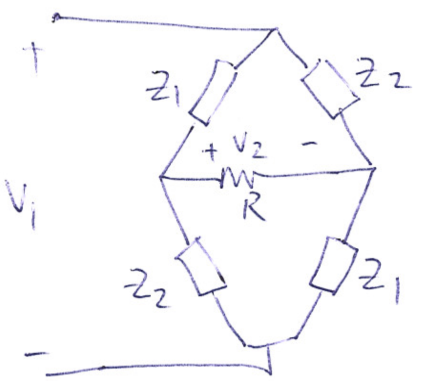
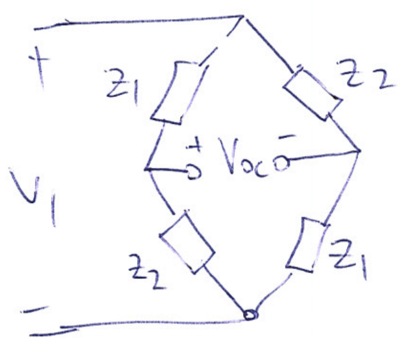
2^{nd} Order All-Pass Circuit:



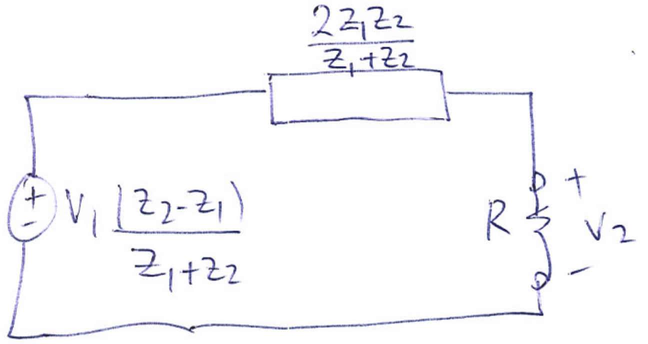
i) Z_{Th} seen by $R \Rightarrow (Z_1 \| Z_2)2$

ii)



$$V_{oc} = V_1\left(\frac{Z_2 - Z_1}{Z_1 + Z_2}\right)$$

Then.



$$\frac{V_2(s)}{V_1(s)} = ?$$

$$\frac{V_2(s)}{V_1(s)} = \frac{(Z_2(s) - Z_1(s))}{Z_1(s) + Z_2(s)} \cdot \frac{R}{R + \frac{2Z_1Z_2}{Z_1+Z_2}} = (Z_2 - Z_1) \cdot \frac{R}{R(Z_1+Z_2) + 2Z_1Z_2}$$

$$= \frac{(Z_2 - Z_1)}{(Z_1 + Z_2) + \frac{2Z_1Z_2}{R}} \rightarrow$$

$\Rightarrow \boxed{\text{Let } Z_1 Z_2 = R^2}$ $\xrightarrow{\text{Then}}$ $$= \frac{(Z_2 - Z_1)}{(Z_1 + Z_2) + 2R}$$

$$= \frac{Z_2(Z_2 - Z_1)}{Z_2\{(Z_1 + Z_2) + 2R\}}$$

$$\boxed{A(s) = \frac{Z_2 - R}{Z_2 + R}} = \frac{(Z_2 - R)(Z_2 + R)}{(Z_2 + R)^2} = \frac{Z_2^2 - R^2}{Z_2^2 + 2RZ_2 + R^2}$$

1B

then

$$H(j\omega) = \frac{Z_2(j\omega) - R}{Z_2(j\omega) + R}.$$

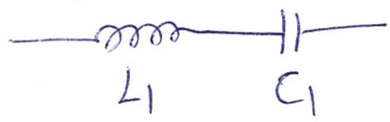
then 1) $Z_2(j\omega)$: purely real $\forall \omega \xrightarrow[Z_2 = R_2]{} H(j\omega) = \frac{R_2 - R}{R_2 + R}.$

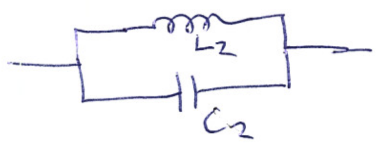
$\rightarrow$ $|H(j\omega)|$ is not function of ω $\rightarrow$ but system does not contain any dynamic elements. Therefore it is just a voltage divider, not a filter.

