EE 531

Suggested Problems for MT2 Preparation

Papoulis:	**Therrien:**
10.2	4.1
10.4	4.2
10.8	4.5
10.13	4.6
10.22	4.11
10.26a	5.4
10.54	5.11
	5.25a

Also, make sure to examine the details of the proofs presented during the lectures.

[Papoulis] : A. Papoulis, Probability, Random Variables, and Stochastic Processes, 3rd edition, McGraw Hill, 1991.

[Therrien] : Therrien, Charles W. , Discrete random signals and statistical signal processing, Prentice Hall, c1992

Given an RV $\mathbf{x}$ and a transformation T as above, we form the random process

$$\mathbf{x}_0 = \mathbf{x} \qquad \mathbf{x}_n = T^n\mathbf{x} \qquad n = -\infty, \ldots, \infty \tag{10B-5}$$

It follows from (10B-4) that the random variables $\mathbf{x}_n$ so formed have the same distribution. We can similarly show that their joint distributions of any order are invariant to a shift of the origin. Hence the process $\mathbf{x}_n$ so formed is SSS.

It can be shown that the converse is also true: Given an SSS process $\mathbf{x}_n$, we can find an RV $\mathbf{x}$ and a one-to-one measuring preserving transformation of the space $\mathscr{S}$ into itself such that for all essential purposes, $\mathbf{x}_n = T^n\mathbf{x}$. The proof of this difficult result will not be given.

PROBLEMS

10-1. In the fair-coin experiment, we define the process $\mathbf{x}(t)$ as follows: $\mathbf{x}(t) = \sin \pi t$ if heads shows, $\mathbf{x}(t) = 2t$ if tails shows. (*a*) Find $E\{\mathbf{x}(t)\}$. (*b*) Find $F(x, t)$ for $t = 0.25$, $t = 0.5$, and $t = 1$.

10-2. The process $\mathbf{x}(t) = e^{\mathbf{a}t}$ is a family of exponentials depending on the RV $\mathbf{a}$. Express the mean $\eta(t)$, the autocorrelation $R(t_1, t_2)$, and the first-order density $f(x, t)$ of $\mathbf{x}(t)$ in terms of the density $f_a(a)$ of $\mathbf{a}$.

10-3. Suppose that $\mathbf{x}(t)$ is a Poisson process as in Fig. 10-3 such that $E\{\mathbf{x}(9)\} = 6$. (*a*) Find the mean and the variance of $\mathbf{x}(8)$. (*b*) Find $P\{\mathbf{x}(2) \le 3\}$. (*c*) Find $P\{\mathbf{x}(4) \le 5 \mid \mathbf{x}(2) \le 3\}$.

10-4. The RV $\mathbf{c}$ is uniform in the interval $(0, T)$. Find $R_x(t_1, t_2)$ if (*a*) $\mathbf{x}(t) = U(t - \mathbf{c})$, (*b*) $\mathbf{x}(t) = \delta(t - \mathbf{c})$.

10-5. The RVs $\mathbf{a}$ and $\mathbf{b}$ are independent $N(0; \sigma)$ and p is the probability that the process $\mathbf{x}(t) = \mathbf{a} - \mathbf{b}t$ crosses the t axis in the interval $(0, T)$. Show that $\pi p = \arctan T$.

Hint: $p = P\{0 \le \mathbf{a}/\mathbf{b} \le T\}$.

10-6. Show that if

$$R_v(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$$

$\mathbf{w}''(t) = \mathbf{v}(t)U(t)$ and $\mathbf{w}(0) = \mathbf{w}'(0) = 0$, then

$$E\{\mathbf{w}^2(t)\} = \int_0^t (t - \tau)q(\tau)\,d\tau$$

10-7. The process $\mathbf{x}(t)$ is real with autocorrelation $R(\tau)$. (*a*) Show that

$$P\{|\mathbf{x}(t + \tau) - \mathbf{x}(t)| \ge a\} \le 2[R(0) - R(\tau)]/a^2$$

(*b*) Express $P\{|\mathbf{x}(t + \tau) - \mathbf{x}(t)| \ge a\}$ in terms of the second-order density $f(x_1, x_2; \tau)$ of $\mathbf{x}(t)$.

10-8. The process $\mathbf{x}(t)$ is WSS and normal with $E\{\mathbf{x}(t)\} = 0$ and $R(\tau) = 4e^{-2|\tau|}$. (*a*) Find $P\{\mathbf{x}(t) \le 3\}$. (*b*) Find $E\{[\mathbf{x}(t + 1) - \mathbf{x}(t - 1)]^2\}$.

10-9. Show that the process $\mathbf{x}(t) = \mathbf{c}w(t)$ is WSS iff $E\{\mathbf{c}\} = 0$ and $w(t) = e^{j(\omega t + \theta)}$.

10-10. The process $\mathbf{x}(t)$ is WSS and $E\{\mathbf{x}(t)\} = 0$. Show that if $\mathbf{z}(t) = \mathbf{x}^2(t)$, then $C_{zz}(\tau) = 2C_{xx}^2(\tau)$.

10-11. Find $E\{\mathbf{y}(t)\}$, $E\{\mathbf{y}^2(t)\}$, and $R_{yy}(\tau)$ if

$$\mathbf{y}''(t) + 4\mathbf{y}(t) + 13\mathbf{y}(t) = 26 + \boldsymbol{\nu}(t) \qquad R_{\nu\nu}(\tau) = 10\delta(\tau)$$

Find $P\{\mathbf{y}(t) \le 3\}$ if $\boldsymbol{\nu}(t)$ is normal.

10-12. Show that: If $\mathbf{x}(t)$ is a process with zero mean and autocorrelation $f(t_1)f(t_2)w(t_1 - t_2)$, then the process $\mathbf{y}(t) = \mathbf{x}(t)/f(t)$ is WSS with autocorrelation $w(\tau)$. If $\mathbf{x}(t)$ is white noise with autocorrelation $q(t_1)\delta(t_1 - t_2)$, then the process $\mathbf{z}(t) = \mathbf{x}(t)/\sqrt{q(t)}$ is WSS white noise with autocorrelation $\delta(\tau)$.

10-13. Show that $|R_{xy}(\tau)| \le \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$.

10-14. Show that if the processes $\mathbf{x}(t), \mathbf{y}(t)$ are WSS and $E\{|\mathbf{x}(0) - \mathbf{y}(0)|^2\} = 0$, then $R_{xx}(\tau) \equiv R_{xy}(\tau) \equiv R_{yy}(\tau)$.

Hint: Set $\mathbf{z} = \mathbf{x}(t + \tau)$, $\mathbf{w} = \mathbf{x}^*(t)) - \mathbf{y}^*(t)$ in (10-163).

10-15. Show that if $\mathbf{x}(t)$ is a complex WSS process, then

$$E\{|\mathbf{x}(t + \tau) - \mathbf{x}(t)|^2\} = 2\,\mathrm{Re}[R(0) - R(\tau)]$$

10-16. Show that if $\boldsymbol{\varphi}$ is an RV with $\Phi(\lambda) = E\{e^{j\lambda\boldsymbol{\varphi}}\}$ and $\Phi(1) = \Phi(2) = 0$, then the process $\mathbf{x}(t) = \cos(\omega t + \boldsymbol{\varphi})$ is WSS. Find $E\{\mathbf{x}(t)\}$ and $R_x(\tau)$ if $\boldsymbol{\varphi}$ is uniform in the interval $(-\pi, \pi)$.

10-17. Given a process $\mathbf{x}(t)$ with orthogonal increments and such that $\mathbf{x}(0) = 0$, show that (*a*) $R(t_1, t_2) = R(t_1, t_1)$ for $t_1 \le t_2$, and (*b*) if $E\{[\mathbf{x}(t_1) - \mathbf{x}(t_2)]^2\} = q|t_1 - t_2|$ then the process $\mathbf{y}(t) = [\mathbf{x}(t + \varepsilon) - \mathbf{x}(t)]/\varepsilon$ is WSS and its autocorrelation is a triangle with area q and base 2ε.

