# EE 504

Homework #5
Due: May 7, 2004

**P.1: [Griffiths' p-vector algorithm]** (Hayes 9.14)
Griffiths has developed an algorithm for LMS adaptive filters. The method is based on a variation of LMS in which the desired signal is not explicitly present at the filter output but a-priori knowledge of the cross-correlation between the input and the desired output is given. Specifically, recall that the LMS algorithm has the coefficient update equation as follows:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu e(n)\mathbf{x}(n) = \mathbf{w}_n + \mu d(n)\mathbf{x}(n) - \mu \left[\mathbf{w}_n^T \mathbf{x}(n)\right] \mathbf{x}(n)$$

If we replace $d(n)\mathbf{x}(n)$ with its expected value $\mathbf{p} = E\{d(n)\mathbf{x}(n)\}$, we get the p-vector algorithm of Griffiths'.

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu \mathbf{p} - \mu \left[\mathbf{w}_n^T \mathbf{x}(n)\right] \mathbf{x}(n)$$

Note that this update does not require $d(n)$ explicitly.

a) Derive an expression on $\mu$ for convergence in the mean.

b) Develop a leaky p-vector algorithm by writing down the filter coefficient update equation. Determine the range of $\mu$ values for its convergence in the mean and find $\lim_{n\to\infty} E\{\mathbf{w}_n\}$.

c) **[Computer Exercise]** Let $x(n)$ be a process that is generated according to the difference equation

$$x(n) = 1.2728x(n-1) - 0.81x(n-2) + v(n)$$

where $v(n)$ is unit variance white Gaussian noise. Suppose we would like to implement a linear predictor for $x(n)$:

$$\hat{x}(n) = w_n(1)x(n-1) + w_n(2)x(n-2)$$

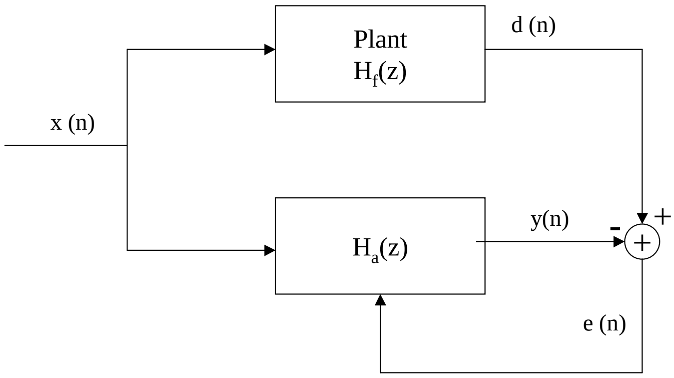
i) Generate a sequence x(n) of length N = 500 and determine a step size so that $\mathbf{w}_n$ converges in 200 iterations (set the time constant $\tau = \frac{1}{\mu \lambda_{min}}$ to 200).

ii) Implement the LMS adaptive predictor and plot $w_n(k)$ versus n for $k = 1, 2$.

iii) Find the p-vector, $\mathbf{p} = E\{d(n)\mathbf{x}(n)\}$. And implement the p-vector algorithm (modify llms.m) Plot $w_n(k)$ and compare with part ii).

iv) Investigate the sensitivity of the p-vector algorithm to the errors in the p-vector. Make some small changes to the p-vector of part iii) and re-run the algorithm.

**P.2: [Feintuch's Algorithm]** (Hayes 9.22)
Consider a system identification problem.




The unknown plant has the transfer function of the form:

$$H_f(z) = \frac{0.05 - 0.4z^{-1}}{1 - 1.1314z^{-1} + 0.25z^{-2}}$$

The adaptive that is used to model $H_f(z)$ has two free parameters $a$ and $b$.

$$H_a(z) = \frac{b}{1 - az^{-1}}$$

The input to the both systems is unit variance white noise. The goal is to find $a$ and $b$ that minimizes $\varepsilon = E\{(e(n))^2\}$ where $e(n) = d(n) - y(n)$. The error function for the minimization has a global minimum at $(b, a) = (-0.311, 0.906)$ and a local minimum at $(b, a) = (0.114, -0.519)$.

