**M E T U**

**Northern Cyprus Campus**

| Math 219 Differential Equations I. Exam 30.10.2008 | | |
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| Last Name :<br>Name :<br>Student No: | Dept./Sec. :<br>Time : 17: 40<br>Duration : 120 *minutes* | Signature |
| 5 QUESTIONS ON 5 PAGES | | TOTAL 100 POINTS |

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**EACH PROBLEM IS 20 POINTS.**

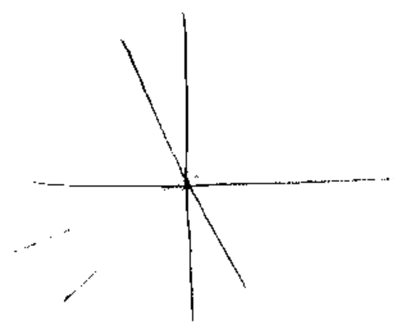
**Question 1.** Consider the following differential equation

$$\frac{dy}{dx} = \frac{y}{x}$$

**(a)** Find all solutions of the equation.

$\frac{dy}{y} = \frac{dx}{x}$ (if $y \neq 0$) $\Rightarrow \ln|y| = \ln|cx| \Rightarrow y = cx$, $c \in \mathbb{R}$

($y = 0$ belongs to the family)



**(b)** Through which points $(x_0, y_0) \in \mathbb{R}^2$ is there only one solution curve? Infinitely many solution curves? No solution curve? Explain your answers.

If $x_0 \neq 0$ then we have a unique solution curve through $(x_0, y_0)$ thanks to the existence uniqueness theorem

There are infinitely many curves through $(0,0)$

If $x = 0$ then the relation $x\frac{dy}{dx} = y$ implies that $y = 0$ So, there is no curve through $(0, k)$ if $k \neq 0$

**Question 2** Consider the differential equations

(i) $(\frac{xy^3}{3} + x^2y + y)dx + (y^2 + x)dy = 0$ (ii)$(x^3y^2 + xy^4 + x^5)dx + (\frac{x^4y}{2} + \cos y + 2x^2y^3)dy = 0$

**(a)** Which one is exact and which one has an integrating factor depending only on $x$?

(i) $M = \frac{xy^3}{3} + x^2y + y$, $N = y^2 + x$, $M_y = xy^2 + x^2 + 1 \neq N_x = y^2$

(ii) $M = x^3y^2 + xy^4 + x^5$, $N = \frac{x^4y}{2} + \cos(y) + 2x^2y^3$

$M_y = 2x^3y + 4xy^3 = N_x = 2x^3y + 4xy^3$ !

**(b)** Solve the equation (i). Put $\bar{M} = \mu(x)M$, $\bar{N} = \mu(x)N$. Then

$\bar{M}_y = \mu(xy^2 + x^2 + 1) = \mu'(y^2 + x) + \mu = \bar{N}_x$. It follows that

$\mu' = x\cdot\mu \Rightarrow \ln|\mu| = \frac{x^2}{2} \Rightarrow \mu(x) = e^{x^2/2}$ is an integrating factor

So, there is $F(x,y)$ s.t. $F_x = \bar{M}$, $F_y = \bar{N}$, or

$F_y = e^{x^2/2}(y^2 + x) \Rightarrow F = e^{x^2/2}\frac{y^3}{3} + xe^{x^2/2}y + h(x)$

$F_x = xe^{x^2/2}\frac{y^3}{3} + x^2e^{x^2/2}y + e^{x^2/2}y + h'(x)$

So, we can take $h = 0$, and $F = e^{x^2/2}(\frac{y^3}{3} + xy) = C$

**(c)** Solve the equation (ii). Since it is exact, it follows that there is $F(x,y)$ s.t. $F_x = M$, $F_y = N$ or

$F_x = x^3y^2 + xy^4 + x^5 \Rightarrow F = \frac{x^4y^2}{4} + \frac{x^2y^4}{2} + \frac{x^6}{6} + h(y) \Rightarrow$

$F_y = \frac{x^4y}{2} + 2x^2y^3 + h'(y) = \frac{x^4y}{2} + 2x^2y^3 + \cos(y) \Rightarrow h'(y) = \cos(y)$

or $h(y) = \sin(y)$ Hence

$$\frac{x^4y^2}{4} + \frac{x^2y^4}{2} + \frac{x^6}{6} + \sin(y) = C \,!$$

**Question 3.** Consider the differential equation $ay'' + by' + cy = g(t)$ with the constants $a > 0$, $b > 0$ and $c > 0$. If $Y_1(t)$ and $Y_2(t)$ are solutions of this equation, then show that $\lim_{t\to\infty}(Y_1(t) - Y_2(t)) = 0$.