10-18. Show that if $R_{xx}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$ and $\mathbf{y}(t) = \mathbf{x}(t) * h(t)$ then

$$E\{\mathbf{x}(t)\mathbf{y}(t)\} = h(0)q(t)$$

10-19. The process $\mathbf{x}(t)$ is normal with $\eta_x = 0$ and $R_x(\tau) = 4e^{-3|\tau|}$. Find a memoryless system $g(x)$ such that the first-order density $f_y(y)$ of the resulting output $\mathbf{y}(t) = g[\mathbf{x}(t)]$ is uniform in the interval $(6, 9)$.

Answer: $g(x) = 3\mathbb{G}(x/2) + 6$.

10-20. Show that if $\mathbf{x}(t)$ is an SSS process and $\boldsymbol{\varepsilon}$ is an RV independent of $\mathbf{x}(t)$, then the process $\mathbf{y}(t) = \mathbf{x}(t - \boldsymbol{\varepsilon})$ is SSS.

10-21. Show that if $\mathbf{x}(t)$ is a stationary process with derivative $\mathbf{x}'(t)$, then for a given t the RVs $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are orthogonal and uncorrelated.

10-22. Given a normal process $\mathbf{x}(t)$ with $\eta_x = 0$ and $R_x(\tau) = 4e^{-2|\tau|}$, we form the RVs $\mathbf{z} = \mathbf{x}(t + 1)$, $\mathbf{w} = \mathbf{x}(t - 1)$, (*a*) find $E\{\mathbf{zw}\}$ and $E\{(\mathbf{z} + \mathbf{w})^2\}$, (*b*) find

$$f_z(z) \qquad P\{\mathbf{z} < 1\} \qquad f_{zw}(z, w)$$

10-23. Show that if $\mathbf{x}(t)$ is normal with autocorrelation $R(\tau)$, then

$$P\{\mathbf{x}'(t) \le a\} = \mathbb{G}\left[\frac{a}{\sqrt{-R''(0)}}\right]$$

10-24. Show that if $\mathbf{x}(t)$ is a normal process with zero mean and $\mathbf{y}(t) = \operatorname{sgn}\mathbf{x}(t)$, then

$$R_y(\tau) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[J_0(n\pi) - (-1)^n\right]\sin\left[n\pi\frac{R_x(\tau)}{R_x(0)}\right]$$

where $J_0(x)$ is the Bessel function.

Hint: Expand the arcsine in (10-71) into a Fourier series.

10-25. Show that if $\mathbf{x}(t)$ is a normal process with zero mean and $\mathbf{y}(t) = Ie^{a\mathbf{x}(t)}$, then

$$\eta_y = = I\exp\left\{\frac{a^2}{2}R_x(0)\right\} \qquad R_y(\tau) = I^2\exp\{a^2[R_x(0) + R_x(\tau)]\}$$

10-26. Show that (*a*) if

$$\mathbf{y}(t) = a\mathbf{x}(ct) \qquad \text{then} \quad R_y(\tau) = a^2R_x(c\tau)$$

(*b*) if $R_x(\tau) \to 0$ as $\tau \to \infty$ and

$$\mathbf{z}(t) = \lim_{\varepsilon\to\infty}\sqrt{\varepsilon}\,\mathbf{x}(\varepsilon t) \qquad \text{then} \quad R_z(\tau) = q\delta(\tau) \qquad q = \int_{-\infty}^{\infty}R_x(\tau)\,d\tau$$

10-27. Show that if $\mathbf{x}(t)$ is white noise, $h(t) = 0$ outside the interval $(0, T)$, and $\mathbf{y}(t) = \mathbf{x}(t) * h(t)$ then $R_{yy}(t_1, t_2) = 0$ for $|t_1 - t_2| > T$.

10-28. Show that if

$$R_{xx}(t_1, t_2) = q(t_1)\delta(t_1 - t_2) \qquad E\{\mathbf{y}^2(t)\} = I(t)$$

and

$$(a) \qquad \mathbf{y}(t) = \int_0^t h(t,\alpha)\mathbf{x}(\alpha)\,d\alpha \qquad \text{then} \quad I(t) = \int_0^t h^2(t,\alpha)q(\alpha)\,d\alpha$$

$$(b) \qquad \mathbf{y}'(t) + c(t)\mathbf{y}(t) = \mathbf{x}(t) \qquad \text{then} \quad I'(t) + 2c(t)I(t) = q(t)$$

10-29. Find $E\{\mathbf{y}^2(t)\}$ (*a*) if $R_{xx}(t) = 5\delta(\tau)$ and

$$\mathbf{y}'(t) + 2\mathbf{y}(t) = \mathbf{x}(t) \qquad \text{all} \quad t \tag{i}$$

(*b*) if (i) holds for $t > 0$ only and $\mathbf{y}(t) = 0$ for $t \le 0$.

Hint: Use (10-90).

10-30. The input to a linear system with $h(t) = Ae^{-\alpha t}U(t)$ is a process $\mathbf{x}(t)$ with $R_x(\tau) = N\delta(\tau)$ applied at $t = 0$ and disconnected at $t = T$. Find and sketch $E\{\mathbf{y}^2(t)\}$.

Hint: Use (10-90) with $q(t) = N$ for $0 < t < T$ and 0 otherwise.

10-31. Show that if

$$\mathbf{s} = \int_0^{10}\mathbf{x}(t)\,dt \qquad \text{then} \quad E\{\mathbf{s}^2\} = \int_{-10}^{10}(10 - |\tau|)R_x(\tau)\,d\tau$$

Find the mean and variance of $\mathbf{s}$ if $E\{\mathbf{x}(t)\} = 8$, $R_x(\tau) = 64 + 10e^{-2|\tau|}$.

10-32. The process $\mathbf{x}(t)$ is WSS with $R_{xx}(\tau) = 5\delta(\tau)$ and

$$\mathbf{y}'(t) + 2\mathbf{y}(t) = \mathbf{x}(t) \tag{i}$$

Find $E\{\mathbf{y}^2(t)\}$, $R_{xy}(t_1, t_2)$, $R_{yy}(t_1, t_2)$ (*a*) if (i) holds for all t, (*b*) if $\mathbf{y}(0) = 0$ and (i) holds for $t \ge 0$.

10-33. Find $S(\omega)$ if (*a*) $R(\tau) = e^{-\alpha\tau^2}$, (*b*) $R(\tau) = e^{-\alpha\tau^2}\cos\omega_0\tau$.

10-34. Show that the power spectrum of an SSS process $\mathbf{x}(t)$ equals

$$S(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1x_2G(x_1, x_2; \omega)\,dx_1\,dx_2$$

where $G(x_1, x_2; \omega)$ is the Fourier transform in the variable τ of the second-order density $f(x_1, x_2; \tau)$ of $\mathbf{x}(t)$.

10-35. Show that if $\mathbf{y}(t) = \mathbf{x}(t+a) - \mathbf{x}(t-a)$, then

$$R_y(\tau) = 2R_x(\tau) - R_x(\tau + 2a) - R_x(\tau - 2a) \qquad S_y(\omega) = 4S_x(\omega)\sin^2 a\omega$$

10-36. Using (10-122), show that

$$R(0) - R(\tau) \ge \frac{1}{4^n}[R(0) - R(2^n\tau)]$$

Hint:

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2} \ge 2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac{1}{4}(1 - \cos 2\theta)$$

10-37. The process $\mathbf{x}(t)$ is normal with zero mean and $R_x(\tau) = Ie^{-\alpha|\tau|}\cos\beta\tau$. Show that if $\mathbf{y}(t) = \mathbf{x}^2(t)$, then $C_y(\tau) = I^2e^{-2\alpha|\tau|}(1 + \cos 2\beta\tau)$. Find $S_y(\omega)$.