2) $Z_2(j\omega)$: purely imaginary $\forall \omega \xrightarrow[Z_2 = jX_2]{} H(j\omega) = \frac{jX_2 - R}{jX_2 + R}$

$\rightarrow$ $|H(j\omega)| = 1$ $\rightarrow$ we have dynamic system with an all-pass structure. (Q: If $Z_2 = jX_2$, can $Z_1 Z_2 = R^2$ be satisfied? (No!))

Example: How to select Z_1 and Z_2 such that $Z_1 Z_2 = R^2$ is satisfied?

Z_1: 

Z_2: 

$$\begin{cases} Z_1(s) = sL_1 + 1/sC_1 \\ Z_2(s) = \left(\frac{1}{sL_2} + sC_2\right)^{-1} \end{cases}$$

$$Z_1 \cdot Z_2 = R^2 \rightarrow Z_1 = \frac{R^2}{Z_2} \rightarrow sL_1 + \frac{1}{sC_1} = R^2\left(\frac{1}{sL_2} + sC_2\right)$$

$\downarrow$

Let

$$\boxed{\begin{aligned} L_1 &= R^2 C_2 \\ C_1 &= L_2/R^2 \end{aligned}}$$

Then $$H(s) = \frac{Z_2 - R}{Z_2 + R} = \frac{1 - RY_2}{1 + RY_2} \qquad \left(Z_2^{-1} = Y_2 = \frac{1}{sL_2} + sC_2 \right)$$

$$= \frac{1 - \frac{R}{sL_2} - sRC_2}{1 + \frac{R}{sL_2} + sRC_2}$$

$$= - \frac{s^2 - \frac{1}{RC_2}s + \frac{1}{L_2C_2}}{s^2 + \underbrace{\frac{1}{RC_2}}_{2\alpha w_0}s + \underbrace{\frac{1}{L_2C_2}}_{w_0^2}}$$

$$H(s) = - \frac{s^2 - 2\alpha w_0 s + w_0^2}{s^2 + 2\alpha w_0 s + w_0^2}$$

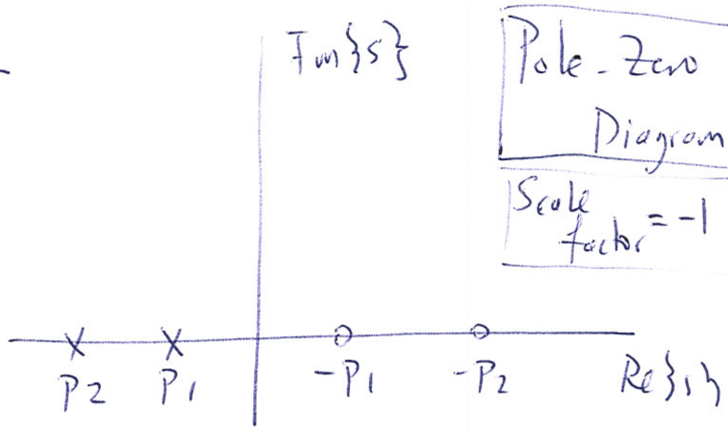
Note: If s_1, s_2 are roots of $s^2 + 2\alpha w_0 s + w_0^2 = 0$.
then $-s_1, -s_2$ are the roots of $s^2 - 2\alpha w_0 s + w_0^2 = 0$.
(Why?: (Hint: $\mp 2\alpha w_0$ = sum of roots ; w_0^2 : product of roots))

Then.

