigma(A) = \{-1^{②}, 1^{①}\}$.

The generalized eigenvectors for $\lambda = -1$: $\vec{\xi}^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{\xi}^{(2)} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$

An eigenvector for $\lambda = 1$: $\vec{\eta} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. So,

$$T = [\vec{\xi}^{(1)} \ \vec{\xi}^{(2)} \ \vec{\eta}] = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$e^{Jt} = \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}, \text{ where } J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{- Jordan}$$

matrix of A.

The fundamental matrix: $\Psi(t) = Te^{Jt} = \begin{bmatrix} 0 & \frac{1}{2}e^{-t} & 0 \\ e^{-t} & te^{-t} & e^{t} \\ 0 & 0 & 2e^{t} \end{bmatrix}$.

The general solution: $\vec{x}(t) = \Psi(t)\vec{c}$.

I.V.P. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{x}(0) = \Psi(0)\vec{c} \Rightarrow \vec{c} = \begin{bmatrix} 1/2 \\ 2 \\ 1/2 \end{bmatrix}$. Hence

$$\vec{x}(t) = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t} + 2\begin{bmatrix} 1/2 \\ t \\ 0 \end{bmatrix} e^{-t} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} e^{t}$$ is the

sought solution.

Question 2 (20 pts.)
Find the general solution of the equation

$$y''' - 2y'' + y' = t^3$$

using the method of undetermined coefficients.

Homogenous equation: $r^3 - 2r^2 + r = r(r-1)^2 = 0$

So, $y_h(t) = C_1 + C_2 e^t + C_3 t e^t$.

Special solution of nonhom. equation:

$Y(t) = t(At^3 + Bt^2 + Ct + D)$

(there is a duplication)

$Y'(t) = 4At^3 + 3Bt^2 + 2Ct + D$

$Y''(t) = 12At^2 + 6Bt + 2C$

$Y'''(t) = 24At + 6B$. Thus

$(24At + 6B) - 2(12At^2 + 6Bt + 2C) + 4At^3 + 3Bt^2 +$
$+ 2Ct + D = t^3$ or

$t^3 = (4A)t^3 + (3B - 24A)t^2 + (24A - 12B + 2C)t + (6B - 4C + D)$

It follows that $A = \frac{1}{4}$, $B = 2$, $C = 9$, $D = 24$. Hence

$Y(t) = \frac{1}{4}t^4 + 2t^3 + 9t^2 + 24t$ is a special solution and

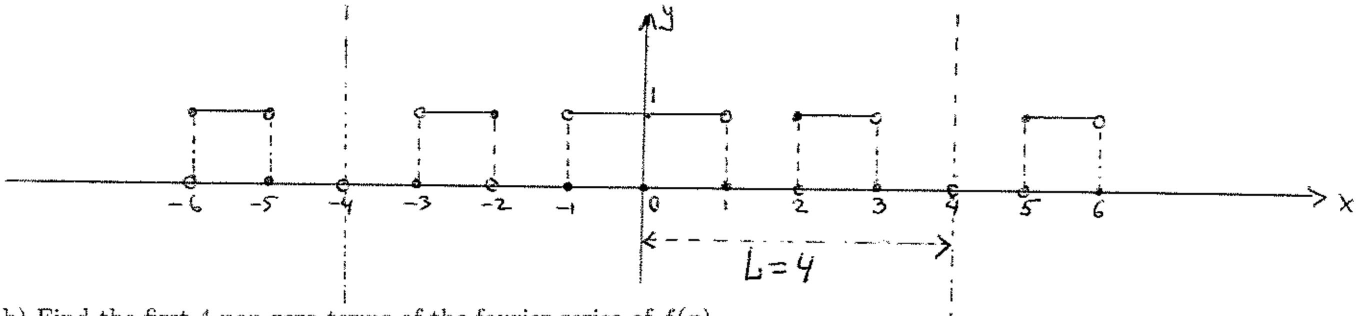
$y = y_h + Y = C_1 + C_2 e^t + C_3 t e^t + \frac{1}{4}t^4 + 2t^3 + 9t^2 + 24t$

is the general solution.

Question 3 (20 pts.)

$$\text{Let } f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x < 2 \\ 1 & \text{if } 2 \le x < 3 \\ 0 & \text{if } 3 \le x < 4 \end{cases}$$

a) Extend $f(x)$ as an even function of period 8 to $\mathbb{R}$, and sketch its graph for $-6 \le x \le 6$.



b) Find the first 4 non-zero terms of the fourier series of $f(x)$.