10-38. Show that if $R(\tau)$ is the inverse Fourier transform of a function $S(\omega)$ and $S(\omega) \ge 0$, then, for any a_i,

$$\sum_{i,k} a_i a_k^* R(\tau_i - \tau_k) \ge 0$$

Hint:

$$\int_{-\infty}^{\infty} S(\omega)\left|\sum_i a_i e^{j\omega\tau_i}\right|^2 d\omega \ge 0$$

10-39. Find $R(\tau)$ if (*a*) $S(\omega) = 1/(1+\omega^4)$, (*b*) $S(\omega) = 1/(4+\omega^2)^2$.

10-40. Show that, for complex systems, (10-136) and (10-181) yield

$$\mathbf{S}_{yy}(s) = \mathbf{S}_{xx}(s)\mathbf{H}(s)\mathbf{H}^*(-s^*) \qquad \mathbf{S}_{yy}(z) = \mathbf{S}_{xx}(z)\mathbf{H}(z)\mathbf{H}^*(1/z^*)$$

10-41. The process $\mathbf{x}(t)$ is normal with zero mean. Show that if $\mathbf{y}(t) = \mathbf{x}^2(t)$, then

$$S_y(\omega) = 2\pi R_x^2(0)\delta(\omega) + 2S_x(\omega) * S_x(\omega)$$

Plot $S_y(\omega)$ if $S_x(\omega)$ is (*a*) ideal LP, (*b*) ideal BP.

10-42. The process $\mathbf{x}(t)$ is WSS with $E\{\mathbf{x}(t)\} = 5$ and $R_{xx}(\tau) = 25 + 4e^{-2|\tau|}$. If $\mathbf{y}(t) = 2\mathbf{x}(t) + 3\mathbf{x}'(t)$, find η_y, $R_{yy}(\tau)$, and $S_{yy}(\omega)$.

10-43. The process $\mathbf{x}(t)$ is WSS and $R_{xx}(\tau) = 5\delta(\tau)$. (*a*) Find $E\{\mathbf{y}^2(t)\}$ and $S_{yy}(\omega)$ if $\mathbf{y}'(t) + 3\mathbf{y}(t) = \mathbf{x}(t)$. (*b*) Find $E\{\mathbf{y}^2(t)\}$ and $R_{xy}(t_1, t_2)$ if $\mathbf{y}'(t) + 3\mathbf{y}(t) = \mathbf{x}(t)U(t)$. Sketch the functions $R_{xy}(2, t_1)$ and $R_{xy}(t_1, 3)$.

10-44. Given a complex process $\mathbf{x}(t)$ with autocorrelation $R(\tau)$, show that if $|R(\tau_1)| = 1$, then

$$R(\tau) = e^{j\omega\tau}w(\tau) \qquad \mathbf{x}(t) = e^{j\omega t}\mathbf{y}(t)$$

where $w(\tau)$ is a periodic function with period τ_1 and $\mathbf{y}(t)$ is an MS periodic process with the same period.

10-45. Show that (*a*) $E\{\mathbf{x}(t)\check{\mathbf{x}}(t)\} = 0$, (*b*) $\check{\check{\mathbf{x}}}(t) = -\mathbf{x}(t)$.

10-46. (*Stochastic resonance*) The input to the system

$$\mathbf{H}(s) = \frac{1}{s^2 + 2s + 5}$$

is a WSS process $\mathbf{x}(t)$ with $E\{\mathbf{x}^2(t)\} = 10$. Find $S_x(\omega)$ such that the average power $E\{\mathbf{y}^2(t)\}$ of the resulting output $\mathbf{y}(t)$ is maximum.

Hint: $|\mathbf{H}(j\omega)|$ is maximum for $\omega = \sqrt{3}$.

10-47. Show that if $R_x(\tau) = Ae^{j\omega_0\tau}$, then $R_{xy}(\tau) = Be^{j\omega_0\tau}$ for any $\mathbf{y}(t)$.

Hint: Use (10-167).

10-48. Given a system $H(\omega)$ with input $\mathbf{x}(t)$ and output $\mathbf{y}(t)$, show that (*a*) if $\mathbf{x}(t)$ is WSS and $R_{xx}(\tau) = e^{j\alpha\tau}$, then

$$R_{yx}(\tau) = e^{j\alpha\tau}H(\alpha) \qquad R_{yy}(\tau) = e^{j\alpha\tau}|H(\alpha)|^2$$

(*b*) if $R_{xx}(t_1, t_2) = e^{j(\alpha t_1 - \beta t_2)}$, then

$$R_{yx}(t_1, t_2) = e^{j(\alpha t_1 - \beta t_2)}H(\alpha) \qquad R_{yy}(t_1, t_2) = e^{j(\alpha t_1 - \beta t_2)}H(\alpha)H^*(\beta)$$

10-49. Show that if $S_{xx}(\omega)S_{yy}(\omega) \equiv 0$, then $S_{xy}(\omega) \equiv 0$.

10-50. Show that if $\mathbf{x}[n]$ is WSS and $R_x[1] = R_x[0]$, then $R_x[m] = R_x[0]$ for every m.

10-51. Show that if $R[m] = E\{\mathbf{x}[n+m]\mathbf{x}[n]\}$, then

$$R[0]R[2] > 2R^2[1] - R^2[0]$$

10-52. Given an RV $\boldsymbol{\omega}$ with density $f(\omega)$ such that $f(\omega) = 0$ for $|\omega| > \pi$, we form the process $\mathbf{x}[n] = Ae^{jn\boldsymbol{\omega}}\pi$. Show that $S_x(\omega) = 2\pi A^2 f(\omega)$ for $|\omega| < \pi$.

10-53. (*a*) Find $E\{\mathbf{y}^2(t)\}$ if $\mathbf{y}(0) = \mathbf{y}'(0) = 0$ and

$$\mathbf{y}''(t) + 7\mathbf{y}'(t) + 10\mathbf{y}(t) = \mathbf{x}(t) \qquad R_x(\tau) = 5\delta(\tau)$$

(*b*) Find $E\{\mathbf{y}^2[n]\}$ if $\mathbf{y}[-1] = \mathbf{y}[-2] = 0$ and

$$8\mathbf{y}[n] - 6\mathbf{y}[n-1] + \mathbf{y}[n-2] = \mathbf{x}[n] \qquad R_x[m] = 5\delta[m]$$

10-54. The process $\mathbf{x}[n]$ is WSS with $R_{xx}[m] = 5\delta[m]$ and

$$\mathbf{y}[n] - 0.5\mathbf{y}[n-1] = \mathbf{x}[n] \qquad \text{(i)}$$

Find $E\{\mathbf{y}^2[n]\}$, $R_{xy}[m_1, m_2]$, $R_{yy}[m_1, m_2]$ (*a*) if (i) holds for all n, (*b*) if $\mathbf{y}[-1] = 0$ and (i) holds for $n \geq 0$.

