Minimum Error Calculation for Non-Causal IIR Wiener Filter

Previously we have shown that when

$$x[n] = d[n] + v[n]$$

where $d[n]$ is a WSS random process with $r_d[k] = \alpha^{|k|}$ and v[n] is white noise with variance σ_v^2, the mean square error for the two tap causal FIR filter, that is

$$\widehat{d}[n] = w_0 x[n] + w_1 x[n-1],$$

is equal to $r_d[0] - w_0 r_d[0] - w_1 r_d[1]$ where w_0 and w_1 are the optimal FIR Wiener filter coefficients.

For $\alpha = 0.8$ and $\sigma_v^2 = 1$, we have shown that the optimal coefficients are $w_0 = 0.4080$ and $w_1 = 0.2381$ and the minimum MSE error is $J_{\min} = 0.4048$.

Following Hayes Example 7.3.1, we will evaluate the error for the non-causal IIR filter for the same example.

The optimal non-causal IIR filter is

$$H^{NC}(z) = \frac{P_d(z)}{P_d(z)+P_v(z)}$$

where $P_d(z) = Z\{\alpha^{|k|}\} = \frac{1-\alpha^2}{(1-\alpha z^{-1})(1-\alpha z)}$ and $P_v(z) = \sigma_v^2$. When $P_d(z)$ and $P_v(z)$ is inserted in $H^{NC}(z)$ we get,

$$H^{NC}(z) = \frac{\frac{1-\alpha^2}{(1-\alpha z^{-1})(1-\alpha z)}}{\frac{1-\alpha^2}{(1-\alpha z^{-1})(1-\alpha z)}+\sigma_v^2} = \frac{1-\alpha^2}{1-\alpha^2+\sigma_v^2(1-\alpha z^{-1})(1-\alpha z)}$$

For $\alpha = 0.8$ and $\sigma_v^2 = 1$, the filter reduces to

$$\begin{aligned} H^{NC}(z) = \left[\frac{1-\alpha^2}{1-\alpha^2+\sigma_v^2(1-\alpha z^{-1})(1-\alpha z)}\right]_{\alpha=0.8,\sigma_v^2=1} &= \frac{0.36}{(1-0.8z)\left(1-\frac{0.8}{z}\right)+0.36} \\ &= \frac{9/40}{\left(1-\frac{1}{2}z\right)\left(1-\frac{1}{2}z^{-1}\right)} \\ &= \frac{0.3(1-1/4)}{\left(1-\frac{1}{2}z\right)\left(1-\frac{1}{2}z^{-1}\right)} \end{aligned}$$

Comparing the last equation with the Z-transform of $\alpha^{|k|}, Z\{\alpha^{|k|}\} = \frac{1-\alpha^2}{(1-\alpha z^{-1})(1-\alpha z)}$; we get the impulse response as follows:

$$h^{NC}[n] = 0.3\left(\frac{1}{2}\right)^{|n|}$$

The minimum MSE for non-causal Wiener filter is $J_{\min} = r_d[0] - \frac{1}{2\pi}\int_{-\pi}^{-\pi} H(e^{j\omega})P_{dx}(e^{j\omega})d\omega$.

Specific to the filtering application, we have $P_{dx}(e^{j\omega}) = P_d(e^{j\omega})$. The error becomes

$$\begin{aligned} J_{\min} &= \frac{1}{2\pi}\int_{-\pi}^{-\pi} P_d(e^{j\omega})d\omega - \frac{1}{2\pi}\int_{-\pi}^{-\pi} H^{NC}(e^{j\omega})P_d(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{-\pi}(1 - H^{NC}(e^{j\omega})P_d(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{-\pi}\left(1 - \frac{P_d(e^{j\omega})}{P_d(e^{j\omega})+P_v(e^{j\omega})}\right)P_d(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{-\pi}\frac{P_v(e^{j\omega})}{P_d(e^{j\omega})+P_v(e^{j\omega})}P_d(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{-\pi} H^{NC}(e^{j\omega})P_v(e^{j\omega})d\omega \end{aligned}$$

When $v[n]$ is white noise, that is $P_v(e^{j\omega}) = \sigma_v^2$, then the error is

$$\begin{aligned} J_{\min} &= \sigma_v^2 \frac{1}{2\pi}\int_{-\pi}^{-\pi} H^{NC}(e^{j\omega})d\omega \\ &= \sigma_v^2\, h^{NC}[0]. \end{aligned}$$

For the given example, the minimum error for the non-causal IIR filter is then $J_{\min} = 0.3$.