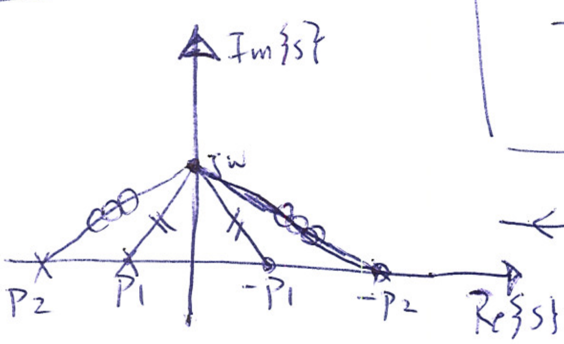
$$H(s) = - \frac{(s - P_1)(s - P_2)}{(s + P_1)(s + P_2)}$$

Pole-Zero Diagram

Scale factor $= -1$

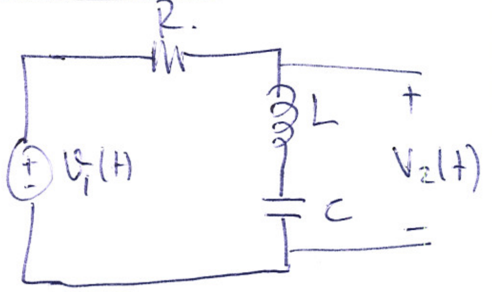


Note:



← this picture shows that system is all-pass!

Series RLC Band-Stop Filter

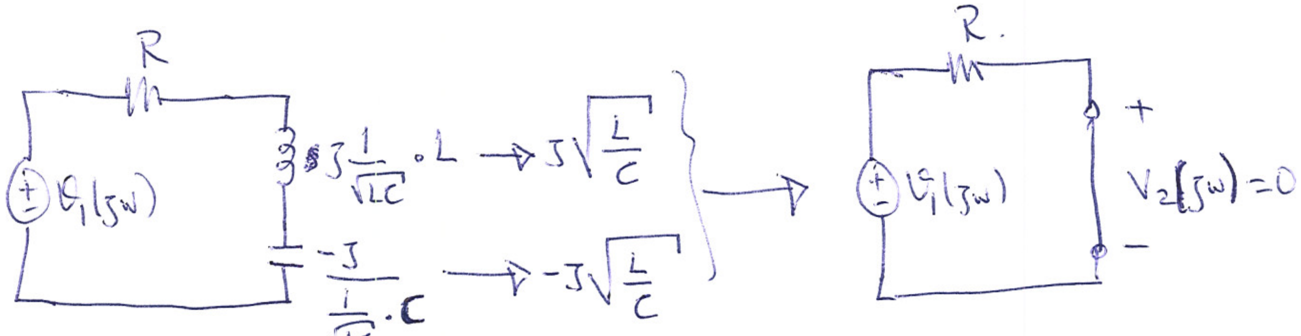


$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{sL + 1/sC}{sL + 1/sC + R}$$

$$= \frac{s^2 + 1/LC}{s^2 + \frac{R}{L}s + 1/LC}$$

$$H(j\omega) = \frac{\left(1/LC - \omega^2\right)}{\left(1/LC - \omega^2\right) + j\omega\frac{R}{L}} \longrightarrow \text{Note: } |H(j\omega)| = 0 \text{ for } \omega = \left\{\pm\sqrt{1/LC}\right\}$$

Let's redraw the circuit at $\omega = 1/\sqrt{LC}$

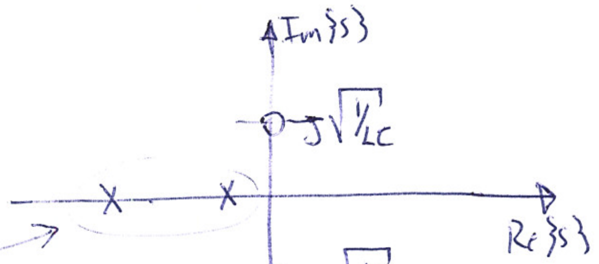


$\omega = \frac{1}{\sqrt{LC}}$

Note that $v_1(j\omega)$ source sees purely resistive circuit at $\omega = \frac{1}{\sqrt{LC}}$

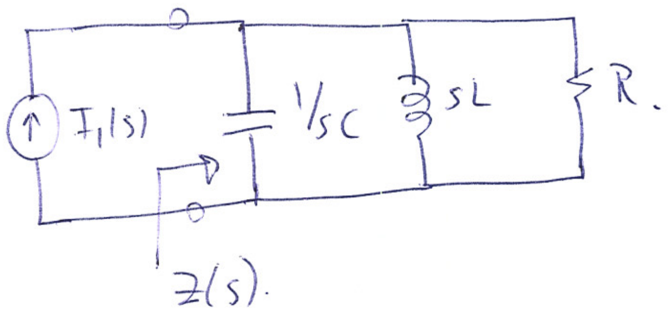
This phenomena is called <u>resonance</u>. (more on this later).