10-55. Show that (*a*) if $R_x[m_1, m_2] = q[m_1]\delta[m_1 - m_2]$ and

$$\mathbf{s} = \sum_{n=0}^{N} a_n \mathbf{x}[n] \qquad \text{then} \quad E\{\mathbf{s}^2\} = \sum_{n=0}^{N} a_n^2 q[n]$$

(*b*) If $R_{xx}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$ and

$$\mathbf{s} = \int_0^T a(t)\mathbf{x}(t)\,dt \qquad \text{then} \quad E\{\mathbf{s}^2\} = \int_0^T a^2(t)q(t)\,dt$$

14. Henry Stark and John W. Woods. *Probability, Random Processes, and Estimation Theory for Engineers*. Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1986.
15. Harold J. Larson and Bruno O. Schubert. *Probabilistic Models in Engineering Sciences*. Volume II. John Wiley & Sons, New York, 1979.
16. Kenneth S. Miller. *Complex Stochastic Processes*. Addison-Wesley, Reading, Massachusetts, 1974.
17. Jerry M. Mendel. Tutorial on higher-order statistics (spectra) in signal processing and system theory: theoretical results and some applications. *Proceedings of the IEEE*, 79(3):278–305, March 1991.
18. Jerry M. Mendel. Use of higher order statistics in signal processing and system theory: a short perspective. In *Proceedings of 8th International Conference on Mathematical Theory of Networks and Systems*, Phoenix, Arizona, June 1987.
19. Jerry M. Mendel. Use of higher order statistics in signal processing and system theory: an update. In *Proceedings of SPIE Conference on Advanced Algorithms and Architectures for Signal Processing*, San Diego, California, August 1988.
20. Chrysostomos L. Nikias. Higher-order spectral analysis. In Simon Haykin, editor. *Advances in Spectrum Analysis and Array Processing*, pages 326–365. Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1992.
21. Chrysostomos L. Nikias and Mysore R. Raghuveer. Bispectrum estimation: a digital signal processing framework. *Proceedings of the IEEE*, 75(7):869–891, July 1987.
22. D. R. Brillinger. An introduction to polyspectra. *Annals of Mathematical Statistics*, 36:1351–1374, 1965.
23. Chrysostomos L. Nikias. *Higher Order Moments in Digital Signal Processing*. Prentice Hall, Inc., Englewood Cliffs, New Jersey.
24. Murray Rosenblatt. *Stationary Sequences and Random Fields*. Birkhäuser, Cambridge, Massachusetts, 1985.

PROBLEMS

4.1. **(a)** Determine the mean and autocorrelation function for the random process

$$x[n] = v[n] + 3v[n-1]$$

where $v[n]$ is a sequence of independent random variables with mean μ and variance σ^2.

(b) Is $x[n]$ stationary?

4.2. Random processes $x[n]$ and $y[n]$ are defined by

$$x[n] = v_1[n] + 3v_2[n-1]$$

and

$$y[n] = v_2[n+1] + 3v_1[n-1]$$

where $v_1[n]$ and $v_2[n]$ are independent white noise processes each with variance equal to 0.5.

(a) What are the autocorrelation functions of x and y? Are these processes wide-sense stationary?

(b) What is the cross-correlation function $R_{xy}[n_1, n_0]$? Are these processes jointly stationary (in the wide sense)?

4.3. Beginning with (4.16), give an alternative simpler proof of property (4.18) when R_x is a real-valued correlation function.

4.4. **(a)** Show that if a correlation function can be decomposed and written as

$$R_x[l] = h[l] * h^*[-l]$$

for some suitable nonzero sequence h, then it satisfies condition (4.16) with strict inequality, i.e., it is *positive definite*.

(b) Show that the correlation function in Example 4.4 can be written in the form above where

$$h[l] = \sqrt{3}(-0.5)^l u[l]$$

where $u[l]$ is the unit step function. Therefore prove that the correlation function in the example is positive definite.

4.5. Random processes $v_1[n]$ and $v_2[n]$ are independent and have the same correlation function

$$R_v[n_1, n_0] = 0.5\delta[n_1 - n_0]$$

(a) What is the correlation function of the random process

$$x[n] = v_1[n] + 2v_1[n+1] + 3v_2[n-1]?$$

Is this random process wide-sense stationary?

(b) Find the correlation matrix for a random vector consisting of eight consecutive samples of $x[n]$.

4.6. In the following, $x[n]$ is a sample of a real stationary zero-mean random process. The random vector $\boldsymbol{x}$ is the vector of samples

$$\boldsymbol{x} = [x[0]\ x[2]\ x[3]\ x[4]\ x[5]]^T$$

Note that $x[1]$ is missing. Say if the following statements are correct or incorrect and why.

(a) The covariance matrix for this random vector is a *Toeplitz* matrix.

(b) The covariance matrix for this random vector is a *symmetric* matrix.

(c) If the covariance matrix of the vector $\boldsymbol{x}$ is given, we can determine from it the covariance matrix for the vector

$$\boldsymbol{x}' = [x[0]\ x[1]\ x[2]\ x[3]\ x[4]\ x[5]]^T$$

where $x[1]$ is not missing.

4.7. The density function for a stationary Gaussian random process is described by its mean vector and covariance matrix. Since the process is stationary, the covariance matrix is Toeplitz. It may occur to you that if the inverse of this Toeplitz matrix is also Toeplitz, then the computation of the quadratic product in the exponent of the Gaussian density function would be simplified. Is the inverse of a symmetric Toeplitz matrix Toeplitz? If the answer is "yes," then you must prove it. If the answer is "no," then show it by counterexample.

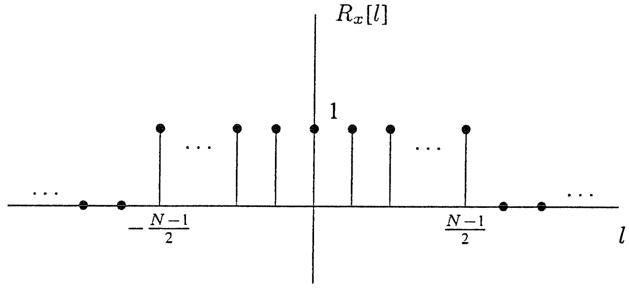
4.8. By considering appropriate combinations of $x[n]$ and $y[n-l]$, prove the bounds (4.36) and (4.37) on the cross-correlation function.

4.9. State whether or not each of the following sequences could represent a legitimate correlation function for a random process. Show why or why not.

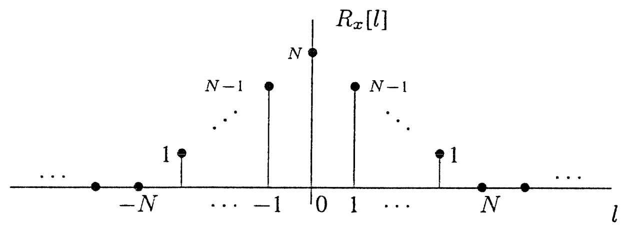
(a)

$$R_x[l] = e^{-l^2}$$

(b)



(c)



4.10. By direct use of the Fourier transform, show that the power spectral density function of the exponential correlation function (4.46) is given by (4.48). Show also that the formula can be derived by evaluating the complex spectral density function in Table 4.2 on the unit circle.

4.11. Find the power spectral density functions corresponding to the following correlation functions and verify that they are real and nonnegative.

(a)

$$R_x[l] = \begin{cases} 3 - |l| & -3 \le l \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$R_x[l] = 2(-0.6)^{|l|} + \delta[l]$$

4.12. Find the complex spectral density functions corresponding to the correlation functions in Problem 4.11 and their regions of convergence.

4.13. Find the correlation function corresponding to the following complex spectral density functions using contour integration.

(a)

$$S_x(z) = \frac{1.5}{z + 2.5 + z^{-1}}$$

(b)

$$S_x(z) = \frac{40z}{-99z^2 + 202z - 99}$$

(c)

$$S_x(z) = \frac{0.6z - 2 + 0.6z^{-1}}{0.6z - 1.36 + 0.6z^{-1}}$$

(d)

$$S_x(z) = \frac{5}{z^2 + \frac{5}{2} + z^{-2}}$$

(e)

$$S_x(z) = -\frac{z - \frac{26}{5} + z^{-1}}{z^2 + \frac{5}{2} + z^{-2}}$$

4.14. Repeat Problem 4.13 using partial fraction expansion.

4.15. Show that the complex spectral density function $S_x(z)$ satisfies the property (4.57). Show further that when R_x is real, $S_x(z)$ satisfies the property (4.58).

