3. The joint probability mass function of X and Y, $p(x, y)$, is given by

$$p(1,1)=\tfrac{1}{9},\quad p(2,1)=\tfrac{1}{3},\quad p(3,1)=\tfrac{1}{9}$$
$$p(1,2)=\tfrac{1}{9},\quad p(2,2)=0,\quad p(3,2)=\tfrac{1}{18}$$
$$p(1,3)=0,\quad p(2,3)=\tfrac{1}{6},\quad p(3,3)=\tfrac{1}{9}$$

Compute $E[X|Y=i]$ for $i = 1, 2, 3$.

4. In Exercise 3, are the random variables X and Y independent?

5. An urn contains three white, six red, and five black balls. Six of these balls are randomly selected from the urn. Let X and Y denote respectively the number of white and black balls selected. Compute the conditional probability mass function of X given that $Y = 3$. Also compute $E[X|Y = 1]$.

***6.** Repeat Exercise 5 but under the assumption that when a ball is selected its color is noted, and it is then replaced in the urn before the next selection is made.

7. Suppose $p(x, y, z)$, the joint probability mass function of the random variables X, Y, and Z, is given by

$$p(1,1,1)=\tfrac{1}{8},\quad p(2,1,1)=\tfrac{1}{4},$$
$$p(1,1,2)=\tfrac{1}{8},\quad p(2,1,2)=\tfrac{3}{16},$$
$$p(1,2,1)=\tfrac{1}{16},\quad p(2,2,1)=0,$$
$$p(1,2,2)=0,\quad p(2,2,2)=\tfrac{1}{4}$$

What is $E[X|Y = 2]$? What is $E[X|Y = 2, Z = 1]$?

8. An unbiased die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a six and a five. Find (a) $E[X]$, (b) $E[X|Y = 1]$, (c) $E[X|Y = 5]$.

9. Show in the discrete case that if X and Y are independent, then

$$E[X|Y = y] = E[X] \qquad \text{for all } y$$

10. Suppose X and Y are independent continuous random variables. Show that

$$E[X|Y = y] = E[X] \qquad \text{for all } y$$

11. The joint density of X and Y is

$$f(x,y) = \frac{(y^2 - x^2)}{8}e^{-y}, \qquad 0 < y < \infty, \quad -y \leqslant x \leqslant y$$

Show that $E[X|Y = y] = 0$.

12. The joint density of X and Y is given by

$$f(x,y) = \frac{e^{-x/y}e^{-y}}{y}, \qquad 0 < x < \infty, \quad 0 < y < \infty$$

Show $E[X|Y = y] = y$.

***13.** Let X be exponential with mean $1/\lambda$; that is,

$$f_X(x) = \lambda e^{-\lambda x}, \qquad 0 < x < \infty$$

Find $E[X|X > 1]$.

14. Let X be uniform over $(0, 1)$. Find $E[X|X < \frac{1}{2}]$.

15. The joint density of X and Y is given by

$$f(x,y) = \frac{e^{-y}}{y}, \qquad 0 < x < y, \quad 0 < y < \infty$$

Compute $E[X^2|Y = y]$.

16. The random variables X and Y are said to have a bivariate normal distribution if their joint density function is given by

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)} \times \left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, where σ_x, σ_y, μ_x, μ_y, and ρ are constants such that $-1 < \rho < 1$, $\sigma_x > 0$, $\sigma_y > 0$, $-\infty < \mu_x < \infty$, $-\infty < \mu_y < \infty$.

(a) Show that X is normally distributed with mean μ_x and variance σ_x^2, and Y is normally distributed with mean μ_y and variance σ_y^2.
(b) Show that the conditional density of X given that $Y = y$ is normal with mean $\mu_x + (\rho\sigma_x/\sigma_y)(y - \mu_y)$ and variance $\sigma_x^2(1-\rho^2)$.

