METU, Ankara
Fall 2011-12

EE 503
Midterm #1
(Duration: 110 minutes)

(Solutions)

Dr. Çağatay CANDAN
ODTÜ EEMB

1. (10 pts.) In the fair coin experiment, we define the process $x(t)$ as follows:

$$x(t) = \begin{cases} \sin(\pi t), & \text{if heads show} \\ 2t, & \text{if tails show} \end{cases}$$

3 (a) Find $E\{x(t)\}$.

3 (b) Find the first order distribution of $x(t)$ for $t = 1$.

4 (c) Find the joint distribution of $x(t)$ for $t = 1/2$ and $t = 1$.

2. (15 pts.) A real valued random process $y[n]$ is defined as follows

$$y[n] = x[n] - x[n-1]$$

Here $x[n]$ is i.i.d. and takes the values of $\{1, 0\}$ with probability $p$ and $(1-p)$, respectively.

3 (a) Find the first order distribution of $y[n]$.

3 (b) Is the process $y[n]$ stationary?

3 (c) Are the processes $y[n]$ and $x[n]$ jointly stationary?

6 (d) What is the correlation between the samples, i.e. $E\{y[n]y[n-k]\}$ ?

3. (15 pts.) A real valued random process $x[n]$ has the power spectrum density of $S_x(e^{j\omega}) = 3 + 2\cos(\omega)$. This process is filtered with $H(z) = 1 - \frac{1}{2}z^{-2}$.

3 (a) Is the output process stationary in any sense?

3 (b) What is the output autocorrelation?

6 (c) Is the process $z[n] = (x[n])^2$ stationary in any sense ?

(d) Bonus (5 pts.): Find the first order density of $z[n] = (x[n])^2$ when $x[n]$ is a Gaussian process.

4. (20 pts.) The random variables $x_1$ and $x_2$ are defined as follows:

$$\begin{aligned} x_1 &= u + v + w \\ x_2 &= u - v + w \end{aligned}$$

Here $u$ and $v$ are zero mean Gaussian random variables with zero mean and variance of 1 and 2 respectively. The correlation coefficient of $u$ and $v$ is $1/\sqrt{2}$. The random variable $w$ is Gaussian distributed with zero mean and variance of 2. The random variable $w$ is independent from $u$ and $v$.

5 (a) Write the joint pdf of $x_1$ and $x_2$.

7 (b) Find a linear transformation $\mathbf{A}$, i.e.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

such that $y_1$ and $y_2$ are uncorrelated.

8 (c) Find a linear transformation of $x_1$ and $x_2$, i.e. $z = \alpha_1 x_1 + \alpha_2 x_2$, such that $z$ is independent from $u$.

5. (25 pts.) A real valued random process $x[n]$ is defined as follows:

$$x[n] = \rho x[n-1] + w[n], \quad n \geq 0$$

where $|\rho| < 1$ and $w[n]$ is white noise with zero mean and variance of $\sigma_w^2$. The initial condition at $n = -1$ is given as zero, i.e. $x[-1] = 0$.

2 (a) Calculate the $\mu_x[n] = E\{x[n]\}$.

6 (b) Calculate var$\{x[0]\}$. var$\{x[1]\}$ explicitly and generalize to var$\{x[n]\}$. Is the process $x[n]$ stationary for $n \geq 0$ ?

8 (c) Calculate the autocorrelation of $x[n]$, i.e. $R_x[k_1, k_2] = E\{x[k_1]x[k_2]\}$ and check the consistency of the result with part (b). Is the process stationary?

9 (d) Assume that the initial condition of $x[-1]$ is also a random variable. The sample $x[-1]$ has zero mean and variance $\frac{\sigma_w^2}{1-\rho^2}$. Repeat part (b) for the random initilization. Comment on your results.

6. (15 pts.) The process $x(t)$ is zero mean WSS process.