4.16. A nonstationary random process $x[n]$ has the correlation matrix

$$\mathbf{R}_x = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(a) What are the DKLT basis functions?

(b) What is the mean-square error in representing the random process as

$$\hat{x}[n] = \kappa_1 \varphi_1[n]$$

where $\varphi_1[n]$ is the first eigenfunction of the process?

(c) What is the mean-square error in representing the process by

$$\hat{x}[n] = \kappa_1 \varphi_1[n] + \kappa_2 \varphi_2[n]$$

where φ_1 and φ_2 are the first two eigenfunctions? Is there more than one choice for the basis functions that would give the same mean-square error?

4.17. Consider representing a segment of a stationary random process with $N = 3$ time samples. The first three values of the correlation function for this random process are given by

$$R[0] = 3, \quad R[1] = 2, \quad R[2] = 1$$

What is the mean-square error in representing the random sequence as

$$\hat{x}[n] = \kappa_1 \varphi_1[n]$$

where $\varphi_1[n]$ is the eigenvector basis function corresponding to the largest eigenvalue of the correlation matrix? Is this larger or smaller than the mean-square error that would be encountered in representing the sequence by

$$\hat{x}[n] = \kappa_2 \varphi_2[n] + \kappa_3 \varphi_3[n]$$

where $\varphi_2[n]$ and $\varphi_3[n]$ are the remaining two eigenvector basis functions?

4.18. A random process $x[n]$ is defined by

$$x[n] = \mathrm{s}[n] + w[n]$$

where $\mathrm{s}[n]$ is a *deterministic* signal and $w[n]$ is a zero-mean white noise sequence with variance σ^2. Consider taking N samples of the random process and defining the vectors

$$\boldsymbol{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}; \quad \mathbf{s} = \begin{bmatrix} \mathrm{s}[0] \\ \mathrm{s}[1] \\ \vdots \\ \mathrm{s}[N-1] \end{bmatrix}; \quad \boldsymbol{w} = \begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[N-1] \end{bmatrix}$$

(a) What is the correlation matrix $\mathbf{R}_x$?
(b) What are the eigenvectors and eigenvalues of the correlation matrix?
(c) What is the mean-square error if $x[n]$ is represented as a truncated Karhunen-Loève expansion using only the first orthonormal basis function?

4.19. State if the following statements are correct or incorrect and why.
(a) The eigenfunctions corresponding to a stationary random process are always sinusoidal.
(b) The mean-square error involved in the DKLT for data with nonzero mean can always be reduced if the mean is first removed.
(c) The complex cross-spectral density function is always nonnegative on the unit circle.
(d) The third and higher moments of a white noise process are always zero.

4.20. In this problem you will show that the magnitude squared coherence satisfies the upper bound in (4.54).
(a) Define a random process

$$v[n] = a_1^* x[n] + a_2^* y[n]$$

where a_1 and a_2 are any arbitrary complex numbers. Show that the correlation function $R_v[l]$ is given by

$$R_v[l] = \mathbf{a}^{*T} \begin{bmatrix} R_x[l] & R_{xy}[l] \\ R_{yx}[l] & R_y[l] \end{bmatrix} \mathbf{a} \quad \text{where} \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

(b) By taking the Fourier transform of this equation, and observing that $S_v(e^{\jmath\omega}) \geq 0$ for any choice of the a_1 and a_2, show that the power spectral density matrix

$$\begin{bmatrix} S_x(e^{\jmath\omega}) & S_{xy}(e^{\jmath\omega}) \\ S_{yx}(e^{\jmath\omega}) & S_y(e^{\jmath\omega}) \end{bmatrix}$$

is positive semidefinite.
(c) Since the power spectral density matrix is positive semidefinite, show that this implies that

$$|\Gamma(e^{\jmath\omega})|^2 = \frac{|S_{xy}(e^{\jmath\omega})|^2}{S_x(e^{\jmath\omega})S_y(e^{\jmath\omega})} \leq 1$$

4.21. Show that the complex cross-spectral density function satisfies the property (4.64). Thus when $z = e^{\jmath\omega}$ the cross-power spectral density satisfies (4.50). Show further that when R_{xy} is *real*, (4.64) becomes equivalent to (4.65).

4.22. What are the correlation functions for the real and imaginary parts of the exponential correlation function in Table 4.2? What is the cross-correlation function? Determine also the complex spectral density function for the real and imaginary parts of the process and sketch its poles and zeros.

4.23. A certain real random process is defined by

$$x[n] = A\cos\omega_o n + w[n]$$

where A is a Gaussian random variable with mean zero and variance σ_A^2 and $w[n]$ is a white noise process with variance σ_w^2 independent of A.

(a) What is the correlation function of $x[n]$?

(b) Can the power spectrum of $x[n]$ be defined? If so, what is the power spectral density function?

(c) Repeat parts (a) and (b) in the case when the cosine has an independent random phase uniformly distributed between $-\pi$ and π.

4.24. (a) Derive a general expression for the correlation function of the random process defined by (4.103) and show that the process is wide-sense stationary if and only if the amplitudes satisfy the orthogonality condition (4.104).

(b) By decomposing a real sinusoid with real random amplitude and uniform phase into the sum of two complex sinusoids; show that such a random process satisfies the orthogonality condition above and is therefore a stationary random process.

(c) A sampled random square wave with discrete period P has the form

$$x[n] = A\,\mathrm{sqr}(nT - \tau)$$

where A is a real random variable. T is the sampling interval, and τ is a random delay parameter uniformly distributed over $[0. PT]$ and independent of A. This signal is comprised of the fundamental frequency and only odd harmonics. Show that this periodic random process is also stationary.

(d) Give an example of a random process in the form of (4.103) that does *not* satisfy the orthogonality conditions (4.104). Recall that each component of the process is assumed to have uniform phase and amplitude distributed independent of phase.

4.25. A sufficient condition for the random process

$$x[n] = Ae^{j\omega n} = |A|e^{j(\omega n+\phi)}$$

to be wide-sense stationary is that A and ϕ be independent and ϕ be uniformly distributed. This condition also guarrantees that the essential requirement

$$\mathcal{E}\{x[n_1]x[n_0]\} = 0$$

is satisfied for the complex random process. Show that the foregoing condition is only *sufficient* for the stationarity, but not *necessary*. In other words, show by counterexample that $|A|$ and ϕ need not be independent, and that if they are independent, the phase need not be uniformly distributed in order for the random process to be stationary.

4.26. By following a procedure similar to that in Section 4.1.2, prove the positive semidefinite property for the correlation function of a continuous random process. Then by taking the function $a_c(t)$ in Table 4.7 to be an appropriate combination of continuous-time impulses, show that the property (4.130) holds.

4.27. By following an argument similar to that which was given in Section 4.4 for discrete random processes, show that the positive semi-definite property of the correlation function for continuous random processes implies that the power spectrum is always ≥ 0.

4.28. Find an expression for the power spectral density function of a continuous real random process with correlation function

$$R_{x_c}^c(\tau) = \sigma_x^2 e^{-\frac{|\tau|}{\tau_0}}$$

4.29. A bandlimited random signal is sampled at $T = 1$ ms.

(a) If this signal is used to form a discrete random sequence, what is the highest frequency in hertz that can be represented in the power density spectrum $S_x(e^{j\omega})$?

(b) What must be the maximum bandwidth of the process if the continuous-time signal is to be reconstructed from the discrete-time sequence?