Taking into account that extended function $f(x)$ is even, we conclude that it has cosine Fourier series:

$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right)$. Let's find a_n:

$$a_0 = \frac{2}{4}\int_0^4 f(x)\,dx = \frac{1}{2}\left(\int_0^1 dx + \int_2^3 dx\right) = 1,$$

$$a_n = \frac{1}{2}\int_0^4 f(x)\cos\left(\frac{n\pi x}{4}\right)dx = \frac{1}{2}\left(\int_0^1 \cos\left(\frac{n\pi x}{4}\right)dx + \int_2^3 \cos\left(\frac{n\pi x}{4}\right)dx\right)$$

$$= \frac{1}{2}\frac{4}{n\pi}\left(\sin\left(\frac{n\pi x}{4}\right)\Big|_0^1 + \sin\left(\frac{n\pi x}{4}\right)\Big|_2^3\right) = \frac{2}{n\pi}\left(\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\right).$$ Note that

$$a_{2k} = \frac{1+(-1)^k}{\pi k}\sin\left(\frac{\pi k}{2}\right) = 0 \text{ for all } k. \text{ Further}$$

$$a_1 = \frac{2}{\pi}\left(\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right) - \sin\left(\frac{\pi}{2}\right)\right) = \frac{2}{\pi}(\sqrt{2}-1),$$

$$a_3 = \frac{2}{3\pi}\left(\sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{9\pi}{4}\right) - \sin\left(\frac{3\pi}{2}\right)\right) = \frac{2(\sqrt{2}+1)}{3\pi},$$

$$a_5 = \frac{2}{5\pi}\left(\sin\left(\frac{5\pi}{4}\right) + \sin\left(\frac{15\pi}{4}\right) - \sin\left(\frac{5\pi}{2}\right)\right) = \frac{-2}{5\pi}(\sqrt{2}+1).$$

So,

$$S(x) = \frac{1}{2} + \frac{2}{\pi}\left((\sqrt{2}-1)\cos\left(\frac{\pi x}{4}\right) + \frac{\sqrt{2}+1}{3}\cos\left(\frac{3\pi x}{4}\right) - \frac{\sqrt{2}+1}{5}\cos\left(\frac{5\pi x}{4}\right)\right.$$

$$+ \dots$$

Question 4 (20 pts.) Using separation of variables derive the solution of the heat equation

$$u_t = u_{xx}, \qquad 0 < x < 1, \quad t > 0$$

with the boundary conditions

$$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t \geq 0$$

and with the initial condition

$$u(x,0) = 2\sin\frac{\pi}{2}x - \sin\frac{3\pi}{2}x \quad 0 < x < 1.$$

Put $u(x,t) = X(x)\,T(t)$. Then $X \cdot T' = X'' T \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$

$\Rightarrow \begin{cases} X'' + \lambda X = 0 \\ T' + \lambda T = 0 \end{cases}$. But $0 = u(0,t) = X(0)\,T(t) \Rightarrow X(0) = 0$;

$0 = u_x(1,t) = X'(1)\,T(t) \Rightarrow X'(1) = 0$, for $u(x,t) \neq 0$.

So, we have two point B.V.P.:

$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X'(1) = 0 \end{cases} \Rightarrow X(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$

$0 = X(0) = C_1 \Rightarrow X(x) = C_2 \sin(\sqrt{\lambda}\, x)$.

But $0 = X'(1) = C_2 \sqrt{\lambda} \cos(\sqrt{\lambda}) \Rightarrow \sqrt{\lambda_n} = \frac{\pi}{2} + n\pi$, $n \geq 0$

are eigenvalues and $X_n(x) = \sin(\sqrt{\lambda_n}\, x)$ are eigenfunctions.

If $\lambda = 0$ then $X(x) = C_1 + C_2 x \Rightarrow 0 = X(0) = C_1$, $0 = X'(1) = C_2$

$\Rightarrow X(x) = 0$, that is, $\lambda = 0$ is not an eigenvalue.

If $\lambda < 0$ then $X(x) = C_1 e^{\sqrt{-\lambda}\,x} + C_2 e^{-\sqrt{-\lambda}\,x} \Rightarrow 0 = X(0) = C_1 + C_2$,

$0 = X'(1) = \sqrt{-\lambda}\, C_1 e^{\sqrt{-\lambda}} - \sqrt{-\lambda}\, C_2 e^{-\sqrt{-\lambda}} \Rightarrow C_1 = C_2 = 0$, that is,

λ is not an eigenvalue.