Show that, if

$$\mathrm{s} = \frac{1}{N}\sum_{k=1}^{N} x(kT)$$

then

$$\mathrm{E}\{\mathrm{s}^2\} = \frac{1}{2\pi N^2}\int_{-\infty}^{\infty} S_x(\omega)\frac{\sin^2(N\omega T/2)}{\sin^2(\omega T/2)}\,d\omega$$

MIDDLE EAST TECHNICAL UNIVERSITY
Electrical and Electronics Engineering Department
Examination Answer Book

EE 503 MT#1 Fall 2011-12.
Solutions

Lastname, Name :
Date :

| Question | Page | Grade |
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| 1 | | |
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| Total | | |

Instructions:

1. Number the pages and write your last name on each page
2. On the first page, write the page number of each answer
3. Start a new question on a new page

Start writing below this line, plase!

① a) $E\{x(t)\} = \frac{1}{2}\sin \pi t + \frac{1}{2}2t$

b) $x(1) = \begin{cases} 0 & \text{heads} \\ 2 & \text{tails} \end{cases} \longrightarrow f_{x(1)}(x_1) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-2)$

c)

| $x(1) \backslash x(\frac{1}{2})$ | 1 | 1 |
|---|---|---|
| 0 | H | H |
| 2 | T | T |

$f_{x(1),x(\frac{1}{2})}(x_1, x_{1/2}) = \delta(x_{1/2} - 1)\left[\frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-2)\right]$.

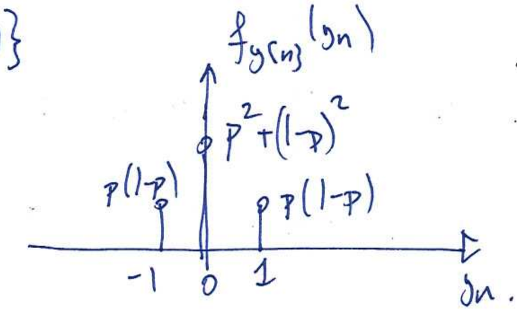
② $x[n] = \{1, 0\}$

a) $y[n] = x[n] - x[n-1]. \longrightarrow y[n] = \{1, 0, -1\}$

$p(y[n] = 0) = p^2 + (1-p)^2$

$p(y[n] = 1) = p(1-p)$

$p(y[n] = -1) = (1-p)p$




b) Yes, $1^{st}$ order pdf stays the same for all shifts.

$2^{nd}$ order pdf $\longrightarrow y[n], y[n-\Delta] \longrightarrow \Delta > 1$ $y[n], y[n-\Delta]$ are independent

$\hookrightarrow \Delta = 1$ not independent but not a func. of time.

$3^{rd}$ order pdf $\longrightarrow$ not a func. time ✓

c) $f_{x[n],y[n]}(x_n, y_n) = f_{y[n]|x[n]}(y_n|x_n)\ f_{x_n}(x_n)$

$$= \frac{1}{2}\left[\delta(x_n-1) + \delta(x_n)\right] \cdot \left[p\,\delta(y_n-1) + (1-p)\,\delta(y_n)\right]$$

Clearly joint pdf not a func. of "n".

Stationary.

d) $E\{y[n]\,y[n-k]\} = E\{(x[n]-x[n-1])(x[n-k]-x[n-k-1])\}$

$$= r_x(k) - r_x(k+1) - r_x(k-1) + r_x(k)$$

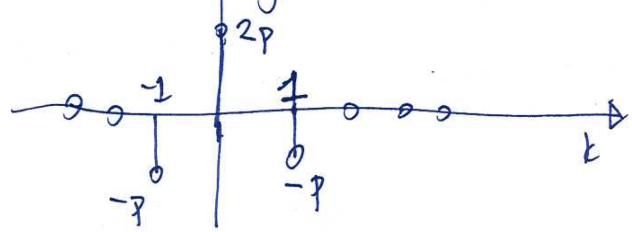
$r_x[k] = \cancel{A}\,\delta[k]$ $\qquad$ $\cancel{+ 2A\delta[k] - A\delta[k+1] - A\delta[k-1]}$

$E\{x^2[n]\} = p$ $\qquad$ $\cancel{A = E\{x^2[n]\} = p}$

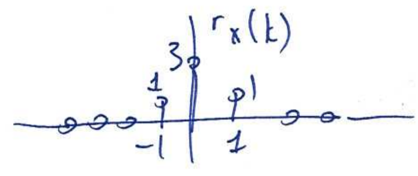
$E\{x[n]\,x[n-k]\} = p^2 \quad k \neq 0$