(c) If the continuous random process has a correlation function of the form

$$R_{x_c}^c(\tau) = \sigma_x^2 e^{-\frac{|\tau|}{\tau_0}}$$

what is the smallest value of the parameter τ_0 such that the continuous spectrum at the bandlimit corresponding to the sampling interval above is 40 dB below its value at $f = 0$? This gives a measure of the "correlation time" for a random process that can be adequately represented by samples at the given sampling interval.

4.30. **(a)** Starting with the definition (4.162)(c), show that the real third-order cumulant satisfies the symmetry properties (4.167) and has the symmetry regions depicted in Fig. 4.24(b).

(b) Given the symmetry properties (4.167), show that the bispectrum for a real random process has the symmetry properties (4.170) and (4.171) and that when these are combined, it leads to the 12 regions of symmetry depicted in Fig. 4.25(b).

COMPUTER ASSIGNMENTS

4.1. On the enclosed diskette you will find four real-valued data sets called

S00.DAT
S01.DAT
S02.DAT
S03.DAT

Each of these represents a random sequence with 512 samples. The format of each file is similar; you can look at the file with an editor. The first line contains the integer 512, representing the number of data points. Successive lines of the file contain the floating-point values of the sequence, four numbers per line. (The first four numbers are the first four values of the sequence, the next four numbers are the next four values, and so on.)

(a) Plot each of the sequences and generate hard copies of the plots. Tell whether you think each of these sequences is positively correlated, negatively correlated, or more-or-less uncorrelated.

(b) Compute the mean of each sequence as a signal average. Do the results of your computation seem reasonable based on the plots?

PROBLEMS

5.1. A linear shift-invariant system has the impulse response shown in Fig. PR5.1.

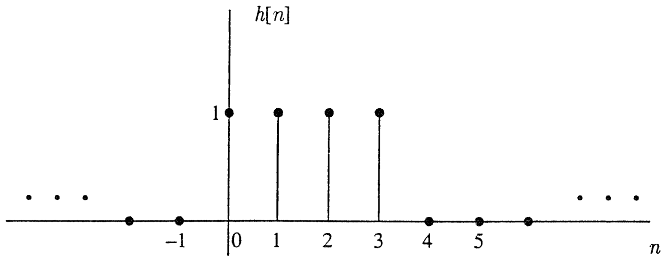


Figure PR5.1

If the system is driven with stationary white noise with variance σ_o^2,
(a) What is the mean of the output?
(b) What is the cross-correlation function between input and output?
(c) What is the correlation function of the output?

5.2. A random process with correlation function

$$R_x[l] = \rho^{|l|}$$

is passed through a linear shift-invariant system with impulse response

$$h[n] = \delta[n] - \delta[n-1]$$

(a) Compute and sketch the cross-correlation function $R_{yx}[l]$.
(b) Compute and sketch the output correlation function $R_y[l]$.

5.3. A white noise process is passed through an ideal lowpass filter whose frequency response is sketched in Fig. PR5.3.

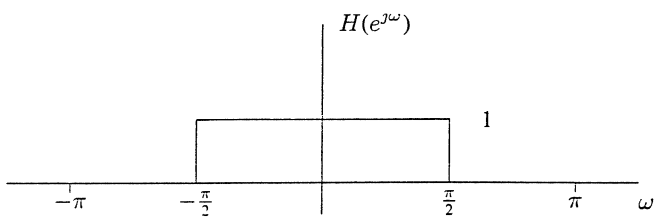


Figure PR5.3

(a) What is the mean of the output of the filter?

(b) What is the correlation function of the output?

5.4. A linear system is defined by

$$y[n] = 0.7y[n-1] + x[n] - x[n-1]$$

(a) Compute the first four values of $R_{yx}[l]$ if it is known that $R_x[l] = \delta[l]$.
(b) What is $R_{xy}[l]$ for $-3 \le l \le 3$?
(c) What is the power spectral density function $S_y(e^{\jmath\omega})$?

5.5. A causal linear shift-invariant system is described by the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n-1]$$

Observe that the correlation and cross-correlation functions satisfy the difference equations

$$R_{yx}[l] - \frac{5}{6}R_{yx}[l-1] + \frac{1}{6}R_{yx}[l-2] = R_x[l-1]$$

$$R_y[l] - \frac{5}{6}R_y[l-1] + \frac{1}{6}R_y[l-2] = R_{yx}[1-l]$$

(a) Show that if the input x is a white noise process with unit variance, then the solution to the first equation is

$$R_{yx}[l] = 6\left(\left(\frac{1}{2}\right)^l - \left(\frac{1}{3}\right)^l\right)u[l]$$

where $u[l]$ is the unit step function.

(b) The function above is now used as an input to the second equation. Since the equation is driven with the sum of two exponentials, for $l < 0$ it is reasonable to assume that the response will be of the form

$$R_y[l] = c_1\left(\frac{1}{2}\right)^{-l} + c_2\left(\frac{1}{3}\right)^{-l}; \qquad l < 0$$

Since there is no input for $l > 0$ it is reasonable to assume that the only response will be the transient response, which has a similar form:

$$R_y[l] = c_1'\left(\frac{1}{2}\right)^{l} + c_2'\left(\frac{1}{3}\right)^{l}; \qquad l > 0$$

With these considerations, find the solution to the second equation.

(c) What is the system function $H(z)$ of the original system? Use this to find the z-transform of the correlation function $R_y[l]$.

(d) What is the power spectrum $S_y(e^{\jmath\omega})$?

(e) Find the impulse response of the original system and use the convolution relationships (5.13)–(5.15) to find the output correlation function $R_y[l]$. Check your answer with part (c).

5.6. Find a general closed-form expression for the correlation function of the random process $y[n]$ described by the first-order difference equation

$$y[n] + ay[n-1] = x[n] + bx[n-1]$$

when the input $x[n]$ is white noise with variance σ_o^2.

5.7. A signal with correlation function $R[l] = \left(\frac{1}{2}\right)^{|l|}$ is applied to a linear shift-invariant system with impulse response $h[n] = \delta[n] + \delta[n-1]$.
(a) Compute the correlation function of the output.
(b) Compute the power spectrum of the *input*.
(c) Compute the power spectrum of the *output*.

5.8. A random signal $x[n]$ is passed through a linear system with impulse response

$$h[n] = \delta[n] - 2\delta[n-1]$$

(a) Find the cross-correlation function between input and output $R_{xy}[l]$ if the input is white noise with variance σ_o^2.
(b) Find the correlation function of the output $R_y[l]$.
(c) Find the output power spectral density $S_y(e^{j\omega})$.

5.9. The impulse response of a linear shift-invariant system is given by

$$h[n] = \begin{cases} (-1)^n & 0 \leq n \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

A white noise process with variance $\sigma_o^2 = 1$ is applied to this system. Call the input to the system $x[n]$ and the output $y[n]$.
(a) What is the cross-correlation function $R_{xy}[l]$? Sketch this function.
(b) What is the correlation function $R_y[l]$ of the output? Sketch this neatly.

5.10. A real random process with correlation function

$$R_x[l] = 2(0.8)^{|l|}$$

is applied to a linear shift-invariant system whose difference equation is

$$y[n] = 0.5y[n-1] + x[n]$$

(a) What is the complex spectral density function of the output $S_y(z)$?
(b) What is the correlation function of the output $R_y[l]$?

5.11. A random process $x[n]$ consists of independent random variables each with uniform density

$$f_x(\mathrm{x}) = \begin{cases} \frac{1}{2} & -1 \leq \mathrm{x} \leq +1 \\ 0 & \text{otherwise} \end{cases}$$

This process is applied to a linear shift-invariant system with impulse response

$$h[n] = \begin{cases} (\frac{1}{2})^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Let the output process be denoted by $y[n]$.
(a) Compute $R_{yx}[l]$.
(b) What is $R_y[l]$?
(c) What is $S_y(z)$? Use this to compute $S_y(e^{j\omega})$.