In order for the filter coefficients $a_n$ and $b_n$ in $H_a(z)$ to converge in the mean using Feintuch's algorithm, it is necessary that

$$\lim_{n\to\infty} E\{e(n)x(n)\} = 0$$

and

$$\lim_{n\to\infty} E\{e(n)y(n-1)\} = 0$$

a) Find the values of $E\{e(n)x(n)\}$ and $E\{e(n)y(n-1)\}$ at the global minimum of $\varepsilon$. What does this imply about the Feintuch's Algorithm ?

b) Find the stationary point of the Feintuch's adaptive filter, i.e. $(a, b)$ for which $E\{e(n)x(n)\} = 0$ and $E\{e(n)y(n-1)\} = 0$.

(Hint: $d(n) = \sum_{k=0}^{\infty} h_f(k)x(n-k)$ and $y(n) = \sum_{k=0}^{\infty} h_a(k)x(n-k)$. $x(n)$ is unit variance white noise. $e(n) = d(n) - y(n)$)

**P.3: [Equation Error Method]** (Hayes)

Consider the system identification set-up of Problem 2. Assume that the plant has the transfer function of

$$G(z) = \frac{1}{1 - 0.5z^{-1}}$$

and the constructed adaptive filter is of the form

$$H_a(z) = \frac{b}{1 - az^{-1}}$$

Assume that the plant output $d(n)$ is corrupted with white noise of variance $\rho_v^2$.

The Wiener solution minimizing the error $\varepsilon = E\{(d(n) - y(n))^2\}$ is expected to be $(b, a) = (1, 0.5)$ and the minimum error corresponding to optimal coefficients is $\rho_v^2$.

In this problem, we will see that the equation error method is biased when the plant output is noisy. To find the optimum $(a, b)$ according to the equation error method, we need to solve the equation:

$$\mathbf{R}_u \Theta = \mathbf{p} \tag{1}$$

where $u(n) = [x(n) \;\; d(n-1)]^T$ and

$$\mathbf{R}_u = \begin{bmatrix} r_{xx}(0) & r_{xd}(1) \\ r_{xd}(1) & r_{dd}(0) \end{bmatrix}; \;\; \mathbf{\Theta} = \begin{bmatrix} b \\ a \end{bmatrix}; \;\; \mathbf{p} = \begin{bmatrix} r_{dx}(0) \\ r_{dd}(1) \end{bmatrix}$$

The autocorrelation values can be calculated as:

$r_{dx}(0)$ :

$$\begin{aligned} E\{d(n)x(n)\} &= \frac{1}{2\pi i} \oint P_x(z) G(z) \frac{dz}{z} \\ &= \frac{1}{2\pi i} \oint \frac{1}{1 - 0.5z^{-1}} \frac{dz}{z} \\ &= \frac{1}{2\pi i} \oint \frac{1}{z - 0.5} dz \\ &= \text{residue of } \frac{1}{z - 0.5} \text{ at } z = 0.5 \\ &= 1 \end{aligned}$$

$r_{xd}(1)$ : $\quad E\{x(n)d(n-1)\} = 0$
$r_{dd}(0)$ : $\quad E\{d(n)d(n)\} = 4/3 + \rho_v^2$
$r_{dd}(1)$ : $\quad E\{d(n)d(n-1)\} = 2/3$

Then the solution of equation system is:

$$\mathbf{\Theta} = \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \frac{1}{1 + \rho_v^2 \frac{3}{4}} \end{bmatrix}$$

The solution of the equation error method is therefore biased if $\rho_v^2 \neq 0$. Note that the bias is proportional to the noise variance. For zero-noise conditions the equation error system converges to the known Wiener solution $(b, a) = (1, 0.5)$.

a) Assume that the input to the unknown plant is not white but has the correlation of

$$P_x(e^{jw}) = \frac{1}{|1 - 0.3e^{-jw}|^2}$$

Find the Wiener filter coefficients $(b, a)$ minimizing the error $E\{e^2(n)\}$.

b) Find the values for $a$ and $b$ that are optimum according to the equation error method.

c) Repeat part a) and b) for

$$P_x(e^{jw}) = \frac{1}{|1 - 0.8e^{-jw}|^2}$$

d) What do you observe with respect to the biases in the coefficients $a$ and $b$ ? How are the biases affected by the shape of the power spectrum $P_x(e^{jw})$ ?

e) Write Matlab programs to implement **output error method** and **equation error method** and confirm your results derived in parts (a) - (d).