The quantity ρ is called the correlation between X and Y. It can be shown that

$$\rho = \frac{E[(X-\mu_x)(Y-\mu_y)]}{\sigma_x\sigma_y} = \frac{\text{Cov}(X,Y)}{\sigma_x\sigma_y}$$

17. Let Y be a gamma random variable with parameters (s, α). That is, its density is

$$f_Y(y) = Ce^{-\alpha y} y^{s-1}, \qquad y > 0$$

where C is a constant that does not depend on y. Suppose also that the conditional distribution of X given that $Y = y$ is Poisson with mean y. That is,

$$P\{X = i | Y = y\} = e^{-y} y^i / i!, \qquad i \geqslant 0$$

Show that the conditional distribution of Y given that $X = i$ is the gamma distribution with parameters $(s + i, \alpha + 1)$.

18. Let $X_1, \ldots, X_n$ be independent random variables having a common distribution function that is specified up to an unknown parameter θ. Let $T = T(\mathbf{X})$ be a function of the data $\mathbf{X} = (X_1, \ldots, X_n)$. If the conditional distribution of $X_1, \ldots, X_n$ given $T(\mathbf{X})$ does not depend on θ then $T(\mathbf{X})$ is said to be a *sufficient statistic* for θ. In the following cases, show that $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a sufficient statistic for θ.

(a) The X_i are normal with mean θ and variance 1.
(b) The density of X_i is $f(x) = \theta e^{-\theta x}$, $x > 0$.
(c) The mass function of X_i is $p(x) = \theta^x (1 - \theta)^{1-x}$, $x = 0, 1$, $0 < \theta < 1$.
(d) The X_i are Poisson random variables with mean θ.

***19.** Prove that if X and Y are jointly continuous, then

$$E[X] = \int_{-\infty}^{\infty} E[X | Y = y] f_Y(y)\, dy$$

20. An individual whose level of exposure to a certain pathogen is x will contract the disease caused by this pathogen with probability $P(x)$. If the exposure level of a randomly chosen member of the population has probability density function f, determine the conditional probability density of the exposure level of that member given that he or she

(a) has the disease.
(b) does not have the disease.
(c) Show that when $P(x)$ increases in x, then the ratio of the density of part (a) to that of part (b) also increases in x.

21. Consider Example 3.12 which refers to a miner trapped in a mine. Let N denote the total number of doors selected before the miner reaches safety. Also, let T_i denote the travel time corresponding to the ith choice, $i \geqslant 1$. Again let X denote the time when the miner reaches safety.

(a) Give an identity that relates X to N and the T_i.
(b) What is $E[N]$?
(c) What is $E[T_N]$?
(d) What is $E[\sum_{i=1}^N T_i | N = n]$?
(e) Using the preceding, what is $E[X]$?

22. Suppose that independent trials, each of which is equally likely to have any of m possible outcomes, are performed until the same outcome occurs k consecutive times. If N denotes the number of trials, show that

$$E[N] = \frac{m^k - 1}{m - 1}$$

Some people believe that the successive digits in the expansion of $\pi = 3.14159\ldots$ are "uniformly" distributed. That is, they believe that these digits have all the appearance of being independent choices from a distribution that is equally likely to be any of the digits from 0 through 9. Possible evidence against this hypothesis is the fact that starting with the 24,658,601st digit there is a run of nine successive 7s. Is this information consistent with the hypothesis of a uniform distribution?

To answer this, we note from the preceding that if the uniform hypothesis were correct, then the expected number of digits until a run of nine of the same value occurs is

$$(10^9 - 1)/9 = 111{,}111{,}111$$

Thus, the actual value of approximately 25 million is roughly 22 percent of the theoretical mean. However, it can be shown that under the uniformity assumption the standard deviation of N will be approximately equal to the mean. As a result, the observed value is approximately 0.78 standard deviations less than its theoretical mean and is thus quite consistent with the uniformity assumption.

***23.** A coin having probability p of coming up heads is successively flipped until two of the most recent three flips are heads. Let N denote the number of flips. (Note that if the first two flips are heads, then $N = 2$.) Find $E[N]$.

24. A coin, having probability p of landing heads, is continually flipped until at least one head and one tail have been flipped.

(a) Find the expected number of flips needed.
(b) Find the expected number of flips that land on heads.
(c) Find the expected number of flips that land on tails.
(d) Repeat part (a) in the case where flipping is continued until a total of at least two heads and one tail have been flipped.