Pole-Zero Diagram



Assuming two distinct poles for $H(s)$. → (overdamped system).

Parallel RLC Resonance Circuit:



$$Z(s) = \frac{1}{1/R + 1/sL + sC} = \frac{s/C}{s^2 + \frac{1}{RC}s + 1/LC}$$

Definition: The frequency for which $Z(j\omega)$ is purely real is called the resonance frequency.

$$Z(j\omega) = \frac{j\omega/C}{-\omega^2 + 1/LC + j\omega\, 1/RC} = \frac{\omega/C}{j(\omega^2 - 1/LC) + \omega\, 1/RC}$$

is zero when $\omega = 1/\sqrt{LC}$

Resonance freq.

(We have studied this system under the title of 2nd Order Band-Pass circuits)

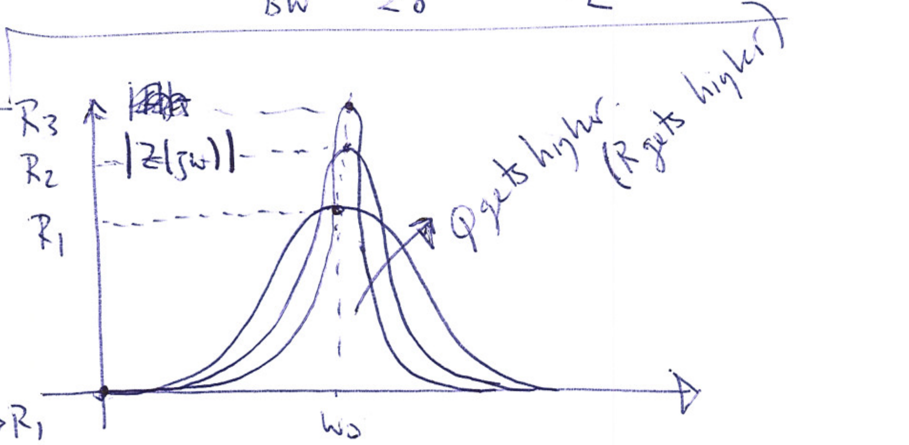
$$s^2 + \underbrace{\frac{1}{RC}}_{2\gamma\omega_0}s + \underbrace{1/LC}_{\omega_0^2} \longrightarrow \quad BW = \omega_2 - \omega_1 = 2\gamma\omega_0 = 1/RC$$

-3dB points

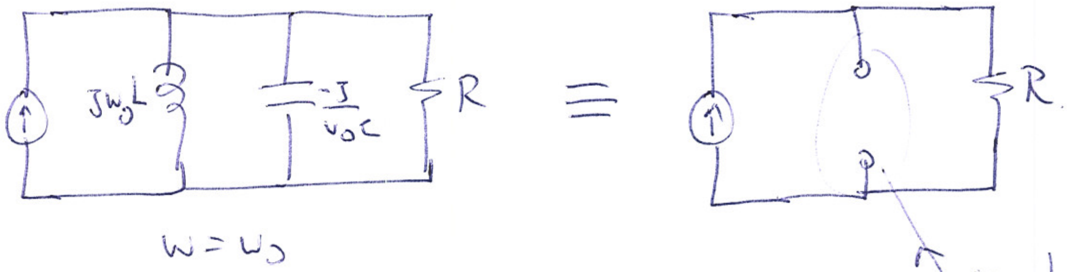
$$Q = \frac{\omega_0}{BW} = \frac{1}{2\gamma} = R\sqrt{\frac{C}{L}}$$

$$\omega_0 = \sqrt{1/LC}\,, \quad 2\gamma = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Picture shows the impedance seen by the source for 3 parallel RLC circuits with $R_3 > R_2 > R_1$ and the same L and C's.