$r_y[k] = 2r_x(k) - r_x(k+1) - r_x(k-1)$

$$r_x(k) = \begin{cases} p & k = 0 \\ p^2 & k \neq 0 \end{cases}$$



3) $S_x(e^{jw}) = 3 + e^{jw} + e^{-jw} \longrightarrow r_x(k) = 3\delta[k] + \delta[k-1] + \delta[k+1]$.



a) Yes, WSS process filtered with LTI system $\rightarrow$ output: WSS.

b) $r_y(k) = h(k) * h(-k) * r_x(k)$

$$h(k) * h(-k) \rightarrow \left(1 - \frac{1}{2}z^{-2}\right)\left(1 - \frac{1}{2}z^{2}\right) = \frac{5}{4} - \frac{1}{2}z^{-2} - \frac{1}{2}z^{2}$$

$$h(k) * h(-k) * r_x(k) \rightarrow \left(\frac{5}{4} - \frac{1}{2}z^{-2} - \frac{1}{2}z^{+2}\right)\left(3 + z^{-1} + z\right) = \begin{cases} 15/4 + z^{-1}\left(\frac{5}{4} - \frac{1}{2}\right) + z^{-2}\left(-\frac{3}{2}\right) + z^{-3}\left(-\frac{1}{2}\right) \\ z^{1}(\;) + z^{2}\left(\frac{3}{2}\right) + z^{3}\left(-\frac{1}{2}\right) \end{cases}$$

$$r_y(k) = \begin{cases} 15/4 & k=0 \\ 3/4 & |k|=1 \\ -3/2 & |k|=2 \\ -1/2 & |k|=3 \\ 0 & \text{other} \end{cases}$$

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c) We can not say that $z[n] = (y(n))^2$ is WSS ~~from~~; since

$$E\{z[n]\,z[n-k]\} = E\{y^2[n]\,y^2[n-k]\}$$ depends on $4^{th}$ order moments.

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But if $x[n]$ is SSS; $z[n]$ is a memoryless mapping therefore $z[n]$ has density only depending on $x[n]$ itself; therefore $z[n]$ has joint pdf not a function of "$n$", i.e. $z[n]$ is SSS if $x[n]$ is SSS.

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d) $x[n] \sim N(\mu, \underbrace{3 + \mu_x^2}_{r_x(k)\,|_{k=0}})$

Then $z(n) = (x[n])^2 \longrightarrow P\{z(n) < z_n\} = P\{|x[n]| \leq \sqrt{z_n}\}$

$$= F_x(\sqrt{z_n}) - F_x(-\sqrt{z_n})$$

$$f_{z(n)}(z_n) = \frac{1}{2\sqrt{z_n}} f_x(\sqrt{z_n}) + \frac{1}{2\sqrt{z_n}} f_x(-\sqrt{z_n}).$$

4) $\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}}_{\underline{\underline{T}}} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\underline{u}} \rightarrow \underline{\underline{R}}_u = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$E\{uv\} = \rho_{uv} \sigma_u \sigma_v = \frac{1}{\sqrt{2}} \cdot 1 \sqrt{2} = 1$

a) $\underline{\underline{R}}_x = \underline{\underline{T}}\, \underline{\underline{R}}_u T^T$

$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 1 & 3 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} 7 & 1 \\ 1 & 3 \end{bmatrix}}_{\underline{\underline{C}}_x} \right) = \frac{1}{2\pi\, |C_x|^{1/2}} e^{-\frac{1}{2}[x_1 \ x_2]\, C_x^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$

b) $\underline{\underline{R}}_x = \begin{bmatrix} 7 & 1 \\ 1 & 3 \end{bmatrix}$

Let's use LU decomp.