5.12. A linear shift-invariant system has the impulse response

$$h[n] = 2\delta[n] + \delta[n-1] - \delta[n-2]$$

(a) What is the correlation function of the output if the correlation function of the input is $\delta[l]$? Sketch it versus l.
(b) What is the complex spectral density function of the output?

5.13. A random process is defined by

$$x[n] = s[n] + \eta[n]$$

where $\eta[n]$ is a unit-variance white noise process and $s[n]$ is defined by

$$s[n] = \rho s[n-1] + w[n]$$

where $w[n]$ is another unit-variance white noise process independent of $\eta[n]$.
(a) What is the correlation function $R_x[l]$?
(b) What is $S_x(z)$?

5.14. A linear shift-invariant FIR filter has impulse response

$$h[n] = \delta[n] - \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$$

The filter is driven by a white noise process $x[n]$ with variance σ_o^2. Call the output process $y[n]$.
(a) Determine the cross-correlation function $R_{xy}[l]$.
(b) Determine the correlation function of the output $R_y[l]$.
(c) Determine the output spectral density function $S_y(e^{j\omega})$.

5.15. You are given a causal linear shift-invariant system described by the difference equation

$$y[n] = 0.2y[n-1] + x[n]$$

What is:
(a) the correlation function of the output $R_y[l]$ when the input is white noise with unit variance?
(b) the complex spectral density function of the output $S_y(z)$?
(c) the power spectral density of the output $S_y(e^{j\omega})$ when the input is a zero-mean random process with correlation function

$$R_x[l] = 2(0.5)^{|l|}$$

5.16. By using the properties of reversal, the fact that the eigenvalue is real, and the fact that $\mathbf{R}_\eta$ is a Toeplitz matrix, show that the eigenvalue $\lambda = \tilde{\mathbf{s}}^T\mathbf{R}_\eta^{-1}\tilde{\mathbf{s}}^*$ can be written as $\mathbf{s}^{*T}\mathbf{R}_\eta^{-1}\mathbf{s}$. Also show that the matched filter (5.56) satisfies (5.50) and when substituted in (5.54) results in the eigenvalue given by (5.55).

5.17. A waveform frequently used in radar and sonar is the CW pulse

$$\cos(\omega_o n)(u[n] - u[n-P])$$

where $u[n]$ is the unit step function.
(a) Find the matched filter corresponding to this signal for the case of white noise with variance σ_o^2.
(b) Find the form of the matched filter for noise with the exponential correlation function

$$R_\eta[l] = \frac{\sigma_o^2}{1-\rho^2}\rho^{|l|}$$

(It is not necessary to evaluate the SNR explicitly for this part.)

5.18. Show that the expressions for the impulse response and SNR for the colored noise matched filter in Example 5.4 generalize to the closed-form expressions given in that example for an arbitrary value of P.

5.19. Compute and sketch the impulse response of the matched filter in Example 5.4 for the following values of a and ρ. Assume that $\sigma_o^2 = 0.25$.

(a) $a = 0.7$, $\rho = 0$
(b) $a = 0.7$, $\rho = 0.2$
(c) $a = 0.7$, $\rho = 0.9$
(d) $a = 0.9$, $\rho = 0.7$
(e) $a = 0.9$, $\rho = -0.4$

5.20. The matched filter for a deterministic signal can be formulated in the frequency domain as follows. Let $S_\eta(e^{\jmath\omega})$ represent the power spectral density function of the noise. Let $\mathrm{S}(e^{\jmath\omega})$ and $H(e^{\jmath\omega})$ represent the Fourier transforms of the signal and the matched filter respectively. Now in order to maximize SNR, impose the constraint $y_s[n_p] = 1$ and *minimize* $\mathcal{E}\left\{|y_\eta[n_p]|^2\right\}$.

(a) By using the method for constrained minimization of a complex quantity (see Appendix A), show that this leads to minimizing the complex Lagrangian

$$\mathcal{L} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big\{|H(e^{\jmath\omega})|^2 S_\eta(e^{\jmath\omega}) + \lambda\left(1 - H(e^{\jmath\omega})\mathrm{S}(e^{\jmath\omega})e^{\jmath\omega n_p}\right) + \lambda^*\left(1 - H^*(e^{\jmath\omega})\mathrm{S}^*(e^{\jmath\omega})e^{-\jmath\omega n_p}\right)\Big\}\, d\omega$$

where λ is a complex Lagrange multiplier.

(b) To find the matched filter it is necessary to choose $H(e^{\jmath\omega})$ to minimize the expression in part (a). Some results from the calculus of variations show that it is possible to take the gradient inside the integral, treating $H(e^{\jmath\omega})$ as if it were an ordinary variable instead of a function. By forming ∇_{H^*} of the integrand and setting it to zero, show that the expression for the optimal filter is given by

$$H(e^{\jmath\omega}) = \frac{\lambda^*\mathrm{S}^*(e^{\jmath\omega})e^{-\jmath\omega n_p}}{S_\eta(e^{\jmath\omega})}$$

Since SNR is independent of any scale factor, we can argue that a suitable filter is obtained if we ignore the constant λ^*, that is,

$$H(e^{\jmath\omega}) = \frac{\mathrm{S}^*(e^{\jmath\omega})e^{-\jmath\omega n_p}}{S_\eta(e^{\jmath\omega})}$$

Note that the resulting filter may be IIR and/or noncausal.

5.21. **(a)** Using the frequency domain expression for the matched filter developed in Problem 5.20, develop the form of the matched filter in colored noise for the simple transient signal of Example 5.4.

(b) Express the matched filter of part (a) in the signal domain and compare it to the matched filter derived in Example 5.4. Since the filter is not constrained to be FIR, the two results are not necessarily identical.

5.22. In this problem we explore an alternative derivation of the matched filter for a deterministic signal starting with the general expression

$$\text{SNR} = \frac{\mathbf{h}^{*T}\tilde{\mathbf{s}}^{*}\tilde{\mathbf{s}}^{T}\mathbf{h}}{\mathbf{h}^{*T}\mathbf{R}_{\eta}\mathbf{h}}$$

(a) For the case of white noise, $\mathbf{R}_{\eta} = \sigma_{o}^{2}\mathbf{I}$. The expression for SNR thus reduces to

$$\text{SNR} = \frac{\mathbf{h}^{*T}\tilde{\mathbf{s}}^{*}\tilde{\mathbf{s}}^{T}\mathbf{h}}{\sigma_{o}^{2}\mathbf{h}^{*T}\mathbf{h}}$$

By application of the Schwartz inequality, show that the value of $\mathbf{h}$ that maximizes SNR is given by

$$\mathbf{h} = K\tilde{\mathbf{s}}^{*}$$

where K is an arbitrary constant.

(b) For colored noise $\left(\mathbf{R}_{\eta} \neq \sigma_{o}^{2}\mathbf{I}\right)$, show by using the transformation

$$x' = \mathbf{R}_{\eta}^{-1/2}x$$

that the colored noise problem reduces to the white noise problem and that the resulting filter for the colored noise case is given by

$$\mathbf{h} = K\mathbf{R}_{\eta}^{-1}\tilde{\mathbf{s}}^{*}$$

5.23. The matched filter can be derived from the point of view of optimal detection theory for Gaussian random processes. Assume that the signal is deterministic and that the noise is a zero-mean complex Gaussian random process. Further assume that the exact starting point n_{o} of the signal is known and that the signal has length P. Then there are two possible hypotheses, namely:

$$\begin{array}{lll} \mathrm{H}_1: & x[n] = \mathrm{s}[n] + \eta[n] & \text{(signal present)} \\ \mathrm{H}_0: & x[n] = \eta[n] & \text{(signal not present)} \end{array}$$

Let x be the vector of the P samples $x[n_{o}], x[n_{o}+1], \ldots, x[n_{p}]$. Then optimal detection theory states that the optimal test for the presence of the signal has the form

$$\ln \frac{f_{x}^{(1)}(\mathbf{x})}{f_{x}^{(0)}(\mathbf{x})} \begin{cases} \geq \vartheta & \text{decide } \mathrm{H}_1 \\ < \vartheta & \text{decide } \mathrm{H}_0 \end{cases}$$

where $f_{x}^{(i)}$ is the density function for x under hypothesis i, and ϑ is some appropriate threshold value. The quantity on the left is called the *log likelihood ratio*.