25. A gambler wins each game with probability p. In each of the following cases, determine the expected total number of wins.

38. Let U be a uniform (0, 1) random variable. Suppose that n trials are to be performed and that conditional on $U = u$ these trials will be independent with a common success probability u. Compute the mean and variance of the number of successes that occur in these trials.

39. A deck of n cards, numbered 1 through n, is randomly shuffled so that all $n!$ possible permutations are equally likely. The cards are then turned over one at a time until card number 1 appears. These upturned cards constitute the first cycle. We now determine (by looking at the upturned cards) the lowest numbered card that has not yet appeared, and we continue to turn the cards face up until that card appears. This new set of cards represents the second cycle. We again determine the lowest numbered of the remaining cards and turn the cards until it appears, and so on until all cards have been turned over. Let m_n denote the mean number of cycles.

(a) Derive a recursive formula for m_n in terms of m_k, $k = 1, \ldots, n-1$.
(b) Starting with $m_0 = 0$, use the recursion to find m_1, m_2, m_3, and m_4.
(c) Conjecture a general formula for m_n.
(d) Prove your formula by induction on n. That is, show it is valid for $n = 1$, then assume it is true for any of the values $1, \ldots, n-1$ and show that this implies it is true for n.
(e) Let X_i equal 1 if one of the cycles ends with card i, and let it equal 0 otherwise, $i = 1, \ldots, n$. Express the number of cycles in terms of these X_i.
(f) Use the representation in part (e) to determine m_n.
(g) Are the random variables $X_1, \ldots, X_n$ independent? Explain.
(h) Find the variance of the number of cycles.

40. A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom.

(a) Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, 0.2, what is the expected number of days until he reaches freedom?
(b) Assuming that the prisoner is always equally likely to choose among those doors that he has not used, what is the expected number of days until he reaches freedom? (In this version, for instance, if the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3.)
(c) For parts (a) and (b) find the variance of the number of days until the prisoner reaches freedom.

41. A rat is trapped in a maze. Initially it has to choose one of two directions. If it goes to the right, then it will wander around in the maze for three minutes and will then return to its initial position. If it goes to the left, then with probability $\frac{1}{3}$ it will depart the maze after two minutes of traveling, and with probability $\frac{2}{3}$ it will return to its initial position after five minutes of traveling. Assuming that the rat is at all times equally likely to go to the left or the right, what is the expected number of minutes that it will be trapped in the maze?

42. A total of 11 people, including you, are invited to a party. The times at which people arrive at the party are independent uniform (0, 1) random variables.

(a) Find the expected number of people who arrive before you.
(b) Find the variance of the number of people who arrive before you.

43. The number of claims received at an insurance company during a week is a random variable with mean μ_1 and variance σ_1^2. The amount paid in each claim is a random variable with mean μ_2 and variance σ_2^2. Find the mean and variance of the amount of money paid by the insurance company each week. What independence assumptions are you making? Are these assumptions reasonable?

44. The number of customers entering a store on a given day is Poisson distributed with mean $\lambda = 10$. The amount of money spent by a customer is uniformly distributed over (0, 100). Find the mean and variance of the amount of money that the store takes in on a given day.

45. An individual traveling on the real line is trying to reach the origin. However, the larger the desired step, the greater is the variance in the result of that step. Specifically, whenever the person is at location x, he next moves to a location having mean 0 and variance βx^2. Let X_n denote the position of the individual after having taken n steps. Supposing that $X_0 = x_0$, find

(a) $E[X_n]$
(b) $\text{Var}(X_n)$

46. (a) Show that

$$\text{Cov}(X, Y) = \text{Cov}(X, E[Y \mid X])$$

(b) Suppose, that, for constants a and b,

$$E[Y \mid X] = a + bX$$

Show that

$$b = \text{Cov}(X, Y)/\text{Var}(X)$$

***47.** If $E[Y \mid X] = 1$, show that

$$\text{Var}(X Y) \geqslant \text{Var}(X)$$

then what is the probability that the coin flipped on the third day after the initial flip is coin 1?