$\begin{bmatrix} 1 & 0 \\ -\frac{1}{7} & 1 \end{bmatrix} \underline{\underline{R}}_x = \begin{bmatrix} 7 & 1 \\ 0 & \frac{20}{7} \end{bmatrix}$

$\underline{\underline{R}}_x = \begin{bmatrix} 1 & 0 \\ 1/7 & 1 \end{bmatrix} \overbrace{\begin{bmatrix} 7 & 0 \\ 0 & 20/7 \end{bmatrix} \begin{bmatrix} 1 & 1/7 \\ 0 & 1 \end{bmatrix}}$

Then let $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 1/7 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1/7 & 1 \end{bmatrix}$

Then $\underline{\underline{A}}\, \underline{\underline{R}}_x \underline{\underline{A}}^T = \begin{bmatrix} 7 & 0 \\ 0 & 20/7 \end{bmatrix}$

c) $z = [\alpha_1 \quad \alpha_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$E\{z\,u\} = 0 \longrightarrow E\left\{ [\alpha_1 \quad \alpha_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} u \right\} = 0$$

$$\Rightarrow [\alpha_1 \quad \alpha_2]\, E\left\{ \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} u \right\} = 0$$

$$[\alpha_1 \quad \alpha_2] \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} E\left\{ \begin{bmatrix} u^2 \\ uv \\ uw \end{bmatrix} \right\} = 0$$

$$[\alpha_1 \quad \alpha_2] \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$[\alpha_1 \quad \alpha_2] \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0$$

$2\alpha_1 = 0$ for independence.

Then $z = \alpha_2 x_2$ is independent of $u$ for $\forall \alpha_2$.

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5) $x[n] = \rho x[n-1] + w[n], n \geq 0$. $x[-1] = 0$

a) $\mu_x[n] \longrightarrow \mu_x[n] = \rho \mu_x[n-1] + 0 \;, n \geq 0$

$\mu_x[-1] = 0$ $\longrightarrow \mu_x[n] = 0$, $n \geq -1$

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b)

b) $x[0] = w[0]$

$x[1] = w[1] + \rho w[0]$

$x[2] = w[2] + \rho w[1] + \rho^2 v[0]$.

$$Var\{x[0]\} = \sigma_w^2$$

$$Var\{x[1]\} = \sigma_w^2(1 + \rho^2)$$

$$Var\{x[2]\} = \sigma_w^2(1 + \rho^2 + \rho^4)$$

$$\vdots$$

$$Var\{x[n]\} = \sigma_w^2(1 + \rho^2 + \rho^4 + \dots + \rho^{2n})$$

The process is not stationary in any sense, since

$Var\{x[n]\}$ changes by $n$.

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c) $$x[n] = \underbrace{\rho^{n+1} x[-1]}_{\substack{\text{zero-input response} \\ \text{(due resp. to initial cond.)}}} + \underbrace{\sum_{k=0}^{n} \rho^k w[n-k]}_{\substack{\text{zero-state resp.} \\ \text{(due resp. to input)}}} \qquad n \geq 0$$

~~$x[n]x[n-k]$~~

$$x[k_1]x[k_2] = \left(\rho^{k_1+1} x[-1] + \sum_{l_1=0}^{k_1} \rho^{l_1} w[k_1 - l_1]\right)\left(\rho^{k_2+1} x[-1] + \sum_{l_2=0}^{k_2} \rho^{l_2} w[k_2 - l_2]\right)$$

$$E\{x[k_1]x[k_2]\} = \rho^{k_1+k_2+2} E\{(x[-1])^2\} + \sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2} \rho^{l_1+l_2} \sigma_w^2 \delta[k_1 - k_2 + l_2 - l_1]$$

$$R_x[k_1,k_2] = \sigma_w^2 \sum_{l_1=0}^{k_1} \sum_{\substack{l_2=0 \\ \text{s.t.} \\ (l_1-l_2=k_1-k_2)}}^{k_2} \rho^{l_1+l_2}$$

Assume $k_1 > k_2$ without any loss generality.