The direct substitution into the differential equation shows that $Y_1(t) - Y_2(t)$ is a solution of the relevant homogeneous equation $ay'' + by' + cy = 0$ It remains to prove that $\lim_{t\to\infty} y(t) = 0$ for a solution $y(t)$ of the homogeneous equation Consider the characteristic equation $ar^2 + br + c = 0$ with $\Delta = b^2 - 4ac$.

1) $\Delta > 0 \Rightarrow r_1 = \dfrac{-b+\sqrt{\Delta}}{2a}$, $r_2 = \dfrac{-b-\sqrt{\Delta}}{2a}$.

$a,b,c > 0 \Rightarrow b^2 - 4ac < b^2 \Rightarrow \sqrt{\Delta} < b \Rightarrow -b + \sqrt{\Delta} < 0 \Rightarrow r_1 < 0$

$a > 0 \Rightarrow r_2 < 0$. But $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ So, $y(t) \to 0, t \to \infty$

2) $\Delta = 0 \Rightarrow r_1 = r_2 = r = \dfrac{-b}{2a} < 0$ and $y(t) = C_1 e^{rt} + C_2 t e^{rt}$

Since $\lim_{t\to\infty} t e^{rt} = 0$, we have $y(t) \to 0$ as $t \to \infty$

3) $\Delta < 0 \Rightarrow r_{1,2} = \lambda \pm i\mu$, $\lambda = -\dfrac{b}{2a} < 0$, $\mu = \dfrac{\sqrt{-\Delta}}{2a}$, and

$$y(t) = C_1 e^{\lambda t}\cos(\mu t) + C_2 e^{\lambda t}\sin(\mu t)$$

But $|e^{\lambda t}\cos(\mu t)| \le e^{\lambda t} \to 0$, $|e^{\lambda t}\sin(\mu t)| \le e^{\lambda t} \to 0$ as $t \to \infty$ Therefore $y(t) \to 0$ as $t \to \infty$

**Question 4.** For $t > 0$ the function $y_1(t) = t$ is a solution of the differential equation $t^2y'' + 2ty' - 2y = 0$. Using Reduction of Order Method, find a second solution $y_2(t)$ such that $(y_1, y_2)$ is a fundamental set of solutions.

Set $y = v(t) \cdot t$. Then $y' = v't + v$, $y'' = v''t + v' + v' =$
$= v''t + 2v'$. It follows that

$$t^2(tv'' + 2v') + 2t(v't + v) - 2tv = t^3v'' + 2t^2v' + 2t^2v' +$$
$$+ 2tv - 2tv = t^3v'' + 4t^2v' = 0 \Rightarrow tv'' = -4v'$$
$$\Rightarrow \frac{dv'}{v'} = -\frac{4}{t} \; (t>0) \Rightarrow \ln|v'| = \ln\left|\frac{c}{t^4}\right| \Rightarrow v' = \frac{c}{t^4}$$
$$\Rightarrow v = -\frac{c}{3}\frac{1}{t^3} + k \Rightarrow y(t) = C\frac{-3}{t^2} + kt.$$

But the term $kt$ is a multiple of $y_1$, so we put $k = 0$, and $C = 1$, that is, $y_2(t) = \frac{-3}{t^2}$ is the sought second solution. Indeed,

$$W(y_1, y_2)(t) = \begin{vmatrix} t & \frac{-3}{t^2} \\ 1 & \frac{6}{t^3} \end{vmatrix} = \frac{6}{t^2} + \frac{3}{t^2} = \frac{9}{t^2} \neq 0 \; (t>0)!$$

**Question 5.** **(a)** Find the general solution of the homogeneous equation

$$9y'' - 12y' + 4y = 0$$

Consider the characteristik equation $9r^2 - 12r + 4 = 0$,

$\Delta = 12^2 - 4\cdot 9\cdot 4 = 3^2\cdot 4^2 - 4^2\cdot 9 = 0 \Rightarrow r_1 = r_2 = r = \frac{12}{2\cdot 9} = \frac{2}{3}$

So, $y = C_1 e^{\frac{2}{3}t} + C_2 t e^{\frac{2}{3}t}$ is the general solution

**(b)** Using Method of Undetermined Coefficients, find the general solution of the nonhomogeneous equation

$$9y'' - 12y' + 4y = 8t^2 + 3$$

We set $Y(t) = t^s(At^2 + Bt + C)$. First, we put $s = 0$

Then $Y'(t) = 2At + B$, $Y''(t) = 2A \Rightarrow 18A - 24At - 12B +$
$+ 4At^2 + 4Bt + 4C = 4At^2 + (4B - 24A)t + 18A - 12B + 4C =$
$= 8t^2 + 3$ It follows that $A = 2, B = 12, C = \frac{111}{4}$. Whence

$Y(t) = 2t^2 + 12t + \frac{111}{4}$ is a special solution of the nonhomogneous equation In particular,

$y(t) = C_1 e^{2/3 t} + C_2 t e^{2/3 t} + 2t^2 + 12t + \frac{111}{4}$

is its general solution!