9. Suppose in Exercise 8 that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

10. In Example 4.3, Gary is currently in a cheerful mood. What is the probability that he is not in a glum mood on any of the following three days?

11. In Example 4.3, Gary was in a glum mood four days ago. Given that he hasn't felt cheerful in a week, what is the probability he is feeling glum today?

12. For a Markov chain $\{X_n, n \geq 0\}$ with transition probabilities $P_{i,j}$, consider the conditional probability that $X_n = m$ given that the chain started at time 0 in state i and has not yet entered state r by time n, where r is a specified state not equal to either i or m. We are interested in whether this conditional probability is equal to the n stage transition probability of a Markov chain whose state space does not include state r and whose transition probabilities are

$$Q_{i,j} = \frac{P_{i,j}}{1 - P_{i,r}}, \qquad i, j \neq r$$

Either prove the equality

$$P\{X_n = m | X_0 = i, X_k \neq r, k = 1, \ldots, n\} = Q^n_{i,m}$$

or construct a counterexample.

13. Let $\mathbf{P}$ be the transition probability matrix of a Markov chain. Argue that if for some positive integer r, $\mathbf{P}^r$ has all positive entries, then so does $\mathbf{P}^n$, for all integers $n \geq r$.

14. Specify the classes of the following Markov chains, and determine whether they are transient or recurrent:

$$\mathbf{P}_1 = \begin{Vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{Vmatrix} \qquad \mathbf{P}_2 = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

$$\mathbf{P}_3 = \begin{Vmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{Vmatrix} \qquad \mathbf{P}_4 = \begin{Vmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

15. Prove that if the number of states in a Markov chain is M, and if state j can be reached from state i, then it can be reached in M steps or less.

***16.** Show that if state i is recurrent and state i does not communicate with state j, then $P_{ij} = 0$. This implies that once a process enters a recurrent class of states it can never leave that class. For this reason, a recurrent class is often referred to as a *closed* class.

17. For the random walk of Example 4.15 use the strong law of large numbers to give another proof that the Markov chain is transient when $p \neq \frac{1}{2}$.

> **Hint:** Note that the state at time n can be written as $\sum_{i=1}^{n} Y_i$ where the Y_is are independent and $P\{Y_i = 1\} = p = 1 - P\{Y_i = -1\}$. Argue that if $p > \frac{1}{2}$, then, by the strong law of large numbers, $\sum_1^n Y_i \to \infty$ as $n \to \infty$ and hence the initial state 0 can be visited only finitely often, and hence must be transient. A similar argument holds when $p < \frac{1}{2}$.

18. Coin 1 comes up heads with probability 0.6 and coin 2 with probability 0.5. A coin is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other one.

(a) What proportion of flips use coin 1?
(b) If we start the process with coin 1 what is the probability that coin 2 is used on the fifth flip?

19. For Example 4.4, calculate the proportion of days that it rains.

20. A transition probability matrix $\mathbf{P}$ is said to be doubly stochastic if the sum over each column equals one; that is,

$$\sum_i P_{ij} = 1, \qquad \text{for all } j$$

If such a chain is irreducible and aperiodic and consists of $M + 1$ states $0, 1, \ldots, M$, show that the limiting probabilities are given by

$$\pi_j = \frac{1}{M+1}, \qquad j = 0, 1, \ldots, M$$

***21.** A particle moves on a circle through points which have been marked 0, 1, 2, 3, 4 (in a clockwise order). At each step it has a probability p of moving to the right (clockwise) and $1 - p$ to the left (counterclockwise). Let X_n denote its location on the circle after the nth step. The process $\{X_n, n \geq 0\}$ is a Markov chain.

(a) Find the transition probability matrix.
(b) Calculate the limiting probabilities.

22. Let Y_n be the sum of n independent rolls of a fair die. Find

$$\lim_{n\to\infty} P\{Y_n \text{ is a multiple of } 13\}$$

Hint: Define an appropriate Markov chain and apply the results of Exercise 20.

23. Trials are performed in sequence. If the last two trials were successes, then the next trial is a success with probability 0.8; otherwise the next trial is a success with probability 0.5. In the long run, what proportion of trials are successes?

24. Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls, and 2 blue balls; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage, a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to that urn. In the long run, what proportion of the selected balls are red? What proportion are white? What proportion are blue?

25. Each morning an individual leaves his house and goes for a run. He is equally likely to leave either from his front or back door. Upon leaving the house, he chooses a pair of running shoes (or goes running barefoot if there are no shoes at the door from which he departed). On his return he is equally likely to enter, and leave his running shoes, either by the front or back door. If he owns a total of k pairs of running shoes, what proportion of the time does he run barefooted?

26. Consider the following approach to shuffling a deck of n cards. Starting with any initial ordering of the cards, one of the numbers $1, 2, \ldots, n$ is randomly chosen in such a manner that each one is equally likely to be selected. If number i is chosen, then we take the card that is in position i and put it on top of the deck—that is, we put that card in position 1. We then repeatedly perform the same operation. Show that, in the limit, the deck is perfectly shuffled in the sense that the resultant ordering is equally likely to be any of the $n!$ possible orderings.

***27.** Determine the limiting probabilities π_j for the model presented in Exercise 1. Give an intuitive explanation of your answer.

28. For a series of dependent trials the probability of success on any trial is $(k+1)/(k+2)$ where k is equal to the number of successes on the previous two trials. Compute $\lim_{n\to\infty} P\{\text{success on the } n\text{th trial}\}$.

29. An organization has N employees where N is a large number. Each employee has one of three possible job classifications and changes classifications (independently) according to a Markov chain with transition probabilities

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

What percentage of employees are in each classification?

30. Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

31. A certain town never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same the next day, and if it changes then it is equally likely to be either of the other two possibilities. In the long run, what proportion of days are sunny? What proportion are cloudy?

***32.** Each of two switches is either on or off during a day. On day n, each switch will independently be on with probability

$$[1 + \text{number of on switches during day } n-1]/4$$

For instance, if both switches are on during day $n-1$, then each will independently be on during day n with probability 3/4. What fraction of days are both switches on? What fraction are both off?

33. A professor continually gives exams to her students. She can give three possible types of exams, and her class is graded as either having done well or badly. Let p_i denote the probability that the class does well on a type i exam, and suppose that $p_1 = 0.3$, $p_2 = 0.6$, and $p_3 = 0.9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. What proportion of exams are type $i, i = 1, 2, 3$?

34. A flea moves around the vertices of a triangle in the following manner: Whenever it is at vertex i it moves to its clockwise neighbor vertex with probability p_i and to the counterclockwise neighbor with probability $q_i = 1 - p_i$, $i = 1, 2, 3$.

(a) Find the proportion of time that the flea is at each of the vertices.
(b) How often does the flea make a counterclockwise move which is then followed by five consecutive clockwise moves?

35. Consider a Markov chain with states 0, 1, 2, 3, 4. Suppose $P_{0,4} = 1$; and suppose that when the chain is in state i, $i > 0$, the next state is equally likely to be any of the states $0, 1, \ldots, i-1$. Find the limiting probabilities of this Markov chain.

36. The state of a process changes daily according to a two-state Markov chain. If the process is in state i during one day, then it is in state j the following day with probability $P_{i,j}$, where

$$P_{0,0} = .4 \quad P_{0,1} = .6 \quad P_{1,0} = .2 \quad P_{1,1} = .8$$

Every day a message is sent. If the state of the Markov chain that day is i then the message sent is "good" with probability p_i and is "bad" with probability $q_i = 1 - p_i$, $i = 0, 1$.

(a) If the process is in state 0 on Monday, what is the probability that a good message is sent on Tuesday?
(b) If the process is in state 0 on Monday, what is the probability that a good message is sent on Friday?
(c) In the long run, what proportion of messages are good?
(d) Let Y_n equal 1 if a good message is sent on day n and let it equal 2 otherwise. Is $\{Y_n, n \geqslant 1\}$ a Markov chain? If so, give its transition probability matrix. If not, briefly explain why not.