$$= \sigma_w^2 \sum_{l_1=k_1-k_2}^{k_1} \rho^{l_1+l_2} \Big|_{l_2=l_1-(k_1-k_2)}$$

$$= \sigma_w^2 \sum_{l_1=k_1-k_2}^{k_1} \rho^{2l_1-(k_1-k_2)}$$

$$= \sigma_w^2 \sum_{l_1=0}^{k_2} \rho^{2l_1+(k_1-k_2)}$$

$$R_x[k_1,k_2] = \sigma_w^2 \rho^{k_1-k_2} \cdot \frac{1-\rho^{2(k_2+1)}}{1-\rho^2} \quad \text{for } k_1 > k_2.$$

$$\boxed{\text{Remember. } R_x[k_1,k_2] = R_x[k_2,k_1]}$$

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Note Take $k_1-k_2=\Delta$ and make $k_2 \to \infty$

$$R_x[k_1,k_2] = \sigma_w^2 \frac{\rho^{k_1-k_2}}{1-\rho^2}$$ ← Stationary depends only on $\Delta$.

This should be a familiar result from auto-corr. of single pole system.

d) $x[0] = \rho x[-1] + w(0)$

$x[1] = \rho^2 x[-1] + w(1) + \rho w[0]$

$x(2) = \rho^3 x[-1] + w(2) + \rho w(1) + \rho^2 w(0)$

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$$Var\{x[0]\} = \rho^2 \cdot \frac{\sigma_w^2}{1-\rho^2} + \sigma_w^2 = \sigma_w^2 \cdot \frac{1}{1-\rho^2}$$

$$Var\{x[1]\} = \rho^4 \frac{\sigma_w^2}{1-\rho^2} + \sigma_w^2(1+\rho^2) = \sigma_w^2 \cdot \frac{1}{1-\rho^2}$$

$\vdots$

$$Var\{x[2]\} = \rho^6 \frac{\sigma_w^2}{1-\rho^2} + \sigma_w^2 \underbrace{(1+\rho^2+\rho^4)}_{\frac{1-\rho^6}{1-\rho^2}} = \sigma_w^2 \frac{1}{1-\rho^2}$$

$\vdots$

$$Var\{x[n]\} = \rho^{2(n+1)} \frac{\sigma_w^2}{1-\rho^2} + \sigma_w^2 \underbrace{(1+\rho^2+\rho^4+ \cdots +\rho^{2n})}_{\frac{1-\rho^{2n+2}}{1-\rho^2}} = \sigma_w^2 \frac{1}{1-\rho^2}$$

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The process with random initilization is initialized with the value that it should reach at "steady-state." So for all $n \geq 0$, the process is stationary (WSS) with the described random initilization.

(b) $s = \frac{1}{N}\sum_{k=1}^{N} x(kT)$

$$E\{s^2\} = \frac{1}{N^2}\sum_{k_1=1}^{N}\sum_{k_2=1}^{N} E\{x(k_1T)\,x(k_2T)\}.$$

$$= \frac{1}{N^2}\sum_{k_1}\sum_{k_2} r_x((k_1-k_2)T)$$

$$= \frac{1}{N^2}\sum_{k_1}\sum_{k_2}\frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(j\omega)\, e^{j\omega(k_1-k_2)T}\,d\omega$$

$$= \frac{1}{2\pi N^2}\int_{-\infty}^{\infty} S_x(e^{j\omega})\left(\sum_{k_1=1}^{N} e^{j\omega T k_1}\right)\left(\sum_{k_2=1}^{N} e^{-j\omega T k_2}\right)d\omega.$$

$$= \frac{1}{2\pi N^2}\int_{-\infty}^{\infty} S_x(e^{j\omega})\left(\frac{1-e^{j\omega TN}}{1-e^{j\omega T}}\cdot e^{j\omega T}\right)\left(\frac{1-e^{-j\omega TN}}{1-e^{-j\omega T}}\cdot e^{-j\omega T}\right)d\omega$$

$$= \frac{1}{2\pi N^2}\int_{-\infty}^{\infty} S_x(e^{j\omega})\left|\frac{1-e^{j\omega TN}}{1-e^{j\omega T}}\right|^2 d\omega$$

$$= \frac{1}{2\pi N^2}\int_{-\infty}^{\infty} S_x(e^{j\omega})\left(\frac{\sin(N\omega T/2)}{\sin(\omega T/2)}\right)^2 d\omega$$

Dirichlet function!