The maximum impedance is seen at resonance freq. (for parallel RLC) and it is equal to R. Note that



combination of L and C at $w = w_0$.

Another Interpretation for Q: ← Quality factor.

For parallel RLC circuit, the average energy stored in capacitor is $E_C = \frac{1}{2} C V_{eff}^2$; similarly the average magnetic energy stored in inductor is $E_L = \frac{1}{2} L I_{eff}^2 \rightarrow$

Let's calculate the total energy stored in L and C.

$$E_T = E_C + E_L = \frac{1}{2} C V_{eff}^2 + \frac{1}{2} L \left(\frac{v_{eff}}{wL}\right)^2$$

$$= \frac{1}{2}\left(C + \frac{1}{w^2 L}\right) V_{eff}^2$$

$$= \frac{1}{2} C \left(1 + \underbrace{\frac{1}{w^2 LC}}_{w_0^2}\right) v_{eff}^2$$

$$= \frac{1}{2} C \left(1 + \frac{w_0^2}{w^2}\right) v_{eff}^2$$

$$= \frac{1}{2} C \frac{w_0}{w} \left(\frac{w}{w_0} + \frac{w_0}{w}\right) v_{eff}^2$$

We will now show that at $w = w_0$

4A

$$Q = 2\pi \cdot \frac{E_T}{T P_R} \quad \left(\begin{matrix} Q: \\ \text{unitless} \end{matrix}\right) \text{ where } \quad T = \frac{2\pi}{w_0}$$

where P_R is the average ~~power~~ power consumed by R and $T . P_R$ is the energy dissipated over R in a period.

Now

$$2\pi \frac{E_T}{T . P_R} = \frac{w E_T}{P_R} = \frac{w \frac{1}{2} C \cancel{\frac{w_0}{w}} \left(\frac{w}{w_0} + \frac{w_0}{w}\right) \cancel{v_{eff}^2}}{\frac{\cancel{v_{eff}^2}}{R}}$$

$1/\sqrt{LC}$

$$= \frac{R C w_0}{2}\left(\frac{w}{w_0} + \frac{w_0}{w}\right)$$

$$= \frac{R\sqrt{\frac{C}{L}}}{2}\left(\frac{w}{w_0} + \frac{w_0}{w}\right)$$

$$= \frac{Q}{2}\left(\frac{w}{w_0} + \frac{w_0}{w}\right)$$

for $w = w_0 \longrightarrow 2\pi \frac{E_T}{T . P_R} = Q$, $\quad T = \frac{2\pi}{w_0}$.

<u>Note</u>: Resonant freq. is the freq. where maximum of $|Z(jw)|$ for parallel RLC circuit; BUT this is not true in general. If w_m is the freq. of maxima of $|Z(jw)|$ $w_m \neq w_0$ in general.

5A

$$Z(s)\Big|_{s=j\omega} = \frac{j\omega/C + r_1/LC}{\left(\underbrace{1/LC}_{\omega_a^2} - \omega^2\right) + j\omega \underbrace{R_1/L}_{\omega_a/Q}} = \frac{\frac{1}{C}\left(j\omega + \overbrace{r_1/L}^{\omega_a/Q}\right)}{(\omega_a^2 - \omega^2) + j\omega\, \omega_a/Q}$$

$$= \frac{\frac{1}{C}\left(j\omega + \omega_a/Q\right)}{(\omega_a^2-\omega^2) + j\omega\, \omega_a/Q}$$

for $Z(j\omega)$ to be purely real, angle of numerator and denominator of $Z(j\omega)$ should be same $\rightarrow$ so

$$\frac{\cancel{\omega}}{\omega_a/Q} = \frac{\cancel{\omega}\, \omega_a/Q}{\omega_a^2-\omega^2} \rightarrow \left(\frac{\omega_a}{Q}\right)^2 = \omega_a^2 - \omega^2$$

$$\downarrow$$

$$\omega^2 = \omega_a^2\left(1 - 1/Q^2\right)$$

$$\boxed{\omega = \omega_a\sqrt{1 - 1/Q^2}}$$

then $\boxed{\omega_0 = \omega_a\sqrt{1-1/Q^2}}$ is the resonant freq.

