

## EE 503

### Homework #3

Due: November 10, 2005

#### **MATLAB Problem:**

In this problem, we compare the ensemble averages and time averages of a stationary process. The process  $x[n]$  is defined as follows:

$$x[n] = A \cos(w_o n + \phi)$$

where  $\phi$  is uniformly distributed in  $[0, 2\pi]$  and  $w_o = 2\pi/(10\sqrt{2})$ .

a) Assume that  $A$  and  $\phi$  are uncorrelated r.v.'s; calculate the mean of process at sampling points, derive the formula for the auto-correlation of  $x[n]$ .

b) **Ensemble Averaging:** In this part, we will verify derived results of part a. Assume that  $A = 1$  in this part.

Generate 100 realizations of the process, each having 20 samples taken in the interval  $[1, 20]$ . Calculate the mean of 100 realizations vectors, that is the ensemble average for the ensemble of 100 runs. Compare the calculated mean with its expected value.

We will estimate the auto-correlation of *each* realization by `xcorr` command of Matlab. Use `xcorr` to estimate the auto-correlation lags between -5 and 5, i.e.  $r_x(-5)$  to  $r_x(5)$  (use unbiased option of `xcorr`). Calculate the average of auto-correlation estimates found from each of 100 realizations and compare with the derived formula.

c) **Time Averaging:** In this part, we will use a single realization to estimate the mean and auto-correlation lags. Assume that  $A = 1$  in this part.

Generate a single realization of process  $x[n]$  in the interval of  $[1, 20000]$ . We will segment  $x[n]$  and use the segments of varying sizes for the auto-correlation estimation.

The first segment is formed by taking first 200 samples of  $x[n]$ , the second segment contains the first 2000, the third contains the first 4000, the fourth contains first 8000, the fifth contains first 12000, the sixth contains first 16000, the seventh contains all samples of  $x[n]$ , that is all 20000 samples.

For each segment estimate the auto-correlation lags in between -5 and 5 using the `xcorr` command as we did in part b.

Calculate the estimation error vector by subtracting the true auto-correlation values from the estimated ones. Examine the norm (Euclidian norm) of the error vector as the segment size gets bigger.

Repeat the experiment for different realization of 20,000 samples.

d) Repeat part b) and c) when  $A$  is random variable defined as follows:  $A$  is independent from  $\phi$  and normally distributed with zero mean and has the variance of 2. Comment on your results.

### Paper - Pencil Problems:

The problems are taken from the open courseware content of 6.432 Stochastic Processes, Detection, and Estimation course given at MIT. The problems are available online at [www.ocw.mit.edu](http://www.ocw.mit.edu).

#### Problem 1.5

Consider the following  $3 \times 3$  matrices:

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 10 & 3 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} & \mathbf{B} &= \begin{bmatrix} 10 & 5 & 2 \\ 5 & 3 & 3 \\ 2 & 3 & 2 \end{bmatrix} & \mathbf{C} &= \begin{bmatrix} 10 & 5 & 2 \\ 5 & -3 & 3 \\ 2 & 3 & 2 \end{bmatrix} & \mathbf{D} &= \begin{bmatrix} 10 & 5 & 2 \\ -5 & 3 & 1 \\ -2 & -1 & 2 \end{bmatrix} \\ \mathbf{E} &= \begin{bmatrix} 10 & -5 & 2 \\ -5 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix} & \mathbf{F} &= \begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} & \mathbf{G} &= \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \end{aligned}$$

Your answers to the following questions may consist of more than one of the above matrices or none of them. Justify your answers.

- Which of the above could be the covariance matrix of some random vector?
- Which of the above could be the cross-covariance matrix of two random vectors?
- Which of the above could be the covariance matrix of a random vector in which one component is a linear combination of the other two components?
- Which of the above *could be* the the covariance matrix of a vector with statistically independent components? Must a random vector with such a covariance matrix have statistically independent components?