37. Show that the stationary probabilities for the Markov chain having transition probabilities $P_{i,j}$ are also the stationary probabilities for the Markov chain whose transition probabilities $Q_{i,j}$ are given by

$$Q_{i,j} = P_{i,j}^k$$

for some specified positive integer k.

38. Recall that state i is said to be positive recurrent if $m_{i,i} < \infty$, where $m_{i,i}$ is the expected number of transitions until the Markov chain, starting in state i, makes a transition back into that state. Because π_i, the long run proportion of time the Markov chain, starting in state i, spends in state i, satisfies

$$\pi_i = \frac{1}{m_{i,i}}$$

it follows that state i is positive recurrent if and only if $\pi_i > 0$. Suppose that state i is positive recurrent and that state i communicates with state j. Show that state j is also positive recurrent by arguing that there is an integer n such that

$$\pi_j \geqslant \pi_i P_{i,j}^n > 0$$

39. Recall that a recurrent state that is not positive recurrent is called null recurrent. Use the result of Exercise 38 to prove that null recurrence is a class property. That is, if state i is null recurrent and state i communicates with state j, show that state j is also null recurrent.

40. It follows from the argument made in Exercise 38 that state i is null recurrent if it is recurrent and $\pi_i = 0$. Consider the one-dimensional symmetric random walk of Example 4.15.

(a) Argue that $\pi_i = \pi_0$ for all i.
(b) Argue that all states are null recurrent.

***41.** Let π_i denote the long-run proportion of time a given Markov chain is in state i.

(a) Explain why π_i is also the proportion of transitions that are into state i as well as being the proportion of transitions that are from state i.
(b) $\pi_i P_{ij}$ represents the proportion of transitions that satisfy what property?
(c) $\sum_i \pi_i P_{ij}$ represent the proportion of transitions that satisfy what property?
(d) Using the preceding explain why

$$\pi_j = \sum_i \pi_i P_{ij}$$

42. Let A be a set of states, and let A^c be the remaining states.

(a) What is the interpretation of

$$\sum_{i \in A} \sum_{j \in A^c} \pi_i P_{ij}?$$

(b) What is the interpretation of

$$\sum_{i \in A^c} \sum_{j \in A} \pi_i P_{ij}?$$

(c) Explain the identity

$$\sum_{i \in A} \sum_{j \in A^c} \pi_i P_{ij} = \sum_{i \in A^c} \sum_{j \in A} \pi_i P_{ij}$$

43. Each day, one of n possible elements is requested, the ith one with probability P_i, $i \geqslant 1$, $\sum_1^n P_i = 1$. These elements are at all times arranged in an ordered list which is revised as follows: The element selected is moved to the front of the list with the relative positions of all the other elements remaining unchanged. Define the state at any time to be the list ordering at that time and note that there are $n!$ possible states.

(a) Argue that the preceding is a Markov chain.
(b) For any state $i_1, \ldots, i_n$ (which is a permutation of $1, 2, \ldots, n$), let $\pi(i_1, \ldots, i_n)$ denote the limiting probability. In order for the state to be $i_1, \ldots, i_n$, it is necessary for the last request to be for i_1, the last non-i_1 request for i_2, the last non-i_1 or i_2 request for i_3, and so on. Hence, it appears intuitive that

$$\pi(i_1, \ldots, i_n) = P_{i_1} \frac{P_{i_2}}{1 - P_{i_1}} \frac{P_{i_3}}{1 - P_{i_1} - P_{i_2}} \cdots \frac{P_{i_{n-1}}}{1 - P_{i_1} - \cdots - P_{i_{n-2}}}$$

Verify when $n = 3$ that the preceding are indeed the limiting probabilities.

44. Suppose that a population consists of a fixed number, say, m, of genes in any generation. Each gene is one of two possible genetic types. If any generation has exactly i (of its m) genes being type 1, then the next generation will have j type 1 (and $m-j$ type 2) genes with probability

$$\binom{m}{j}\left(\frac{i}{m}\right)^j\left(\frac{m-i}{m}\right)^{m-j}, \qquad j=0,1,\ldots,m$$

Let X_n denote the number of type 1 genes in the nth generation, and assume that $X_0 = i$.