$$Z(j\omega_0) = \frac{1}{C}\,\frac{\cancel{(j\omega_0 + \omega_a/Q)}}{\omega_a/Q\,\cancel{\left(j\omega_0 + \frac{\omega_a^2-\omega^2}{\omega_a/Q}\right)}} = \frac{Q.1}{C\omega_a} \quad \frac{Q.\omega_a^2 L \cancel{C}}{\cancel{C}\omega_a}$$

$\sqrt{\frac{1}{LC}}$

$$= Q.\frac{\omega_a\, r_1}{r_1/L}$$

Q

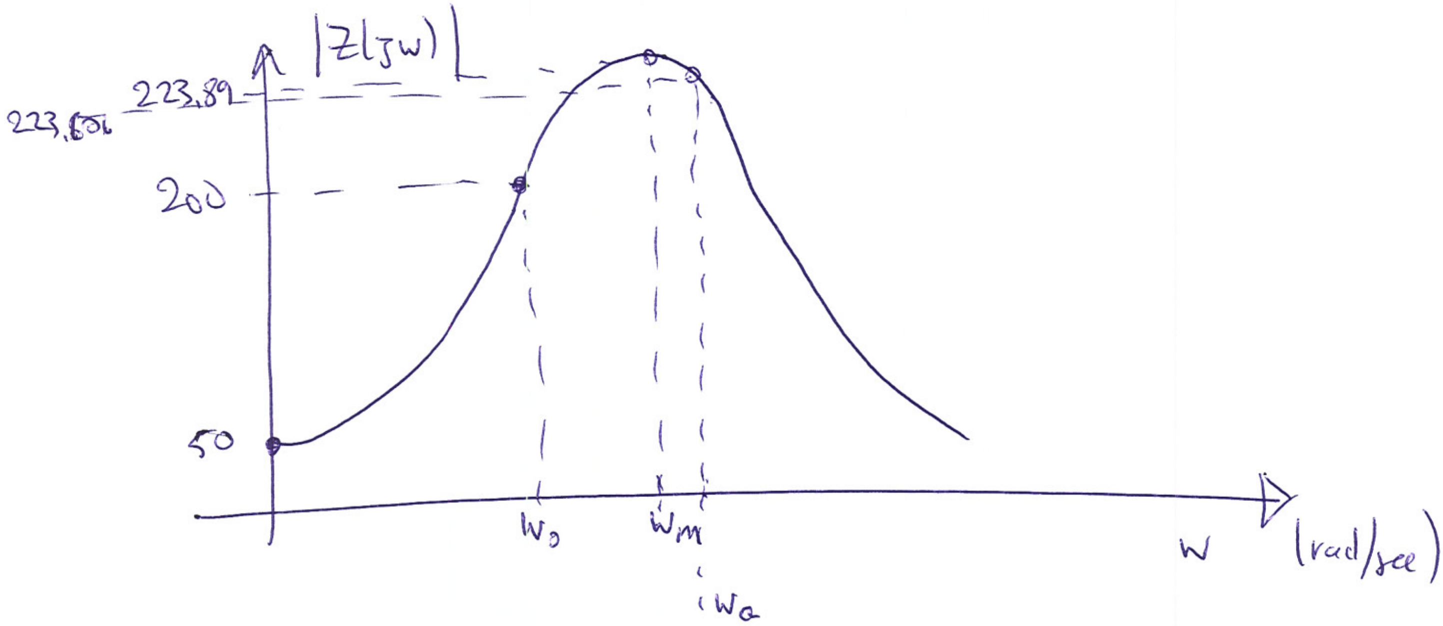
$$= Q^2 . r_1$$

Ex: a) $r_1 = 50\,\Omega$, $L = 10\,mH$, $C = 1\,\mu F$

then
$$\begin{cases} w_a = 1/\sqrt{LC} = 10^4 = 10\,k\ \text{rad/sec.} \\ \varphi = \dfrac{w_a}{r_1/L} = 2 \quad (\text{unitless}). \\ w_o = w_a\sqrt{1 - 1/\varphi^2} = 10^4\sqrt{1 - 1/4} = 8.66\,k\ \text{rad/sec.} \end{cases}$$