**Problem 2.2**

Let  $x_1$  and  $x_2$  be zero-mean jointly Gaussian random variables with covariance matrix

$$\mathbf{\Lambda}_x = \begin{bmatrix} 34 & 12 \\ 12 & 41 \end{bmatrix}.$$

- (a) Verify that  $\mathbf{\Lambda}_x$  is a valid covariance matrix.
- (b) Find the marginal probability density for  $x_1$ . Find the probability density for  $y = 2x_1 + x_2$ .

- (c) Find a linear transformation defining two new variables

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \mathbf{P} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so that  $x'_1$  and  $x'_2$  are statistically independent and so that

$$\mathbf{P}\mathbf{P}^T = \mathbf{I},$$

where  $\mathbf{I}$  is the  $2 \times 2$  identity matrix.

**Problem 2.8**

In the binary communications system shown below, messages  $m = 0$  and  $m = 1$  occur with a priori probabilities  $\frac{1}{4}$  and  $\frac{3}{4}$  respectively. The random variable  $n$  is independent from  $m$  and takes on the values -1, 0, 1 with probabilities  $\frac{1}{8}$ ,  $\frac{3}{4}$ ,  $\frac{1}{8}$  respectively

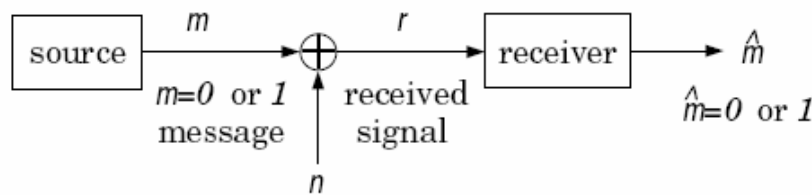


Figure 8-1

Find the receiver which achieves the maximum probability of correct decision. Compute the probability of error for this receiver.

**Problem 4.2**

Let  $x, y, z$  be zero-mean, unit-variance random variables which satisfy

$$\text{var}[x + y + z] = 0$$

Find the covariance matrix of  $(x, y, z)^T$ ; i.e., find the matrix

$$\begin{bmatrix} E[x^2] & E[xy] & E[xz] \\ E[yx] & E[y^2] & E[yz] \\ E[zx] & E[zy] & E[z^2] \end{bmatrix}$$

(Hint: Use vector space ideas.)

**Problem 6.3**

A random process  $x(t)$  is defined as follows:

$$x(t) = \begin{cases} a & t \geq \Theta \\ b & t < \Theta \end{cases}$$

where  $a$ ,  $b$ , and  $\Theta$  are statistically independent unit-variance Gaussian random variables, with means

$$\begin{aligned} E[a] &= 1 \\ E[b] &= -1 \\ E[\Theta] &= 0. \end{aligned}$$

A typical sample function is depicted in Fig. 1-1.

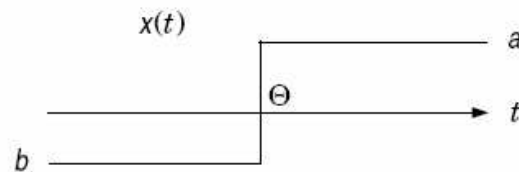


Figure 1-1

Answer each of the following questions concerning this random process, clearly justifying your answer in each case.

- (a) Is  $x(t)$  a Gaussian random process?
- (b) Is  $x(t)$  strict-sense stationary?
- (c) Is  $x(t)$  a Markov process?
- (d) Is  $x(t)$  an independent-increments process?

**Problem 6.7**

Let  $x(t)$  be a non-white, zero-mean, Gaussian, wide-sense stationary, Markov random process; and let

$$y(t) = x(|t|).$$

Answer each of the following questions concerning this new random process  $y(t)$ , clearly justifying your answer in each case.

- (a) Is  $y(t)$  a Gaussian random process?
- (b) Is  $y(t)$  strict-sense stationary?
- (c) Is  $y(t)$  a Markov process?
- (d) Is  $y(t)$  an independent-increments process?