EE 503
Homework 4
Due: Nov. 26th, 2010

The process x[n] is defined as follows:

i. x[0] is 1 or -1 with probability $\frac{1}{2}$.

ii. $x[n] = \begin{cases} x[n-1] & \textbf{with probability } \alpha \\ -x[n-1] & \textbf{with probability } (1-\alpha) \end{cases}$ where $0 < \alpha \le \frac{1}{2}$

1. Show that $p\{x[n+k]=1 \mid x[n]=1\} = \frac{1+(2\alpha-1)^k}{2}$, where k>0. (Hint: You can describe the process as a Markov Chain. If you are not familiar with Markov chains (not taken EE531), then apply induction to show the result.)
2. Is the process strict sense stationary (SSS) in the first order, second order? (considering x[n] for $n \ge 0$.) Is the process SSS for all orders?
3. The process y[n] is defined as follows:

   i. y[0] is 1 or -1 with probability $\frac{1}{3}$ and $\frac{2}{3}$ respectively.

   ii. $y[n] = \begin{cases} y[n-1] & \textbf{with probability } \alpha \\ -y[n-1] & \textbf{with probability } (1-\alpha) \end{cases}$ where $0 < \alpha \le \frac{1}{2}$

   Is the process y[n] strict sense stationary? What can you say about the stationarity of this process as $n \to \infty$?
4. Find the mean and auto-correlation of the process x[n]. Is the process WSS ?
5. **Computer Simulation:**
   a. Generate 3 different realizations of process x[n] and plot the realizations in the same figure using different markers. (Hint: For plotting help, use "help plot" in Matlab.) Set α=0.15.
   b. **Mean Estimation**
      i. **Ensemble Average:** Set α=0.15 and generate a realization of the process, $x_\zeta[n]$ and sample it at the instants n={150,200,250}. Repeat the same sampling for 100 realizations and calculate $\hat{\mu} = \frac{1}{100}\sum_{K=1}^{100} x_{\zeta_K}[n]$, where $x_{\zeta_K}[n]$ is the value at the n'th instant of the K'th realization. Repeat whole procedure for α={0.25, 0.45}.
      ii. **Time Average:** Estimate the mean of the process from a single realization using the estimator, $\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} x[n]$. Try N={10,20,50,100} and α={0.15, 0.25, 0.45}. Repeat the mean estimation procedure for 4 different realizations and present $\hat{\mu}$ in a table. Repeat the mean estimation procedure for 100 different realizations and calculate the average square error on μ (that is $\frac{1}{100}\sum_{k=1}^{100}(\hat{\mu}-\mu)^2$ where μ is the true mean of process x[n]) and present the results in a table for the given values of N and α.

   Comment on the mean estimation results.

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c. **Auto-correlation Estimation:**

i. **Ensemble Average:** Generate a realization of the process, $x_{\zeta}[n]$ and evaluate $x_{\zeta}[n]x_{\zeta}[n-1]$ at the instants n={150,200,250}. Set α=0.15. Repeat the same sampling for 100 realizations and calculate $\hat{r}_x[1] = \frac{1}{100}\sum_{K=1}^{100} x_{\zeta_K}[n]x_{\zeta_K}[n-1]$, where $x_{\zeta_K}[n]$ is the value at the n'th instant of the K'th realization.

ii. **Time Average:** Estimate the first auto-correlation lag of the process from a single realization using the estimator, $\hat{r}_x[1] = \frac{1}{N}\sum_{n=1}^{N} x[n]x[n-1]$ using N={10,20,50,100} and α={0.15, 0.25, 0.45}. Repeat the procedure for 4 different realizations and present $\hat{r}_x[1]$ in a table. Repeat the procedure for 100 different realizations and present the average square error on $r_x[1]$ estimate (that is $\frac{1}{100}\sum_{k=1}^{100}(\hat{r}_x[1] - r_x[1])^2$) for all values of N and α in a table.

Comment on the auto-correlation (first lag) estimation results.

6. The process x[n] is filtered with $H(z) = \frac{1}{2}(1 + z^{-1})$. The output of this filter is called as $x_2[n]$. Using the result of Question 4, analytically find the mean and auto-correlation of the process $x_2[n]$.
7. Find the pdf of $x_2[n]$. Find the joint pdf of $x_2[n]$ and $x_2[n-1]$. Evaluate E{$x_2[n]x_2[n-1]$} and compare your result with the first auto-correlation lag found in Question 6.
8. Find the joint pdf of $x_2[n]$ and $x_2[n-k]$. (Hint: Use Question 1)
9. Find the joint pdf of $x_2[n]$ and $x_2[n-1]$ and $x_2[n-2]$.
10. Is the process $x_2[n]$ WSS and/or SSS? If SSS, then what is its order of stationarity?
11. **Computer Simulation:**
    a. Using α=0.25, generate $x_2[n]$ and estimate its mean and first 3 auto-correlation lags (as discussed previously) using a sample length N={20,50,100}. Repeat the same procedure 100 realizations and report the estimation error variance for N={20,50,100}.
    b. Using α=0.5, generate $x_2[n]$ and estimate its mean and first 3 auto-correlation lags (as discussed previously) using a sample length N={20,50,100}. Repeat the same procedure 100 realizations and report the estimation error variance for N={20,50,100}.
12. Determine the type of the process $x_2[n]$ (MA, AR, ARMA, Periodic Process) when α=0.25 and α=0.5.
13. Generate a Gaussian process with the same auto-correlation and mean with $x_2[n]$ for α=0.25. Analytically express the filter generating such a process. Analytically express the joint density of Gaussian process evaluated at the instants "n" and "n-1". Compare the result of Question 7.

<u>Reading Assignments:</u>

1) You can examine Therrien page 99 (and forward) for an introduction to Markov chains.
2) The process x[n] is closely related to famous random telegraph signal which is described in page 291 of Papoulis. The process x[n] can be considered as the sampled version of the random telegraph signal.
3) The computer simulations show that the average calculated from a single realization can lead to the correct value of the ensemble average. This is an important property of the random processes called the ergodicity. Ergodicity will be discussed later in the course. Interested students can examine ergodicity discussion given in Hayes.

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