Now consider the second equation $T' + \lambda_n T = 0$. Then

$T_n(t) = e^{-\lambda_n t}$ are solutions. Set $u(x,t) = \sum_{n=0}^{\infty} C_n u_n(x,t)$,

$u_n(x,t) = T_n(t)\, X_n(x)$. Using the initial condition, we infer

$$2\sin\left(\frac{\pi}{2}x\right) - \sin\left(\frac{3\pi}{2}x\right) = C_0 \sin\left(\frac{\pi}{2}x\right) + C_1 \sin\left(\frac{3\pi}{2}x\right) + C_2 \sin\left(\frac{5\pi}{2}x\right) + \dots$$

Based upon orthogonality of sin-functions (in $L^2(-1,1)$), we derive that $C_0 = 2$, $C_1 = -1$, $C_n = 0$, $n \geq 2$.

So, the solution of the problem:

$$u(x,t) = 2e^{-\frac{\pi^2 t}{4}} \sin\left(\frac{\pi}{2}x\right) - e^{-\frac{9\pi^2}{4}t} \sin\left(\frac{3\pi}{2}x\right).$$

Question 5 (20 pts.)

a) Find the general solution of

$$x' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} x \qquad A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

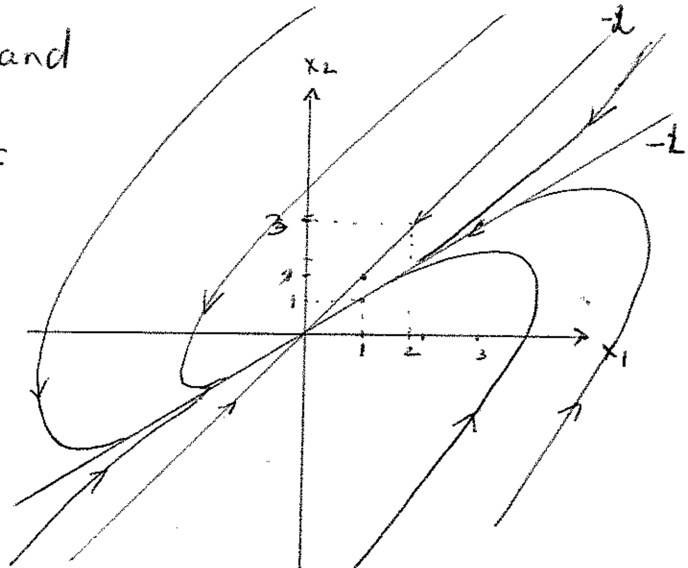
and sketch the phase portrait.

Note that $\sigma(A) = \{-2, -1\}$ and

$$\vec{X}(t) = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} =$$

$$= c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

is the general solution.



the origin is a node

b) Using the solution above, find the solution of the non-homogeneous system

$$x' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} x + \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix} \qquad \vec{g}(t) = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$$

by variation of parameters.

We know that $\Psi(t) = \begin{bmatrix} 2e^{-2t} & e^{-t} \\ 3e^{-2t} & e^{-t} \end{bmatrix}$ is the fundamental matrix of the relevant homogeneous system. To find a special solution of the nonhom. eq., we set $\vec{X}(t) = \Psi(t)\vec{u}(t)$. Then $\Psi(t)\vec{u}'(t) = \vec{g}(t)$ or

$$\begin{bmatrix} 2e^{-2t} & e^{-t} \\ 3e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} \Rightarrow \begin{matrix} u_1' = e^t - e^{3t} \\ u_2' = 3e^{2t} - 2 \end{matrix} \Rightarrow$$

$u_1 = e^t - \frac{e^{3t}}{3} + c_1$, $u_2 = \frac{3}{2}e^{2t} - 2t + c_2$. One can assume that $c_1 = c_2 = 0$. So,

$$\vec{X}_s(t) = \begin{bmatrix} 2e^{-2t} & e^{-t} \\ 3e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} e^t - \frac{e^{3t}}{3} \\ \frac{3}{2}e^{2t} - 2t \end{bmatrix}$$ is a special solution,

and

$\vec{X}(t) = \Psi(t)\vec{c} + \vec{X}_s(t)$ is the general solution.