(a) Find $E[X_n]$.
(b) What is the probability that eventually all the genes will be type 1?

45. Consider an irreducible finite Markov chain with states $0, 1, \ldots, N$.

(a) Starting in state i, what is the probability the process will ever visit state j? Explain!
(b) Let $x_i = P\{\text{visit state } N \text{ before state } 0 | \text{start in } i\}$. Compute a set of linear equations which the x_i satisfy, $i = 0, 1, \ldots, N$.
(c) If $\sum_j jP_{ij} = i$ for $i = 1, \ldots, N-1$, show that $x_i = i/N$ is a solution to the equations in part (b).

46. An individual possesses r umbrellas which he employs in going from his home to office, and vice versa. If he is at home (the office) at the beginning (end) of a day and it is raining, then he will take an umbrella with him to the office (home), provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains at the beginning (end) of a day with probability p.

(i) Define a Markov chain with $r+1$ states which will help us to determine the proportion of time that our man gets wet. (*Note*: He gets wet if it is raining, and all umbrellas are at his other location.)
(ii) Show that the limiting probabilities are given by

$$\pi_i = \begin{cases} \dfrac{q}{r+q}, & \text{if } i = 0 \\[2ex] \dfrac{1}{r+q}, & \text{if } i = 1, \ldots, r \end{cases} \qquad \text{where } q = 1-p$$

(iii) What fraction of time does our man get wet?
(iv) When $r = 3$, what value of p maximizes the fraction of time he gets wet?

***47.** Let $\{X_n, n \geq 0\}$ denote an ergodic Markov chain with limiting probabilities π_i. Define the process $\{Y_n, n \geq 1\}$ by $Y_n = (X_{n-1}, X_n)$. That is, Y_n keeps

track of the last two states of the original chain. Is $\{Y_n, n \geq 1\}$ a Markov chain? If so, determine its transition probabilities and find

$$\lim_{n\to\infty} P\{Y_n = (i, j)\}$$

48. Verify the transition probability matrix given in Example 4.20.

49. Let $P^{(1)}$ and $P^{(2)}$ denote transition probability matrices for ergodic Markov chains having the same state space. Let π^1 and π^2 denote the stationary (limiting) probability vectors for the two chains. Consider a process defined as follows:

(i) $X_0 = 1$. A coin is then flipped and if it comes up heads, then the remaining states $X_1, \ldots$ are obtained from the transition probability matrix $P^{(1)}$ and if tails from the matrix $P^{(2)}$. Is $\{X_n, n \geq 0\}$ a Markov chain? If $p = P\{\text{coin comes up heads}\}$, what is $\lim_{n\to\infty} P(X_n = i)$?
(ii) $X_0 = 1$. At each stage the coin is flipped and if it comes up heads, then the next state is chosen according to $P^{(1)}$ and if tails comes up, then it is chosen according to $P^{(2)}$. In this case do the successive states constitute a Markov chain? If so, determine the transition probabilities. Show by a counterexample that the limiting probabilities are not the same as in part (i).

50. In Exercise 8, if today's flip lands heads, what is the expected number of additional flips needed until the pattern t, t, h, t, h, t, t occurs?

51. In Example 4.3, Gary is in a cheerful mood today. Find the expected number of days until he has been glum for three consecutive days.

52. A taxi driver provides service in two zones of a city. Fares picked up in zone A will have destinations in zone A with probability .6 or in zone B with probability .4. Fares picked up in zone B will have destinations in zone A with probability .3 or in zone B with probability .7. The driver's expected profit for a trip entirely in zone A is 6; for a trip entirely in zone B is 8; and for a trip that involves both zones is 12. Find the taxi driver's average profit per trip.

53. Find the average premium received per policyholder of the insurance company of Example 4.23 if $\lambda = 1/4$ for one-third of its clients, and $\lambda = 1/2$ for two-thirds of its clients.

54. Consider the Ehrenfest urn model in which M molecules are distributed among two urns, and at each time point one of the molecules is chosen at random and is then removed from its urn and placed in the other one. Let X_n denote the number of molecules in urn 1 after the nth switch and let $\mu_n = E[X_n]$.