(a) Find the densities $f_{x}^{(1)}$ and $f_{x}^{(0)}$ assuming that the signal s$[n]$ has a known deterministic form.

(b) Evaluate the log likelihood ratio and express the test in simplest form, bringing any constant terms to the right side of the equation and incorporating them into a new threshold ϑ'.

(c) Show that the optimal decision rule has the general form

$$\text{Re}\left[\tilde{\mathbf{h}}^{T}\mathbf{x}\right] \geq \vartheta'$$

which can be implemented with a matched filter. What is the formula for the filter $\mathbf{h}$?

(d) Now assume that s is a zero-mean Gaussian random process, independent of the noise, with correlation matrix $\mathbf{R}_s$. Show that in this case the likelihood ratio test *cannot* be implemented directly as a matched filter.

5.24. Following the derivations that were given in Section 5.1 for discrete random processes, show that for continuous random processes, if $y_c(t) = h_c(t) * x_c(t)$, where $*$ denotes continuous convolution, then

$$m_{y_c}^c = m_{x_c}^c \int_{-\infty}^{\infty} h_c(\tau)d\tau$$

and

$$R_{y_c x_c}^c(\tau) = h_c(\tau) * R_{x_c}^c(\tau)$$
$$R_{x_c y_c}^c(\tau) = h_c^*(-\tau) * R_{x_c}^c(\tau)$$
$$R_{y_c}^c(\tau) = h_c(\tau) * R_{x_c y_c}^c(\tau) = h_c(\tau) * h_c^*(-\tau) * R_{x_c}^c(\tau)$$

Also show that

$$S_{y_c}^c(f) = |H_c(f)|^2 S_{x_c}^c(f)$$

where $H_c(f)$ is the frequency response of the continuous filter $h_c(t)$.

5.25. A causal linear shift-invariant system is described by the difference equation

$$y[n] - 0.6y[n-1] = x[n] + 1.25x[n-1]$$

The input to this system has a power spectral density function

$$S_x(e^{j\omega}) = \frac{1}{1.64 + 1.6\cos\omega}$$

(a) What is the power spectral density function of the output $S_y(e^{j\omega})$?

(b) Is the system represented by the difference equation minimum-phase? If not, give the difference equation for the system with the same magnitude frequency response that *is* minimum-phase.

5.26. In Section 5.5.2 it was shown that the Paley–Wiener condition (5.73) is sufficient to define a regular process. This problem investigates the *necessity* of the condition. In other words, it shows that if a spectral density function can be factored as

$$S_x(z) = \mathcal{K}_o H_{ca}(z) H_{ca}^*(1/z^*)$$

which implies that

$$S_x(e^{j\omega}) = \mathcal{K}_o |H_{ca}(e^{j\omega})|^2$$

then it satisfies the Paley–Wiener condition. Note that if the last equation is written for simplicity as

$$S_x(e^{j\omega}) = |H'_{ca}(e^{j\omega})|^2$$

where $H'_{ca}(e^{j\omega}) = \sqrt{\mathcal{K}_o} H_{ca}(e^{j\omega})$, then the Paley–Wiener condition can be written as

$$\int_{-\pi}^{\pi} |\ln S_x(e^{j\omega})|\, d\omega = 2\int_{-\pi}^{\pi} |\ln |H'_{ca}(e^{j\omega})||\, d\omega < \infty$$

Thus if $\gamma(\omega)$ is defined as

$$\gamma(\omega) = \ln|H'_{ca}(e^{j\omega})|$$

then it is equivalent to show that

$$\int_{-\pi}^{\pi} |\gamma(\omega)|\, d\omega < \infty$$

(a) Define the function

$$\gamma^+(\omega) = \begin{cases} \gamma(\omega) & \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that

$$\int_{-\pi}^{\pi} e^{2\gamma(\omega)} d\omega > 2\int_{-\pi}^{\pi} \gamma^+(\omega) d\omega$$

Further show that since

$$\int_{-\pi}^{\pi} S_x(e^{j\omega}) d\omega < \infty$$

this implies that

$$\int_{-\pi}^{\pi} \gamma^+(\omega) < \infty$$

(b) Show also that

$$\int_{-\pi}^{\pi} \gamma(\omega) d\omega < \infty$$

[Assume that $H'_{ca}(z)$ has no zeros on the unit circle.]

(c) Finally, show from parts (a) and (b) that

$$2\int_{-\pi}^{\pi} |\gamma(\omega)| d\omega = 2\int_{-\pi}^{\pi} |\ln|H'_{ca}(e^{j\omega})|| d\omega < \infty$$

5.27. Factor the following complex spectral density functions into minimum- and maximum-phase components.

(a)

$$S_x(z) = \frac{-16}{12z - 25 + 12z^{-1}}$$

(b)

$$S_x(z) = -\frac{3 - 10z^{-2} + 3z^{-4}}{3 + 10z^{-2} + 3z^{-4}}$$

5.28. A random process has the complex spectral density function

$$S_x(z) = \frac{z - 2.5 + z^{-1}}{z - 2.05 + z^{-1}}$$

(a) Factor this function into minimum- and maximum-phase terms. What are the poles and zeros?

(b) Find the innovations representation of the random process.

5.29. Find the innovations representation for the random processes whose complex spectral density functions are given by

(a)

$$S_x(z) = \frac{54z - 180 + 54z^{-1}}{72z - 145 + 72z^{-1}}$$

(b)

$$S_x(z) = \frac{-5z^{-2}}{1 - \frac{5}{2}z^{-2} + z^{-4}}$$

(c)

$$S_x(z) = \frac{121}{110z^4 + 221 + 110z^{-4}}$$

5.30. Combine (5.87) and (5.88) with (5.90) and simplify to derive the relation (5.91). You will need to be careful about your choice of summation indices when combining these equations.

5.31. A random process whose third-order cumulant is

$$C_x^{(3)}[l_1, l_2] = \beta_o \delta[l_1]\delta[l_2]$$

is input to the linear shift-invariant system of Problem 5.1.

(a) What is the third-order cumulant of the output? Check to see that it has the proper symmetry.

(b) What is the bispectrum of the output process? Specify magnitude and phase.

5.32. Consider the first-order difference equation

$$y[n] - \rho y[n-1] = x[n]$$

By generalizing the procedure in Section 5.2, show how to compute the third-order cumulant $C_y^{(3)}[l_1, l_2]$ from $C_x^{(3)}[l_1, l_2]$ by solving a series of three two-dimensional difference equations. Define any cross-cumulants such as $C_{xyy}^{(3)}[l_1, l_2]$ that you need in order to do this.

COMPUTER ASSIGNMENTS

5.1. The sequence S00.DAT on the enclosed disk is a realization of a white noise process with $\sigma_o^2 = 1$. Call this sequence $x[n]$. Generate another sequence $y[n]$ by applying this sequence to a first-order filter that is described by the difference equation

$$y[n] = \rho y[n-1] + x[n]$$

Generate three separate filtered sequences using each of the values $\rho = 0.95$, $\rho = 0.7$, and $\rho = -0.95$. Plot each of these sequences and the white noise sequence. What differences do you observe?