$Z(j0) = 50\,\Omega$, $Z(jw_o) = \varphi^2 \cdot r_1 = 200\,\Omega$; $Z(jw_a) = 223.606\angle{-26^\circ}$

$Z(jw_m) = 223.89\angle{-23}$



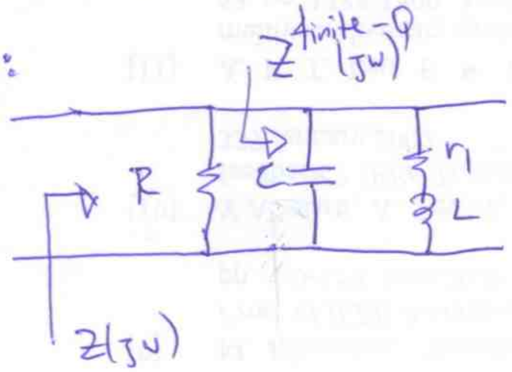
$w_a = \sqrt{1/LC}$

w_m: → should be found by taking derivative of $|Z(jw)|$ wrt. w.

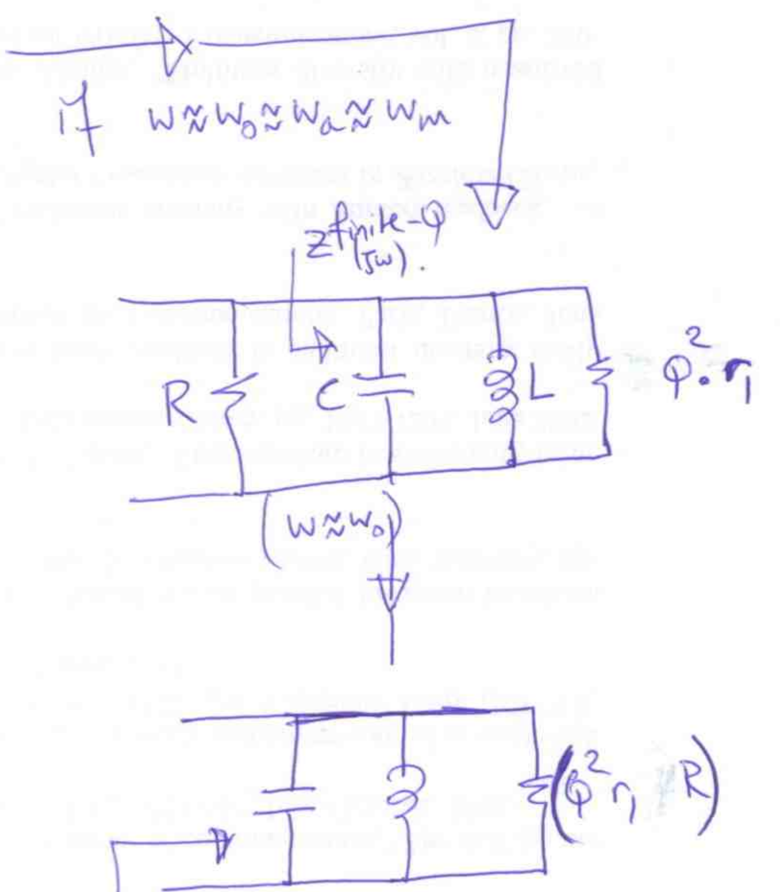
$w_o = w_a\sqrt{1 - 1/\varphi^2}$ ← resonant freq. of finite-φ circuit.

We observe that for $Q=2$; w_a is sufficiently close to w_m. As $Q \to \infty$, w_a and w_m approaches w_0. But note that $Q=2$ is close enough for many purposes in this example.

Ex:



Find $Z(j\omega)$, and assume that $\omega \approx$ resonat freq of this system.



Parallel
Standart resonance circuit!