EE 503

Homework #1
Due: September 29, 2005

Pr.1: The operator $\mathcal{L}$ is defined in the space of 2nd degree polynomials as follows:

$$\mathcal{L}\{f(x)\}(x) = (x+2)\frac{d}{dx}\{f(x)\}$$

a) Find the matrix equivalent of the operator in the canonical basis, i.e. $e_0 = 1$, $e_1 = x$ and $e_2 = x^2$.

b) Find a coordinate system that $\mathcal{L}$ is represented by a diagonal matrix.

c) Determine the 2nd degree polynomial solutions of the following differential equation. Use results of part b.

$$(x+2)\frac{d}{dx}f(x) = \lambda f(x)$$

For which values of λ, $f(x)$ satisfying the differential equation is different from zero.

Pr.2: Operators $\mathcal{P}$ and $\mathcal{Q}$ are defined in the space of 2nd degree polynomials. $\mathcal{P}$ is the multiplication by x operator. $\mathcal{Q}$ is the differentiation by x.

a) Using the matrix representation of the operators, show that $\mathcal{QP} - \mathcal{PQ} = \mathcal{I}$ where $\mathcal{I}$ is the identity operator. (The operator $\mathcal{QP} - \mathcal{PQ}$ is called the commutator of $\mathcal{P}$ and $\mathcal{Q}$ and it is denoted by $[\mathcal{Q}, \mathcal{P}]$.)

b) Show that the result of identity commutator is valid in any coordinate system.

Pr.3: Let $f(x)$ be a second degree polynomial, $f(x) = a + bx + cx^2$. The operator $\mathcal{L}_{x_k}$ is defined as the evaluation operator at $x = x_k$, i.e. $\mathcal{L}_{x_k}\{f(x)\} = f(x_k)$. An evaluation vector of length 3 is defined by the concatenation evaluations, $[\mathcal{L}_{x_k}\{f(x)\}\ \mathcal{L}_{x_l}\{f(x)\}\ \mathcal{L}_{x_m}\{f(x)\}]^T$.

a) Find the mapping between the vector representation of $f(x)$ in the canonical coordinate system and the evaluation vector. For this part and the rest of the problem assume that the evaluations are taken at -1, 0 and 1.

b) Is the mapping found in part a invertible ? In other words, is it possible to determine the polynomial given its evaluation vector ?

c) Find the change of basis matrix from the canonical basis to the $\beta_1(x) = x^2 + x$, $\beta_2(x) = x^2 - x$ and $\beta_3(x) = x^2 - 1$.

c) Using the change of basis matrix, express the mapping found in part a) in β basis. Which representation results in a matrix easier to invert ? That is which representation is easier to use if the evaluation vector is given and the polynomial $f(x)$ is asked to be determined ?

Pr.4: Permutation matrices interchange the entries of a vector. Page 14 in the hand-out distributed contains more information on permutation matrices. In this problem, we examine some fundamental matrix operations using permutation matrices.

a) Show that square of a permutation matrix is the identity matrix, $\mathbf{P}\mathbf{P}^T = \mathbf{I}$.

b) Using the result of part a, conclude that the eigenvalues of permutation matrices are either 1 or -1.

c) Write the permutation matrix $\mathbf{J}$ mapping kth entry of a length P vector to the $(-k)_P$th entry. $(-k)_P$ denotes $-k \equiv$ mod P. Take $P = 5$ for the rest of the problem. And to be consistent with DSP literature assume that the first entry of the vector has the index 0. That is, the vector indices are numbered as $0, 1, 2, 3, 4$. And the mapping described maps the indices $(0, 1, 2, 3, 4) \rightarrow (0, 4, 3, 2, 1)$.

d) Find the eigenvectors of $\mathbf{J}$ by inspection. (No calculation is required, find a vector which remains the same (or negated) after the permutation)

e) Define $\mathbf{P_e} = \frac{1}{2}(\mathbf{I} + \mathbf{J})$ or $\mathbf{P_o} = \frac{1}{2}(\mathbf{I} - \mathbf{J})$. Show that $\mathbf{P}$ matrices have the eigenvalues of $\{$ 0,1$\}$ using part b.

f) Determine $\mathbf{P_e^2}$ and $\mathbf{P_o^2}$ from their definitions given in part c) to verify that eigenvalues are indeed $\{0, 1\}$.

g) Show that any vector can be written as

$$\mathbf{x} = \mathbf{P_e}\mathbf{x} + \mathbf{P_o}\mathbf{x}$$

h) Find the eigenvectors of $\mathbf{P_e}$ and $\mathbf{P_o}$. Are they the same as the ones in part d ?

i) Use the eigen decomposition of $\mathbf{P}$ matrices to express $\mathbf{P_e}$ and $\mathbf{P_o}$ in the form

$$\mathbf{P} = \sum_{k=1}^{5} \mathbf{v_k} \lambda_k \mathbf{v_k}^T$$

j) Insert eigen decomposition of $\mathbf{P}$ matrices in $\mathbf{x} = \mathbf{P_e}\mathbf{x} + \mathbf{P_o}\mathbf{x}$ and interpret the result in terms of inner products, projections. Establish the link between the even - odd decomposition of a vector and the problem ?

Electrical Engin. Department,
METU, Fall 2005
CC.